Open Coloring Axiom and Forcing Axioms

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(This is still a work in progress: any comments or suggestions are recommended!)

The following text is primarily thought as a private note on what is known about Open Coloring Axiom (OCA) and its relations to some Forcing Axioms. None of the results treated here are mine. On the other hand, some people might find the present note useful since it contains some proofs which are either unpublished or not easily accessible.

In section 3, I give a detailed and (hopefully) self-contained presentation of the consistency proof of OCA from ZFC.

In sections 10 and ... we present (equi-)consistency proofs of some weak forcing axioms with relatively small large cardinals.

Everything in sections 1–5 is already stated (at least implicitly) in [12] unless claimed otherwise. I also learned some of the ideas from [3], [13], [14], etc. In section 8, we present a result of Farah in [5] that the isomorphism of $\aleph_1$-dense sets does not follow from OCA. Section 10.1 is based on [7].

Credits for some other results may be missing or incomplete. If this is the case then it is simply because of my present location: the nearest reasonable mathematics library is about 322km (200 miles) far from my place.

Contents

1 Open Coloring Axiom

Suppose that X is a topological space and $[X]^2 = K_0 \mathbin{\dot\cup} K_1$. In the following, we often identify $[X]^2$ with $X^2 \setminus \Delta_X$ and K_i with $\tilde{K}_i = \{\langle x,y\rangle : \{x,y\} \in K_i\}$. A partition $[X]^2 = K_0 \mathbin{\dot\cup} K_1$ is said to be an *open coloring* if $\tilde{K}_0$ is an open (symmetric) subset of $X^2 \setminus \Delta_X$. $Y \subseteq X$ is said to be *i-homogeneous* if $[Y]^2 \subseteq K_i$.

For a topological space X let $\mathrm{OCA}(X)$ denote the following assertion:

$\mathrm{OCA}(X)$: For any open coloring $[X]^2 = K_0 \mathbin{\dot\cup} K_1$, one of the following two cases holds:

(a) X is the union of countably many 1-homogeneous sets;

(b) There is an uncountable 0-homogeneous $Y \subseteq X$.

The Open Coloring Axiom (OCA) is the statement: $\mathrm{OCA}(X)$ *for all* $X \subseteq \mathbb{R}$.

If (a) above holds, then we say that X is *σ-1-homogeneous* (with respect to the coloring $[X]^2 = K_0 \mathbin{\dot\cup} K_1$).

Lemma 1 *Suppose that X, Y are topological spaces. If Y is T_0 and there is a continuous bijection $f : X \to Y$ then $\mathrm{OCA}(X)$ implies $\mathrm{OCA}(Y)$.* □

Lemma 2 *If Y is a T_2 second countable space then there is $X \subseteq {}^\omega 2$ and a continuous surjection $f : X \to Y$.*

Proof Let O_n, $n \in \omega$ be open basis of Y, For $y \in Y$ let $g_y : \omega \to 2$ be defined by

$$g_y(n) = \begin{cases} 1, & \text{if } y \in O_n \\ 0, & \text{otherwise.} \end{cases}$$

By assumption, for y, $y' \in Y$ if $y \neq y'$ then $g_y \neq g_{y'}$. Let $X = \{g_y : y \in Y\}$ and let $f : X \to Y$ be defined by $f(g_y) = y$. Then f is well-defined and continuous. □ (Lemma 2)

Corollary 3 *The following are equivalent:*

(a) OCA;

(b) $\mathrm{OCA}(X)$ *for all* $X \subseteq \mathbb{R}^n$ *for some/any* $n \geq 1$;

(c) $\mathrm{OCA}(X)$ *for all* $X \subseteq {}^\omega\omega$;

(d) $\mathrm{OCA}(X)$ *for all* $X \subseteq {}^\omega 2$;

(e) $\mathrm{OCA}(X)$ *for all separable metric space* X. □

2 A proper p.o. forcing OCA

Let $[X]^2 = K_0 \mathbin{\dot\cup} K_1$ and let θ be sufficiently large. Let $P = \langle P, \leq\rangle$ be the partial ordering defined by:

$$p \in P \quad \Leftrightarrow \quad p = \langle X_p, \mathcal{M}_p\rangle \textit{ such that:}$$

(1) *$X_p \in [X]^{<\aleph_0}$ and X_p is 0-homogeneous;*

(2) *$\mathcal{M}_p$ is a finite non-empty set of countable elementary submodels M of $\mathcal{H}(\theta)$ with X, $\mathcal{O}$, K_1, $K_2 \in M$ such that elements of $\mathcal{M}_p$ build an $\in$-chain;*

(3) *For each $x \in X_p$ there is $M \in \mathcal{M}_p$ such that $x \notin M$;*

(4) *For each distinct x, $y \in X_p$, x and y are separated by an element of $\mathcal{M}_p$. I.e. there is $M \in \mathcal{M}_p$ such that $x \in M \wedge y \notin M$ or $x \notin M \wedge y \in M$ holds;*

(5) *For $M \in \mathcal{M}_p$ and $x \in X_p \setminus M$, $x \notin \bigcup\{H \in M : M \models$"$H$ is 1-homogeneous"$\}$;*

(6) *For p, $q \in P$, $p \leq q \Leftrightarrow X_p \supseteq X_q$ and $\mathcal{M}_p \supseteq \mathcal{M}_q$.*

Note that, by (2) and (4), there is a natural linear ordering on X_p defined by: $x \lhd y \Leftrightarrow x \in M \;\wedge\; y \notin M$ for some $M \in \mathcal{M}_p$.

Theorem 4 *Suppose that X is a separable metric space with a countable open base $\mathcal{O}$ of X and*

($*$) *$[X]^2 = K_0 \,\dot\cup\, K_1$ is an open partition such that no nonempty $O \in \mathcal{O}$ is a union of countably many 1-homogeneous sets.*

Suppose further that θ and $\langle P, \leq\rangle$ are as above. Then:

(a) *For any $p \in P$ and countable $M \prec \mathcal{H}(\theta)$ such that $\bigcup \mathcal{M}_p \in M$, there is $x \in X \setminus M$ such that $\langle X_p \cup \{x\}, \mathcal{M}_p \cup \{M\}\rangle \in P$.*

(b) *P is proper.*

Proof First note that, from ($*$), it follows that

($**$) For any $O \in \mathcal{O}$ and any countable family $\mathcal{F}$ of 1-homogeneous sets, $O \setminus \bigcup \mathcal{F}$ is uncountable.

For (a), suppose that $p \in P$, say $p = \langle X_p, \mathcal{M}_p\rangle$, and that $M \prec \mathcal{H}(\theta)$ is countable such that $\bigcup \mathcal{M}_p \in M$.

First let us assume that $X_p \neq \emptyset$, say $X_p = \{x_1, \ldots, x_n\}$ where we assume that $x_1, \ldots, x_n$ are enumerated in increasing order[1] and separated by $M_1, \ldots, M_{n-1} \in \mathcal{M}_p$. Since X_p is 0-homogeneous by (1) and K_0 is open by assumption, there is an $O \in \mathcal{O}$ such that $x_n \in O$ and $\{x_i\} \times O \subseteq K_0$ for all $1 \leq i < n$.

Let

$$S = \{x \in O : \{x\} \times (O \setminus \{x\}) \subseteq K_1\}.$$

Since $O \in M_{n-1}$ by (2) (recall that $\mathcal{O}$ is countable), $S \in M_{n-1}$ and $M_{n-1} \models$"S is 0-homogeneous". Hence $x_n \notin S$ by (5). Thus we have $\{x_n\} \times (O \setminus \{x_n\}) \not\subseteq K_1$. Since K_0 is open, there is some non-empty rational open interval $O' \subseteq O$ such that $\{x_n\} \times O' \subseteq K_0$. Let

$$H^* = \bigcup\{H \in M : M \models \text{"}H \subseteq X \;\wedge\; H \text{ is 1-homogeneous"}\}.$$

[1] With respect to the natural linear ordering of elements X_p induced from the separation by elements of $\mathcal{M}_p$.

By $(**)$, we can take an $x \in (O' \setminus H^*) \setminus M$. Clearly

$$p^* = \langle X_p \cup \{x\}, \mathcal{M}_p \cup \{M\}\rangle \in P.$$

The case $X_p = \emptyset$ can be treated similarly by taking

$$H^* = \bigcup\{H \in M \,:\, M \models \text{“}H \subseteq X \ \wedge\ H \text{ is 1-homogeneous”}\}$$

and $x \in (X \setminus H^*) \setminus M$.

(b): Let λ be a sufficiently large regular cardinal, in particular $\theta < \lambda$. Let $M \prec \mathcal{H}(\lambda)$ be countable with θ, X, $\mathcal{O}$, K_0, $K_1 \in M$ and let $p \in P \cap M$. To prove that P is proper, it is enough to show that there is M-generic $p' \leq p$.

Note that $P \in M$. Since $\mathcal{H}(\theta) \in \mathcal{H}(\lambda)$, we have that $M \cap \mathcal{H}(\theta) \prec \mathcal{H}(\theta)$. Hence, letting $M_0 = M \cap \mathcal{H}(\theta)$,

$$p' = \langle X_p, \mathcal{M}_p \cup \{M_0\}\rangle$$

is an element[2] of P and $p' \leq p$. We show that p' is an M-generic element of P.

To show this, let $D \in M$ be an open dense subset of P. We shall prove that $D \cap M$ is predense below p', i.e. for an arbitrary $p'' \leq p'$ we show that there is $q \in D \cap M$ compatible with p''.

Now let D be as above and $p'' \leq p'$. Since D is dense there is $q \in D$ with $q \leq p''$. Suppose that $X_q \cap M_0 = \{z_1, \ldots, z_m\}$ and $X_q \setminus M_0 = \{x_1, \ldots, x_n\}$ where the enumeration $z_1, \ldots, z_m, x_1, \ldots, x_n$ is done in increasing order. By (1) and since K_0 is open, there are open sets B_i, $i = 1, \ldots, n$ such that $x_i \in B_i$ for all $1 \leq i \leq n$, $\{z_1, \ldots, z_m\} \times \bigcup_{1 \leq i \leq n} B_i \subseteq K_0$ and $B_i \times B_j \subseteq K_0$ for all $1 \leq i < j \leq n$.

In the following we show that we can choose $y_i \in M_0 \cap B_i$ so that $\{y_1, \ldots, y_n\} \cup \{x_1, \ldots, x_n\}$ is 0-homogeneous, $\{x_i, y_i\} \in K_0$ for all $1 \leq i \leq n$ and there is $q' \in D \cap M$ such that $X_{q'} = \{z_1, \ldots, z_m\} \cup \{y_1, \ldots, y_n\}$ and $\{M \in \mathcal{M}_{q'} \,:\, y_1 \notin M\} = \mathcal{M}_q \cap M_0$. For such q', letting

$$q'' = \langle X_q \cup X_{q'}, \mathcal{M}_q \cup \mathcal{M}_{q'}\rangle,$$

we have $q'' \in P$ [3] and $q'' \leq q, q'$. Hence $q' \in D \cap M$ and $p'' \leq q'$ are compatible, and this completes the proof.

To find $y_1, \ldots, y_n$, first, let

$$\begin{aligned} \overline{D} = \{\langle u_1, \ldots, u_n\rangle \,:\, & \{z_1, \ldots, z_m, u_1, \ldots, u_n\} = X_r \text{ for some } r \in D \text{ such that} \\ & \{M \in \mathcal{M}_r \,:\, u_1 \notin M\} = \mathcal{M}_q \cap M_0 \text{ and the enumeration} \\ & z_1, \ldots, z_m, u_1, \ldots, u_n \text{ is in increasing order with respect to } \mathcal{M}_r\}. \end{aligned}$$

Then $\overline{D} \in M_0$: $\overline{D} \in M$ since $D \in M$. On the other hand we have also $\overline{D} \in \mathcal{H}(\theta)$. Note that $\overline{D} \neq \emptyset$ since $\langle x_1, \ldots, x_n\rangle \in \overline{D}$.

[2]Note that $\bigcup \mathcal{M}_p \in M_0$ since $\bigcup \mathcal{M}_p \in M$ and $\bigcup \mathcal{M}_p \in \mathcal{H}(\theta)$.

[3]Note that x_i and y_j are separated by M_0.

Let

$$T' = \{t \,:\, t \text{ is an initial segment of an element of } \overline{D} \cap B_1 \times \cdots \times B_n\}.$$

Then $T' \in M_0$. For $1 \le i < n$ let M_i be an element of $\mathcal{M}_q$ which separates x_i and x_{i+1}.

We shall construct a descending sequence $T' \supseteq T_n \supseteq T_{n-1} \supseteq \cdots \supseteq T_1$ of subtrees of T' of height n such that $T_i \in M_0$ for all $1 \le i \le n$ and, for all $1 \le i \le n$ and $t \in T_i$ of maximal height, $\{t_i, x_i\} \in K_0$.

Having constructed such a sequence, for any maximal branch $\langle y_1, \ldots, y_n\rangle \in T_1 \cap M$, if $q' \in D \cap M$ is such that $X_{q'} = \{z_1, \ldots, z_m, y_1, \ldots, y_n\}$ and $\{M \in \mathcal{M}_{q'} \,:\, y_1 \notin M\} = \mathcal{M}_q \cap M_0$, then q' is as desired. That we can take such a q' in $D \cap M$ follows from the fact that T_0 is a subtree of T' and by elementarity M.

For $i = 0, \ldots, n-1$, let $t_i = \langle x_1, \ldots, x_i\rangle$. Note that $t_i \in M_i$. First, let

$$T'_n = \{y \in X \,:\, t_{n-1} {}^\frown y \in T'\}.$$

Then $T' \in M_n$. The set

$$\{x \in T'_n \,:\, \{x\} \times (T'_n \setminus \{x\}) \subseteq K_1\}$$

is a 1-homogeneous element of M_{n-1}. Hence by (5), x_n is not an element of this set. Thus $\{x_n\} \times (T'_n \setminus \{x_n\}) \not\subseteq K_1$ and there is a $w \in T'_n$ such that $\{x_n, w\} \in K_0$. Choose disjoint rational open intervals B'_n, $C_n \subseteq B_n$ such that $x_n \in B'_n$, $w \in C_n$ and $B'_n \times C_n \subseteq K_0$. Let

$$\begin{aligned} T_n = \{t \,:& t \text{ is an initial segment of some } \tilde{t} \in T' \text{ of length } n \\ & \text{whose last component is in } C_n\}. \end{aligned}$$

$T_n \ne \emptyset$ since T_n has an element of the form $\langle \ldots, w\rangle$. We have also that $T_n \in M_0$.

Letting $T'_{n-1} = \{y \,:\, t_{n-1} {}^\frown y \in T_n\}$, similar construction yields T_{n-1}. Continuing in this way we obtain all of T_i, $i = n, n-1, \ldots, 1$ as desired. ❑ (Theorem 4)

Corollary 5 PFA *implies* OCA.

Proof Suppose that PFA holds and $X \subseteq \mathbb{R}$. Let $\mathcal{O}$ be a countable open base of X. Then there is $X' \subseteq X$ such that $|\, X \setminus X' \,| \le \aleph_0$ and for any $O \in \mathcal{O}$, if $O \cap X' \ne \emptyset$ then $O \cap X'$ is not the union of countably many 1-homogeneous sets. By replacing X with X' we may assume without loss of generality that no non-empty $O \in \mathcal{O}$ is a union of countably many 1-homogeneous sets. Let θ and P be as in Theorem 4. Let $\{x_\xi \,:\, \xi < \omega_1\}$ be injective enumeration of a subset of X of cardinality $\aleph_1$ and for each $\alpha < \omega_1$, let

$$D_\alpha = \{p \in P \,:\, X_p \cap \{x_\xi \,:\, \alpha < \xi < \omega_1\} \ne \emptyset\}.$$

By Theorem 4 (a), D_α is a dense subset of P. P is proper by Theorem 4 (b). Hence, letting $\mathcal{D} = \{D_\alpha \,:\, \alpha < \omega_1\}$, there is a $\mathcal{D}$-generic filter G over P. Clearly $\bigcup\{X_p \,:\, p \in G\}$ is an uncountable 0-homogeneous subset of X. ❑ (Corollary 5)

3 Forcing OCA without large cardinals

Let us begin with the following trivial remark:

Lemma 6 *For an open coloring $[X]^2 = K_0 \dot\cup K_1$, if $Y \subseteq X$ is 1-homogeneous then its topological closure $\overline{Y}$ is also 1-homogeneous.*

Proof Otherwise there are $x, y \in \overline{Y}$ such that $x \neq y$ and $\{x,y\} \in K_0$. Since K_0 is open. there are open sets $O_0, O_1 \subseteq X$ such that $x \in O_0$, $y \in O_1$ and $O_0 \times O_1 \subseteq K_0$. Since x, $y \in \overline{Y}$, there are x', $y' \in Y$ such that $x' \in O_0$ and $y' \in O_1$. It follows that $\{x', y'\} \in K_0$. This is a contradiction to 1-homogeneity of Y. ❑ (Lemma 6)

For a consistency proof of OCA in which no large cardinal is involved, we need another type of partial orderings forcing large 0-homogeneous sets:

Theorem 7 *Suppose that X is a separable metric space and $[X]^2 = K_0 \dot\cup K_1$ is an open coloring. Assume that*

(∗) *X is not the union of $< 2^{\aleph_0}$ 1-homogeneous sets*

with respect to the open coloring. Then:

(1) *there is $Y \subseteq X$ such that with $|Y| = 2^{\aleph_0}$ and*

$$Q = \{q \in [Y]^{<\aleph_0} \,:\, q \text{ is } 0\text{-homogeneous}\}$$

with the reverse inclusion has the $2^{\aleph_0}$-c.c.

(2) *Y as in (1) can be taken so that Q additionally satisfies:*

For all $s \in Q$, there are $2^{\aleph_0}$ different y's, $y \in Y$ such that $s \cup \{y\} \in Q$.

Remark. Note that Q in Theorem 7(with the additional property in (2)) forces an uncountable 0-homogeneous subset of X and that, under CH, Q is of size $\aleph_1$ and has the c.c.c.

Proof of Theorem 7: (1): Let $\mathcal{O}^*$ be a countable open basis of X. We shall refer to elements of $\mathcal{O}^*$ (open subsets O of X^n of the form $O = O_1 \times \cdots \times O_n$ for $O_1, \ldots, O_n \in \mathcal{O}^*$ resp.) as rational intervals (rational open sets resp.).

For $n \in \omega$, $p \in X^n$ and an open set $U \subseteq X^n$ with $p \in U$, let

$$U_p = \{q \in U \,:\, q_i \neq p_i \text{ and } \{p_i, q_i\} \in K_0 \text{ for all } i < n\}.$$

For $A \subseteq X^n$, $f : A \to X$ and $p \in X^n$ let

$$\omega_f(p) = \bigcap\{\overline{f''U_p \cap A} \,:\, U \subseteq X^n \text{ is open and } p \in U\}.$$

Let f_ξ, $\xi < 2^{\aleph_0}$ be an enumeration of countable partial functions from X^n for some $n \in \omega$ to X. Let T_ξ, $\xi < 2^{\aleph_0}$ be an enumeration of closed 1-homogeneous subsets of X.

For $\alpha < 2^{\aleph_0}$, let x_α be defined inductively so that

(a) $x_\alpha \in X \setminus \{x_\xi : \xi < \alpha\}$;
(b) $x_\alpha \notin T_\xi$ for all $\xi < \alpha$;
(c) $x_\alpha \notin \omega_{f_\xi}(p)$, if $\xi < \alpha$, $p \in \{x_\xi : \xi < \alpha\}^n$ for an appropriate $n \in \omega$ and $\omega_{f_\xi}(p)$ is 1-homogeneous.

Note that (b) and (c) are possible by $(*)$.

Let $Y = \{x_\alpha : \alpha < 2^{\aleph_0}\}$. We show that this Y is as desired. By Δ-system lemma argument, it is enough to prove the following by induction on $n > 0$:

$(\dagger)_n$ If $\mathcal{F} \subseteq [Y]^n \cap Q$ is a pairwise disjoint family with $|\mathcal{F}| = 2^{\aleph_0}$, then there are distinct $p, q \in \mathcal{F}$ such that $p \mathbin{\dot\cup} q \in Q$.

For $n = 1$, let $Y' = \{x \in Y : \{x\} \in \mathcal{F}\}$. Since

$$Y'' = \{x \in Y' : (\forall y \in Y' \setminus \{x\})[\{x, y\} \in K_1]\}$$

is 1-homogeneous, $\overline{Y''}$ also is by Lemma 6. Let $\xi < 2^{\aleph_0}$ be such that $\overline{Y''} = T_\xi$. For $x \in Y'$ such that $x = x_\eta$ for some $\eta > \xi$, $x \notin Y''$ by (b). Hence there is $y \in Y' \setminus \{x\}$ such that $\{x, y\} \in K_0$. Thus $\{x\}$, $\{y\} \in \mathcal{F}$ are as desired.

Now suppose that $n > 1$ and we have $(\dagger)_{n'}$ for all $1 \leq n' < n$. Assume that $\mathcal{F}$ is as in $(\dagger)_n$ and for each $s \in \mathcal{F}$ let

$$s = \{x_{s(0)}, \ldots, x_{s(n-1)}\}$$

where we assume that $s(0) < \cdots < s(n-1) < 2^{\aleph_0}$. We may also assume that there are rational open $O_0, \ldots, O_{n-1} \subseteq X$ such that $x_{s(i)} \in O_i$ for all $s \in \mathcal{F}$ and $i < n$, and $O_i \times O_j \subseteq K_0$ for all $i < j < n$. $s \in \mathcal{F}$. For $s \in \mathcal{F}$ let us denote with $s\restriction(n-1)$ the set $\{x_{s(0)}, \ldots, x_{s(n-2)}\}$ which is also identified with $\langle x_{s(0)}, \ldots, x_{s(n-2)}\rangle$. Let g be the partial function from X^{n-1} to X which sends $s\restriction(n-1)$ to $x_{s(n-1)}$ for all $s \in \mathcal{F}$. Let

$$\mathcal{F}_0 = \{s \in \mathcal{F} : x_{s(n-1)} \in \omega_g(s\restriction(n-1))\}.$$

Claim 7.1 $|\mathcal{F} \setminus \mathcal{F}_0| < 2^{\aleph_0}$.

$\vdash$ Assume otherwise. For each $s \in \mathcal{F} \setminus \mathcal{F}_0$, since $x_{s(n-1)} \notin \omega_g(s\restriction(n-1))$, and by definition of $\omega_g(s\restriction(n-1))$, there is a rational open $U^s \subseteq X^{n-1}$ such that $x_{s(n-1)} \notin \overline{g''U^s_{s\restriction(n-1)} \cap \mathrm{dom}(g)}$. Let $I^s \subseteq X$ be an rational open interval such that $s(n-1) \in I^s$ and $I^2 \cap \overline{g''U^s_{s\restriction(n-1)} \cap \mathrm{dom}(g)} = \emptyset$.

Without loss of generality we may assume that I^s and U^s are all the same for $s \in \mathcal{F} \setminus \mathcal{F}_0$, say $I^s = I$ and $U^s = U$. By induction hypothesis there are two distinct s, $t \in \mathcal{F} \setminus \mathcal{F}_0$ such that $s\restriction(n-1) \cup t\restriction(n-1)$ is 0-homogeneous. It follows that $t\restriction(n-1) \in U^s_{s\restriction(n-1)} \cap \mathrm{dom}(g)$. Hence $g(t) \notin I^s$. But $g(t\restriction(n-1)) = t(n-1) \in I = I^s$. This is a contradiction. $\dashv$ (Claim 7.1)

Now let $g_0 \subseteq g$ be countable and dense in g. Then $g_0 = f_\xi$ for some $\xi < 2^{\aleph_0}$.

Claim 7.2 *For all $s \in \mathcal{F}$, $\omega_g(s\restriction(n-1)) = \omega_{f_\xi}(s\restriction(n-1))$.*

$\vdash$ Suppose that $U \subseteq X^{n-1}$ is open and $s\restriction(n-1) \in U$. It is enough to show that $\overline{g(U_{s\restriction(n-1)} \cap \mathrm{dom}(g))} = \overline{f_\xi(U_{s\restriction(n-1)} \cap \mathrm{dom}(f_\xi))}$. "$\supseteq$" is clear. To show "$\subseteq$", it is enough to show that $g(U_{s\restriction(n-1)} \cap \mathrm{dom}(g)) \subseteq \overline{f_\xi(U_{s\restriction(n-1)} \cap \mathrm{dom}(f_\xi))}$. Suppose[4] that $t \in \mathcal{F}$ is such that $t\restriction(n-1) \in U_{s\restriction(n-1)}$.

Let $O_{n-1} \subseteq X$ be an open neighbor of $x_{t(n-1)}$. We have to show that there is some $u \in \mathcal{F}$ such that $u\restriction(n-1) \in U_{s\restriction(n-1)} \cap \mathrm{dom}(f_\xi)$ and $x_{u(n-1)} \in O_{n-1}$. Let $O_0, \ldots, O_{n-2} \subseteq X$ be open such that $t\restriction(n-1) \in O_0 \times \cdots \times O_{n-1}$ and for all $q \in O_0 \times \cdots \times O_{n-1}$, $\{q_i, s_i\} \in K_0$ for $i < n$. Now since f_ξ is dense in g, there is $u \in \mathcal{F}$ such that $u\restriction(n-1) \in O_0 \times \cdots \times O_{n-1} \cap \mathrm{dom}(f_\xi)$ and $x_{u(n-1)} \in O_{n-1}$. Such u is as desired. $\dashv$ (Claim 7.2)

Let By Claim 7.1, we can find an $s \in \mathcal{F}_0$ such that $\xi < s(0)$. Then

$$x_{s(n-1)} \in \omega_{f_\xi}(s\restriction(n-1))$$

by the claim above. By (c), it follows that $\omega_{f_\xi}(s\restriction(n-1))$ is not 1-homogeneous. Hence we can pick $u, v \in \omega_{f_\xi}(s\restriction(n-1))$ such that $\{u, v\} \in K_0$. Let I, $J \subseteq X$ be open intervals such that $u \in I$, $v \in J$ and $I \times J \subseteq K_0$. Let $p^* \in \mathrm{dom}(g_0)$ be such that $p^* \cup s\restriction(n-1)$ is 0-homogeneous and $g_0(p^*) \in I$. Let $U \subseteq X^{n-1}$ be an open neighborhood of s such that for all $q \in U$, $p^* \cup q$ is 0-homogeneous. Now pick $q^* \in U \cap \mathrm{dom}(g_0)$ such that $g_0(q^*) \in J$ — there is such q^* since $v \in \omega_{g_0}(s\restriction(n-1))$ and $v \in J$. Now $p^* \cup \{g_0(p^*)\}$, $q^* \cup \{g_0(q^*)\} \in \mathcal{F}$ and $p^* \cup \{g_0(p^*)\} \cup q \cup \{g_0(q)\}$ is 0-homogeneous.

(2): Suppose that $Y \in [X]^{2^{\aleph_0}}$ and $Q = \{q \in [Y]^{<\aleph_0} : q \text{ is 0-homogeneous}\}$ with the reverse inclusion has the $2^{\aleph_0}$-c.c. Let

$$B = \{p \in Q : |\{y \in Y : p \cup \{y\} \in Q\}| < 2^{\aleph_0}\}.$$

Then $|B| < 2^{\aleph_0}$: Suppose that $|B| = 2^{\aleph_0}$. Then we can construct a sequence $p_\alpha \in B$, $\alpha <^{2^{\aleph_0}}$ such that, for all $\beta < \alpha < 2^{\aleph_0}$,

$$p_\alpha \not\subseteq p_\beta \cup \{y \in Y : p_\beta \cup \{y\} \in Q\}.$$

Clearly p_α, $\alpha < 2^{\aleph_0}$ are pairwise incompatible. But this is a contradiction to the $2^{\aleph_0}$-c.c. of P.

Hence we are done by replacing Y with $Y \setminus (\bigcup B)$.

$\square$ (Theorem 7)

Suppose now that CH holds in the ground model, $\langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha < \omega_2}$ is a finite support iteration of c.c.c. p.o.s and that $|P_\alpha| \leq \aleph_1$ for all $\alpha < \omega_2$. Let P_{ω_2} be the direct limit of the iteration. Without loss of generality, we may assume that the underlying set of each P_α is some ordinal $< \omega_2$ and that of P is ω_2. Note that there is a club set $C \subseteq \omega_2$ such that for all $\alpha \in C \cap S^{\omega_1}_{\omega_2}$, (the underlying set of) $P_\alpha = \alpha$.

Let $\dot{X}$, $\dot{\mathcal{O}}$, $\dot{K}_0$, $\dot{K}_1$ be P_{ω_2}-names such that

$$\begin{aligned}\Vdash_{P_{\omega_2}} \text{“}&\dot{X} \subseteq \omega_2 \ \wedge\ \dot{X} \text{ is a metric space and } \dot{\mathcal{O}} \text{ is a countable open basis}\\ &\wedge\ [\dot{X}]^2 = \dot{K}_0 \mathbin{\dot\cup} \dot{K}_1 \text{ is a open coloring } \wedge\ \dot{X} \text{ is not a union of}\\ &\text{countably many 1-homogeneous sets”}.\end{aligned}$$

[4]Note that elements of $\mathrm{dom}(g)$ are of the form $t\restriction(n-1)$ for some $t \in \mathcal{F}$.

Since P_{ω_2} satisfies the $\aleph_2$-c.c. and the underlying set of P_{ω_2} is ω_2, we can identify $\dot{X}$, $\dot{\mathcal{O}}$, $\dot{K}_0$, $\dot{K}_1$ with $\hat{X} \in {}^{(\omega_2)}([\omega_2]^{\leq\aleph_1})$, $\hat{\mathcal{O}} \in {}^{(\omega_2\times\omega)}([\omega_2]^{\leq\aleph_1})$, $\hat{K}_i \in {}^{([\omega_2]^2)}([\omega_2]^{\leq\aleph_1})$, for $i = 0, 1$ respectively, with the correspondence being given by

$$p \Vdash_P \text{“}\, \alpha \in \dot{X} \,\text{”} \quad \Leftrightarrow \quad (\exists q \leq p)[\, \hat{X}(\alpha) \ni q \,]$$

etc.

For $\alpha < \omega_2$, let $\dot{X}_\alpha$, $\dot{\mathcal{O}}_\alpha$ and $\dot{K}_i^\alpha$ for $i = 0, 1$ be P-names which correspond to $\hat{X}_\alpha = \{\langle \beta, x \cap P_\alpha\rangle \,:\, \beta \in \alpha,\, \langle \beta, x\rangle \in \hat{X}\}$, $\hat{\mathcal{O}}_\alpha = \{\langle\langle\beta, n\rangle, x \cap P_\alpha\rangle \,:\, \beta \in \alpha,\, \langle\langle\beta, n\rangle, x\rangle \in \hat{\mathcal{O}}\}$ and $\hat{K}_i^\alpha = \{\langle\{\beta,\gamma\}, x \cap P_\alpha\rangle \,:\, \{\beta,\gamma\} \in [\alpha]^2,\, \langle\{\beta,\gamma\}, x\rangle \in \hat{K}_i\}$ respectively.

Since a countable family of closed 1-homogeneous sets can be uniformly coded by reals and there are only $\aleph_1$ many such in each V^{P_α}, $\alpha < \omega_2$, we can find a club $C \subseteq \omega_2$ such that for any $\alpha \in C \cap S^{\omega_1}_{\omega_2}$,

$$\begin{aligned}\Vdash_{P_\alpha} \text{“}\, & \dot{X}_\alpha \subseteq \alpha \ \wedge\ \dot{X}_\alpha \text{ is a metric space and } \dot{\mathcal{O}}_\alpha \text{ is a countable open basis} \\ & \wedge\ [\dot{X}_\alpha]^2 = \dot{K}_0^\alpha \,\dot{\cup}\, \dot{K}_1^\alpha \text{ is a open coloring } \wedge\ \dot{X}_\alpha \text{ is not a union of} \\ & \text{countably many 1-homogeneous sets} \,\text{”}.\end{aligned}$$

Building on this observation we now prove:

Theorem 8 *If* ZFC *is consistent then so is* ZFC + OCA.

Proof Assume that $V \models \mathrm{CH}, \diamondsuit(S^{\omega_1}_{\omega_2})$ and $\langle S_\alpha \,:\, \alpha \in S^{\omega_1}_{\omega_2}\rangle$ is a $\diamondsuit$-sequence guessing quintuple $\langle \leq, \dot{X}, \dot{\mathcal{O}}, \dot{K}_0, \dot{K}_1\rangle$ where $\leq$ is a partial ordering on ω_2 and $\dot{X}, \ldots, \dot{K}_1$ are $\langle\omega_2, \leq\rangle$-names — or rather the functions corresponding to them in the sense of $\hat{X}, \ldots, \hat{\dot{K}}_1$ as above.

Now let $\langle P_\alpha, \dot{Q}_\alpha\rangle_{\alpha<\omega_2}$ be a FS-iteration such that, for all $\alpha < \omega_2$, if S_α is the quintuple $\langle \leq', \dot{X}', \dot{\mathcal{O}}', \dot{K}_0', \dot{K}_1'\rangle$, $\langle P_\alpha, \leq_\alpha\rangle = \langle \alpha, \leq'\rangle$ [5] and

$$\begin{aligned}\Vdash_{P_\alpha} \text{“}\, & \dot{X}' \subseteq \alpha \ \wedge\ \dot{X}' \text{ is a metric space and } \dot{\mathcal{O}}' \text{ is a countable open basis} \\ & \wedge\ [\dot{X}']^2 = \dot{K}_0' \,\dot{\cup}\, \dot{K}_1' \text{ is a open coloring } \wedge\ \dot{X}' \text{ is not a union of} \\ & \text{countably many 1-homogeneous sets} \,\text{”},\end{aligned}$$

then let $\dot{Q}_\alpha$ be such that

$$\begin{aligned}\Vdash_{P_\alpha} \text{“}\, & \dot{Q}_\alpha \text{ is the p.o. in Theorem 7} \\ & \text{for the open coloring } [\dot{X}']^2 = \dot{K}_0' \,\dot{\cup}\, \dot{K}_1' \text{ of } \langle \dot{X}', \dot{\mathcal{O}}'\rangle \,\text{”}\end{aligned}$$

Otherwise let $\dot{Q}_\alpha$ be such that $\Vdash_{P_\alpha}$ “$\dot{Q}_\alpha$ is the trivial p.o.”. Note that $\Vdash_{P_\alpha}$ “CH” and hence, by Theorem 7, we have $\Vdash_{P_\alpha}$ “$\dot{Q}_\alpha$ has the c.c.c.”.

Let P_{ω_2} be the direct limit of the system $\langle P_\alpha, \dot{Q}_\alpha\rangle_{\alpha<\omega_2}$.

Claim 8.1 $\Vdash_{P_{\omega_2}}$ “OCA”.

[5] We assume that we are building a bijection $f : P_{\omega_2} \to \omega_2$ along with the iteration and we identify $\langle P_\alpha, \leq_\alpha\rangle$ with $\langle f(P_\alpha), \{\langle f(p), f(q)\rangle \,:\, p, q \in P_\alpha,\, p \leq_\alpha q\}\rangle$.

$\vdash$ Suppose that $\dot{X}$, $\dot{\mathcal{O}}$, $\dot{K}_0$, $\dot{K}_1$ are P_{ω_2}-names such that

$$\begin{aligned}\Vdash_{P_{\omega_2}} \text{“}\, & \dot{X} \subseteq \omega_2 \ \wedge\ \dot{X} \text{ is a metric space with a countable open basis } \dot{\mathcal{O}} \\ & \wedge\ [\dot{X}]^2 = \dot{K}_0 \,\dot{\cup}\, \dot{K}_1 \text{ is a open coloring } \wedge\ \dot{X} \text{ is not a union of} \\ & \text{countably many 1-homogeneous sets”}.\end{aligned}$$

Then there is an $\alpha \in S^{\omega_1}_{\omega_2}$ such that, letting $S_\alpha = \langle \leq', \dot{X}', \dot{\mathcal{O}}', \dot{K}'_0, \dot{K}'_1 \rangle$,

$$\leq_\alpha = \leq',\ \dot{X}_\alpha = \dot{X}',\ \dot{\mathcal{O}}_\alpha = \dot{\mathcal{O}}',\ \dot{K}^\alpha_0 = \dot{K}'_0 \ \text{ and } \ \dot{K}^\alpha_1 = \dot{K}'_1$$

where $\dot{X}_\alpha$, $\dot{\mathcal{O}}_\alpha$, $\dot{K}^\alpha_i$, $i = 0, 1$ are defined as in the remark before Theorem 8, and

$$\begin{aligned}\Vdash_{P_\alpha} \text{“}\, & \dot{X}_\alpha \subseteq \alpha \ \wedge\ \dot{X}_\alpha \text{ is a metric space and } \dot{\mathcal{O}}_\alpha \text{ is a countable open basis} \\ & \wedge\ [\dot{X}_\alpha]^2 = \dot{K}^\alpha_0 \,\dot{\cup}\, \dot{K}^\alpha_1 \text{ is a open coloring } \wedge\ \dot{X}_\alpha \text{ is not a union of} \\ & \text{countably many 1-homogeneous sets”}.\end{aligned}$$

Hence by construction, $\dot{Q}_\alpha$ forces an uncountable 0-homogeneous subset of $\dot{X}_\alpha$ in $V^{P_{\alpha+1}}$ which is also an uncountable 0-homogeneous subset of $\dot{X}$ in $V^P_{\omega_2}$ with respect to $\dot{\mathcal{O}}$ and the open coloring "$[\dot{X}]^2 = \dot{K}_0 \,\dot{\cup}\, \dot{K}_1$". $\dashv$ (Claim 8.1)

□ (Theorem 8)

Modifying the construction in the proof of Theorem 8 slightly, we can also obtain a model of MA + OCA. Hence:

Corollary 9 *If* ZFC *is consistent then so is* ZFC + MA + OCA. □

4 $\aleph_1$-dense sets and Baumgartner's Assertion

Recall that a subset X of $\mathbb{R}$ is said to be κ-dense if X does not have end-points and for any x, $y \in X$, if $x < y$, then $\{z \in X\ :\ x < z < y\}$ is of cardinality κ.

(BA) Any $\aleph_1$-dense subsets of $\mathbb{R}$ are order isomorphic.

In this section we shall prove that (BA) is consistent with ZFC and follows from PFA.

Theorem 10 (Theorem 12.1 in [13]) *Suppose that X, $Y \in [\mathbb{R}]^{2^{\aleph_0}}$. Then there is a 1-1 mapping $f : X \to Y$ such that $\mathcal{P} = \mathcal{P}_f = \{f\restriction S\ :\ S \in [X]^{<\aleph_0},\ f\restriction S$ is increasing$\}$ has the following properties* (1) *and* (2)*:*

(1) *For any $\mathcal{K} \in [\mathcal{P}]^q \in 2^{\aleph_0}$, there are distinct p, $q \in \mathcal{K}$ such that $p \cup q \in \mathcal{P}$ (i.e. $\mathcal{P}_f$ with the reverse inclusion has the $2^{\aleph_0}$-c.c.).*

(2) *For $\mathcal{P}^{<\omega} = \{p \in {}^\omega\mathcal{P}\ :\ |\{i < \omega\ :\ p(i) = \emptyset\}| < \aleph_0\}$ and any $\mathcal{K} \in [\mathcal{P}^{<\omega}]^{2^{\aleph_0}}$, there are distinct p, $q \in \mathcal{K}$ such that $\{\langle i, p(i) \cup q(i)\rangle\ :\ i \in \omega\} \in \mathcal{P}^{<\omega}$ (i.e. $\mathcal{P}_f$ with the reverse inclusion is productively $2^{\aleph_0}$-c.c.)*

In the proof of the theorem we use the following fact:

Fact 11 (a) *For any $n \in \omega$ and $f : \mathbb{R}^n \to \mathbb{R}$, $C_f = \{x \in \mathbb{R}^n : f$ is continuous at $x\}$ is a G_δ-set.*

(b) *For any partial function $f : \mathbb{R}^n \supseteq\!\to \mathbb{R}$ there is a G_δ-set $C \subseteq \mathbb{R}^n$ and a continuous function $g : C \to \mathbb{R}$ such that $C_f \subseteq C$ and $f \restriction C \subseteq g$.*

Proof For (a), just note that

$$C_f = \bigcap_{l\in\omega} \bigcup_{q\in\mathbb{Q}} \underbrace{\bigcup_{m\in\omega} \{x \in \mathbb{R}^n : \forall y \in \mathbb{R}^n\, [\,||x-y|| < \frac{1}{m} \to |f(y) - q| < \frac{1}{l}\,]\}}_{\text{open}}.$$

For (b), let

$$C = \bigcap_{l\in\omega} \bigcup_{q\in\mathbb{Q}} \bigcup_{m\in\omega} \{x \in \mathbb{R}^n : \forall y \in \mathrm{dom}(f)\, [\,||x-y|| < \frac{1}{m} \to |f(y) - q| < \frac{1}{l}\,]\}.$$

Then $C \supseteq C_f$ and C is G_δ. For $x \in C$

$$g(x) = \lim_{\substack{z \to x \\ z \in \mathrm{dom}(f)}} f(z)$$

is well-defined. Clearly g is as desired. □ (Fact 11)

Proof of Theorem 10: Let $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and let $\langle g_\alpha : \alpha < 2^{\aleph_0}\rangle$ be an enumeration of continuous mappings such that

(3) $\mathrm{dom}(g)$ is a G_δ-set of $\mathbb{R}^n$ for some $n \in \omega$;

(4) $\mathrm{rng}(g) \subseteq \mathbb{R}$.

Now let $f : X \to Y$ be defined inductively so that

($*$) $f(x_\alpha) \in Y \setminus \{f(x_\beta) : \beta < \alpha\}$ and

($**$) $$f(x_\alpha) \notin \{g_\xi(p ^\frown x_\alpha) : \xi < \alpha,\ g_\xi \text{ is an } n\text{-place function},\ p \in (\{f(x_\beta), x_\beta : \beta < \alpha\})^{n-1}\}.$$

We show that this f is as desired.

To prove (1), by Δ-system lemma argument, it is enough to show the following:

(1') *For $\mathcal{K} \in [\mathcal{P}_f]^{2^{\aleph_0}}$ such that elements of $\mathcal{K}$ are pairwise disjoint and all of size n, there are two distinct p, $q \in \mathcal{K}$, such that $p \cup q$ is an increasing function.*

Suppose $\mathcal{K} = \{s^\xi : \xi < 2^{\aleph_0}\}$ and $s^\xi = \{\langle x_i^\xi, f(x_i^\xi)\rangle : i < n\}$, where we assume that $x_0^\xi < x_1^\xi < \cdots < x_{n-1}^\xi$. We will prove (1') by induction on $n \geq 1$. Let $\overline{g} : \mathbb{R}^{2n-1} \supseteq\!\to \mathbb{R}$ be defined by:

$$\overline{g} = \{\langle \underbrace{\langle x_0^\xi, f(x_0^\xi), \ldots, x_{n-2}^\xi, f(x_{n-2}^\xi), x_{n-1}^\xi\rangle}_{\overline{s}^\xi}, f(x_{n-1}^\xi)\rangle : \xi < 2^{\aleph_0}\}.$$

By $(**)$, we have $|\,g_\alpha \cap \overline{g}\,| < 2^{\aleph_0}$ for all $\alpha < 2^{\aleph_0}$. By (2) of Fact 11, it follows that $|\,C_{\overline{g}}\,| < 2^{\aleph_0}$. Let

$$\mathcal{K}_0 = \{s^\xi : \xi < 2^{\aleph_0},\, \overline{s}^\xi \in \mathrm{dom}(\overline{g}) \setminus C_{\overline{g}}\}.$$

We have $|\,\mathcal{K}_0\,| = 2^{\aleph_0}$. By thinning out $\mathcal{K}_0$, we may assume that there are disjoint n rational intervals $I_0, I_1,\ldots, I_{n-1}$ and other disjoint n rational intervals $J_0, J_1,\ldots, J_{n-1}$ such that $I_0 < I_1 \cdots < I_{n-1}$, $J_0 < J_1 \cdots < J_{n-1}$ and $\langle x_i^\xi, f(x_i^\xi)\rangle \in I_i \times J_i$ for all $s^\xi \in \mathcal{K}_0$ and $i < n$. Further we may assume that each element of $\{\overline{s}^\xi : s^\xi \in \mathcal{K}_0\}$ is an accumulation point of $\{\overline{s}^\xi : s^\xi \in \mathcal{K}\}$ (we may assume this because there are only countably may isolated points in $\{\overline{s}^\xi : s^\xi \in \mathcal{K}\}$). In particular, for each $s^\xi \in K_0$, we can find $s_i^\xi \in \mathcal{K}$ for $i \in \omega$ such that $\lim_{i\to\infty} \overline{s}_i^\xi = \overline{s}^\xi$ but $\lim_{i\to\infty} \overline{g}(\overline{s}_i^\xi) \neq \overline{g}(\overline{s}^\xi)$. By thinning out $\mathcal{K}_0$ further and by symmetry we may assume that, for some $d \in \mathbb{Q}$, $\lim_{i\to\infty} \overline{g}(\overline{s}_i^\xi) < d < \overline{g}(\overline{s}^\xi)$. By induction hypothesis, there are distinct s^ξ, $s^\eta \in \mathcal{K}_0$ such that $\overline{\overline{s}}^\xi = s^\xi \restriction \{x_0^\xi,\ldots,x_{n-2}^\xi\}$ and $\overline{\overline{s}}^\eta = s^\eta \restriction \{x_0^\eta,\ldots,x_{n-2}^\eta\}$ are compatible, i.e., $\overline{\overline{s}}^\xi \cup \overline{\overline{s}}^\eta$ is an increasing function. Suppose that $x_{n-1}^\xi < x_{n-1}^\eta$. Then for large enough $i_0 \in \omega$, $\overline{\overline{s}}_{i_0}^\xi$ and $\overline{\overline{s}}^\eta$ are compatible and,a for $\nu < 2^{\aleph_0}$ such that $s_{i_0}^\xi = s^\nu$, we have $x_{n-1}^\nu < x_{n-1}^\eta$ and $f(x_{n-1}^\nu) < f(x_{n-1}^\eta)$. For the last inequality, note that we can take sufficiently small $\varepsilon > 0$ so that, for large enough $i_0 < \omega$ and corresponding x^ν, we have:

$$f(x_{n-1}^\nu) = \overline{g}(\overline{s}_{i_0}^\xi) < d + \varepsilon < \overline{g}(\overline{s}^\eta) = f(x_{n-1}^\eta).$$

By the choice of $I_0,\ldots, J_0,\ldots$, it follows that $s_{i_0}^\xi$ $(= s^\nu)$ and s^η are compatible.

For (2), by Δ-system lemma argument, it is enough to prove the following:

(2') *Suppose that $\mathcal{K} \subseteq \mathcal{P}^n$ is such that $|\,\mathcal{K}\,| = 2^{\aleph_0}$ and for all $p = \langle p_0,\ldots,p_{n-1}\rangle \in \mathcal{K}$, $|\,p_{n-1}\,| = m$. Then there are distinct p, $q \in \mathcal{K}$ such that $p_i \cup q_i$ is an increasing function for all $i < n$.*

This assertion can be proved similarly to the case (1) by induction on $(n, m) \in \omega^2$ with respect to the lexicographical ordering. ❑ (Theorem 11)

Let us denote with (†) the following assertion:

(†) For any X, $Y \in [\mathbb{R}]^{\aleph_1}$ there is a 1-1 function $f : X \to Y$ such that $f = \bigcup_{i\in\omega} f_i$ where each f_i is strictly increasing.

Corollary 12 (1) PFA *implies* (†).
(2) *If* ZFC *is consistent then so is* MA + (†).

Proof (1): Suppose P is a σ-closed forcing notion collapsing cardinals $\aleph_1 < \kappa \leq 2^{\aleph_0}$ In V^P let f function in Theorem 10 for X and Y. Let $\dot{Q}$ be the name for $\mathcal{P}^{<\omega}$ for $\mathcal{P}_f$ in V^P. Then a sufficiently generic G over $P * \dot{Q}$ codes a desired function.

(2): Similarly to the proof of Theorem 8. ❑ (Corollary 12)

Theorem 13 (†) + $\mathbf{t} > \aleph_1$ *implies* BA.

Proof Let X and Y be $\aleph_1$ dense subsets of $\mathbb{R}$. For simplicity let us assume that X and Y are dense in $\mathbb{R}$. By (†), for every pair of rational intervals I, J we can find 1-1 functions $f_{I,J} : X \cap I \to Y \cap J$ and $g_{I,J} : Y \cap J \to X \cap I$ and strictly increasing $f^i_{I,J}$ and $g^i_{I,J}$ for $i \in \omega$ such that

$$f_{I,J} = \bigcup_{i\in\omega} f^i_{I,J}, \quad g_{I,J} = \bigcup_{i\in\omega} g^i_{I,J}.$$

Let

$$\begin{aligned} P = \{p \in \mathrm{Fn}(\mathbb{R},\mathbb{R}) \ :& \ p \text{ is strictly increasing and for all } x \in \mathrm{dom}(p), \text{ there are} \\ & i_x \in \omega \text{ and rational intervals } I_x, J_x \text{ such that } x \in I_x, p(x) \in J_x \text{ and} \\ & \text{either } p(x) = f^{i_x}_{I_x,J_x}(x) \text{ or } x = g^{i_x} I_x, J_x(p(x))\} \end{aligned}$$

and consider P as a p.o. with the reverse inclusion.

Claim 13.1 *P is σ-centered.*

$\vdash$ Let n, $I_0,\ldots, I_{n-1}$, $J_0,\ldots, J_{n-1}$, $I'_0,\ldots, I'_{n-1}$, $J'_0,\ldots, J'_{n-1}$, $i_0,\ldots, i_{n-1}$, $k_0,\ldots, k_{n-1}$ be such that

(1) n, $i_0,\ldots, i_{n-1} \in \omega$;

(2) $I_0,\ldots, I_{n-1}$, $J_0,\ldots, J_{n-1}$ are rational intervals;

(3) $I'_0,\ldots, I'_{n-1}$, $J'_0,\ldots, J'_{n-1}$ are rational intervals such that $I'_0 < \cdots < I'_{n-1}$ and $J'_0 < \cdots < J'_{n-1}$;

(4) $k_0,\ldots, k_{n-1} \in 2$.

Let $\mathcal{S} = \{I_0,\ldots,I_{n-1}, J_0,\ldots,J_{n-1}, I'_0,\ldots,I'_{n-1}, J'_0,\ldots,J'_{n-1}, i_0,\ldots,i_{n-1}, k_0,\ldots,k_{n-1}\}$ and

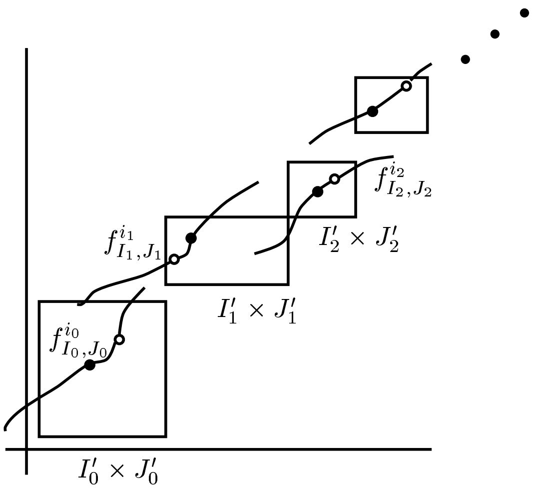
$$P_{\mathcal{S}} = \left\{ \begin{aligned} p \in P \ :& \ \mathrm{dom}(p) = \{x_0,\ldots,x_{n-1}\} \text{ for some } x_0 \in I'_0,\ldots,x_{n-1} \in I'_{n-1} \\ & p(x_0) \in J'_0,\ldots,p(x_{n-1}) \in J'_{n-1} \\ & \text{for all } j < n,\ p(x_j) = \begin{cases} f^{i_j}_{I_j,J_j}(x_j), & \text{if } k_j = 0 \\ (g^{i_j}_{I_j,J_j})^{-1}(x_j), & \text{otherwise} \end{cases} \end{aligned} \right\}.$$

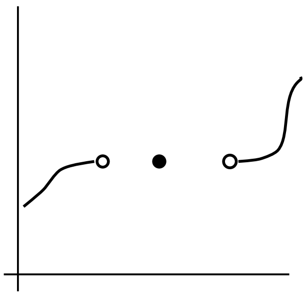
Then $P_{\mathcal{S}}$ is centered (see the figure above). So we are done, since there are only countably many such $\mathcal{S}$ as above. $\dashv$ (Claim 13.1)

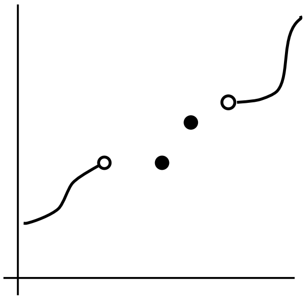
For $x \in X$ let $D^X_x = \{p \in P : x \in \mathrm{dom}(p)\}$ and for $y \in Y$ let $D^Y_y = \{p \in P : y \in \mathrm{rng}(p)\}$. Then D^X_x and D^Y_y are dense subsets of P for any $x \in X$ and $y \in Y$. Let

$$\mathcal{D} = \{D^X_x : x \in X\} \cup \{D^Y_y : y \in Y\}.$$

By Claim 13.1 and $\mathbf{t} > \aleph_1$, there is a $\mathcal{D}$-generic filter G over P. By definition of P and $\mathcal{D}$, $f = \bigcup G$ is an order preserving bijection from X to Y. ❑ (Theorem 13)







5 $2^{\aleph_0}$-dense sets

Using the same idea as in the proof of Theorem 10, we can show that BA for $2^{\aleph_0}$-dense sets is far from being true. First we prove a lemma which is similar to Fact 11:

Lemma 14 *Suppose that $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is strictly monotonous. Then there is a G_δ set $\tilde{X} \subseteq \mathbb{R}$ and a monotonous $\tilde{f} : \tilde{X} \to \mathbb{R}$ such that*

(i) $X \setminus \tilde{X}$ *is countable;*

(ii) $f\restriction X \cap \tilde{X} = \tilde{f}\restriction X \cap \tilde{X}$, *and*

(iii) $\tilde{f}$ *is ≤ 3 to 1 (i.e. for all $y \in \mathbb{R}$, we have $|\, f''\{y\}\,| \leq 3$).*

Proof Let

$$\tilde{X} = \underbrace{\overline{X}}_{\text{closed}} \cap \bigcup_{l\in\omega} \bigcap_{q\in\mathbb{Q}} \underbrace{\bigcup_{m\in\omega} \{x \in \mathbb{R} : \forall y \in X[\,|x-y| < \frac{1}{m} \to |f(y) - q| < \frac{1}{l}\,]\}}_{\text{open}}.$$

$\tilde{X}$ is a G_δ-set. For each $x \in \tilde{X}$, let $\tilde{f}(x) = \lim_{y\to x, y\in X} f(y)$. $\tilde{f}(x)$ is well-defined and $\tilde{f} : \tilde{X} \to \mathbb{R}$ is monotonous. $\tilde{f}$ is ≤ 3 to 1: Suppose that $x, y \in \tilde{X}$, $x < y$ and $\tilde{f}(x) = \tilde{f}(y)$. Then between x and y there can not be more than one element of X.

❑ (Lemma 14)

Theorem 15 (Sierpinski [?]) *There exists a $K \in [\mathbb{R}]^{2^{\aleph_0}}$, such that K is $2^{\aleph_0}$-dense and for any $X, Y \in [K]^{2^{\aleph_0}}$, if $|\,X \setminus Y\,| = 2^{\aleph_0}$, then there is no strictly monotonous $f : X \to Y$.*

Proof Let $\langle f_\alpha : \alpha \in 2^{\aleph_0} \rangle$ be an enumeration of monotonous ≤ 3 to 1 functions f on G_δ-set of $\mathbb{R}$ to $\mathbb{R}$. Let $\langle I_\alpha : \alpha < 2^{\aleph_0} \rangle$ be an enumeration of non-empty rational open intervals such that each non-empty rational open interval I appears $2^{\aleph_0}$ times in the enumeration.

Let x_α, $y_{\alpha,\beta} \in \mathbb{R}$ be defined inductively for $\beta < \alpha < 2^{\aleph_0}$ so that for each $\alpha < 2^{\aleph_0}$:

(1) $x_\alpha \in I_\alpha \setminus \{x_\xi : \xi < \alpha\} \cup \{y_{\xi,\eta} : \eta < \xi \leq \alpha\}$;

(2) $y_{\alpha,\beta} \notin \{x_\xi : \xi \leq \alpha\}$ for all $\xi < \alpha$;

(3) For $\beta < \alpha$, if $f_\beta(x_\alpha) \neq x_\alpha$ then $y_{\alpha,\beta} = f_\beta(x_\alpha)$.

To see that this is possible, suppose that x_ξ, $y_{\xi,\eta}$ for $\eta < \xi < \alpha$ have been constructed such that (1) $\sim$ (3) hold and $x_\xi \neq 0$ for all $\xi < \alpha$. Let

$$S = \{x_\xi : \xi < \alpha\} \cup \{y_{\xi,\eta} : \eta < \xi < \alpha\} \cup \bigcup \{f\eta^{-1\prime\prime}\{x_\xi : \xi < \alpha\} : \eta < \alpha\}.$$

Since $f_\eta' s$ are ≤ 3 to 1, S is of cardinality $< 2^{\aleph_0}$. Let $x_\alpha \in I_\alpha \setminus S$ be such that $x_\alpha \neq 0$. For $\beta < \alpha$ if $f_\beta(x_\alpha) \neq x_\alpha$ then $f_\beta(x_\alpha) \notin \{x_\xi : \xi \leq \alpha\}$ by definition of S. So we may put $y_{\alpha,\beta} = f_\beta(x_\alpha)$ in this case, and $y_{\alpha,\beta} = 0$ otherwise.

We show that $K = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ is as desired. That K is $2^{\aleph_0}$ dense follows from (1). Suppose that $X, Y \in [K]^{2^{\aleph_0}}$,

(*) $|X \setminus Y| = 2^{\aleph_0}$

and $f : X \to Y$ is strictly monotonous. Then by Lemma 14, there is a G_δ-set $\tilde{X} \subseteq \mathbb{R}$ and a monotonous ≤ 3 to 1 function $\tilde{f} : \tilde{X} \to \mathbb{R}$ such that $X \setminus \tilde{X}$ is countable and $f \restriction X \cap \tilde{X} = \tilde{f} \restriction X \cap \tilde{X}$. Let $\beta < 2^{\aleph_0}$ be such that $\tilde{f} = f_\beta$. By (*), there is $\alpha < 2^{\aleph_0}$ such that $\beta < \alpha$, $x_\alpha \in (X \setminus Y) \cap \tilde{X}$. Then $\tilde{f}_\beta(x_\alpha) \neq x_\alpha$. Hence, by construction of K, $f(x) = f_\beta(x) \notin K$. This is a contradiction to $f : X \to Y \subseteq K$. ❑ (Theorem 15)

Lemma 16 (Special case of Fichtenholz-Kantorovich-Hausdorff-Theorem) *For any infinite κ there is a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of cardinality 2^κ such that for each distinct $X, Y \in \mathcal{F}$ we have $|X \setminus Y| = 2^{\aleph_0}$.*

Proof We may replace κ by the free Boolean algebra $\mathrm{Fr}\,\kappa$. Let

$$\mathcal{F} = \{U \in \mathcal{P}(\mathrm{Fr}\,\kappa) : U \text{ is a filter on } \mathrm{Fr}\,\kappa\}.$$

Then $\mathcal{F}$ is clearly as desired. ❑ (Lemma 16)

Theorem 17 *There is a family $\mathcal{F}$ of $2^{\aleph_0}$-dense subsets of $\mathbb{R}$ such that $|\mathcal{F}| = 2^{2^{\aleph_0}}$ and for each distinct $X, Y \in \mathcal{F}$ there is no strictly monotonous $f : X \to Y$ and X and Y are non homeomorphic to each other.*

Proof Let Z be the $2^{\aleph_0}$-dense subset of $\mathbb{R}$ of Theorem 15. Let $Z' \in [Z]^{2^{\aleph_0}}$ be such that $Z \setminus Z'$ is $2^{\aleph_0}$-dense. By Lemma 16, there is a family $\mathcal{F}' \subseteq \mathcal{P}(Z')$ such that $|\mathcal{F}'| = 2^{2^{\aleph_0}}$ and for any distinct $X, Y \in \mathcal{F}'$, $|X \setminus Y| = 2^{\aleph_0}$. Then $\mathcal{F} = \{X \cup (Z \setminus Z') : X \in \mathcal{F}'\}$ is as desired. ❑ (Theorem 17)

6 Some applications of OCA

In most of the applications, the following weaker variants of OCA are enough:

(OCA$^-$) For any uncountable $X \subseteq \mathbb{R}$ and any open coloring $[X]^2 = K_0 \dot\cup K_1$, one of the following two cases holds:

(a) There is an uncountable 1-homogeneous set $Y \subseteq X$;

(b) There is an uncountable 0-homogeneous $Y \subseteq X$.

For a cardinal κ

(OCA$_\kappa$) For any $X \in [\mathbb{R}]^{\leq\kappa}$ and any open coloring $[X]^2 = K_0 \dot\cup K_1$, one of the following two cases holds:

(a) X is the union of countably many 1-homogeneous sets;

(b) There is an uncountable 0-homogeneous $Y \subseteq X$.

Similarly to Corollary 3, $\mathbb{R}$ in (OCA$^-$) or (OCA$_\kappa$) may be replaced by $\mathbb{R}^n$ ${}^\omega 2$, ${}^\omega\omega$ etc. Clearly we have:

$$\text{OCA} \rightarrow \text{OCA}_{\aleph_2} \rightarrow \text{OCA}_{\aleph_1} \rightarrow \text{OCA}^- .$$

Question 1 OCA $\leftrightarrow$ OCA$_{\aleph_2}$ *?*

In a later section we show that OCA$_{\aleph_2}$ and OCA$_{\aleph_1}$ are not equivalent.

Theorem 18 [12], see also [2]
(a) (OCA$^-$) *Every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or an uncountable antichain (chain here means "pairwise incomparable").*

(b) *From the conclusion of* (a), *it follows that, for every function from an uncountable subset X of $\mathbb{R}$ to $\mathbb{R}$, there is an uncountable $X' \subseteq X$ such that f is monotonous on X'.*

(c) *From the conclusion of* (a), *it follows that every uncountable Boolean algebra contains an uncountable antichain (i.e. an uncountable pairwise incomparable subset).*

(d) (OCA$_{\aleph_1}$) *For any uncountable U, $V \subseteq \mathbb{R}$ there is $U' \in [U]^{\aleph_1}$ such that there is a strictly increasing $f : U' \rightarrow V$.*

Proof (a): For uncountable $X \subseteq \mathcal{P}(\omega)$, let $[X]^2 = K_0 \dot\cup K_1$ be the open coloring defined by:

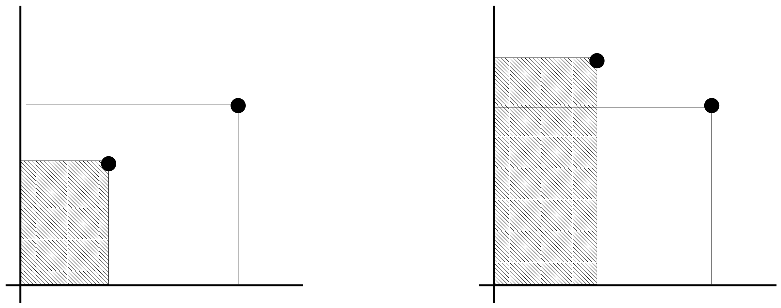
$$K_0 = \{\langle x, y\rangle \in [X]^2 \,:\, x \text{ and } y \text{ are incomparable}\}.$$

By (OCA$^-$) there is uncountable homogeneous set $Y \subseteq X$. Y is pairwise incomparable if Y is 0-homogeneous; Y is a chain if it is 1-homogeneous.

(b): ([2]) Suppose $f : X \rightarrow \mathbb{R}$ for an uncountable $X \subseteq \mathbb{R}$. We may assume that f is 1-1. For $x \in X$, let

$$q_x = \{y \in \mathbb{Q} \,:\, y \leq x\} \times \{z \in \mathbb{Q} \,:\, z \leq f(x)\}.$$

Let $Y = \{q_x \,:\, x \in X\} \subseteq \mathcal{P}(\mathbb{Q} \times \mathbb{Q})$. Then a chain in Y codes an increasing sub-function of f and an antichain of Y a decreasing sub-function of f.



(c): (see [2], Theorem 6.14) For this proof we need the following lemmas. Recall that, for a Boolean algebra B, $Inc(B) = \sup\{S : S \subseteq \text{ is pairwise incomparable}\}$ and $\pi(B) = \inf\{D : D \subseteq B,\ D \text{ is dense in } B\}$.

Lemma 19 (Special Case of a Theorem by Shelah — see 4.25 in [8]) *If $Inc(B) = \aleph_0$ then $\pi(B) = \aleph_0$.*

Proof Suppose that $Inc(B) = \aleph_0$ but $\pi(B) \geq \aleph_1$. From $Inc(B) = \aleph_0$ it follows that B has the c.c.c. Hence there is $a \in B$ such that for any $b \in (B\restriction a)^+$, $\pi(B\restriction b) > \aleph_0$: let $\mathcal{P}$ be a partition of B^+ into homogeneous elements with respect to π-weight. Since $\mathcal{P}$ is countable, there should be an element in $\mathcal{P}$ of homogeneous π-weight $\geq \aleph_1$.

Now let $a = a_0 \dot{+} a_1$. For each $i < 2$ and $\alpha < \omega_1$ let $a_{i,\alpha} \leq a_i$ be such that $a_{i,\beta} \not\leq a_{i,\alpha}$ for all $\beta < \alpha$. Let $a_\alpha = a_{0,\alpha} \dot{+} (a_1 \cdot -a_{1,\alpha})$. Then $\{a_\alpha : \alpha < \omega_1\}$ is pairwise incomparable. This is a contradiction to $Inc(B) = \aleph_0$. ❑ (Lemma 19)

Lemma 20 *If $Inc(B) = \aleph_0$ then B is embeddable into $\mathcal{P}(\omega)$ (or formulated in contraposition: if B is not embeddable into $\mathcal{P}(\omega)$ then $Inc(B) > \aleph_0$).*

Proof We have to show that B is σ-centered. By the previous lemma, there is a countable dense $D \subseteq B$. For each $d \in D$ let $E_d = \{b \in B : d \leq b\}$. Then each E_d is σ-centered and $B^+ = \bigcup_{d \in D} E_d$. ❑ (Lemma 20)

Continuation of the proof of Theorem 18 (c)**:** By Lemma 20, we may assume that $B \subseteq \mathcal{P}(\omega)$ and the Boolean operations of B are just set operations. We have the following two cases:

Case I: There is $b \in B$ such that $B\restriction b$ and $B\restriction (\omega \setminus b)$ are both uncountable. Without loss of generality we may assume that B has cardinality $\aleph_1$. Hence there is a 1-1 mapping f from $B\restriction b$ to $B\restriction (\omega \setminus b)$. Identifying each $b \in B$ with $t(b) = \sum_{i \in b} 3^{-i}$, we obtain a partial mapping $\tilde{f}$ from $\mathbb{R}$ to $\mathbb{R}$ which corresponds to f. Note that with this identification, $b \subseteq b'$ corresponds to the inequality $t(b) \leq t(b')$. By (b) which follows from (a) there is an uncountable subset $\tilde{X}$ of the domain of $\tilde{f}$ such that $\tilde{f}\restriction \tilde{X}$ is monotonous. Let X' be the uncountable subset of $B\restriction b$ which corresponds to $\tilde{X}$. If $\tilde{f}\restriction \tilde{X}$ is increasing, then $\{(b \setminus c) \cup f(c) : c \in X'\}$ is pairwise incomparable. If $\tilde{f}\restriction \tilde{X}$ is decreasing then $\{c \cup f(c) : c \in X'\}$ is pairwise incomparable.

Case II: Otherwise. Then we can construct a sequence $b_\alpha \in B$, $\alpha < \omega_1$ such that $B\restriction b_\alpha$ is countable for all $\alpha < \omega_1$ and $b_\alpha \not\leq b_\beta$ for all $\beta < \alpha < \omega_1$. It follows that $S = \{b_\alpha : \alpha < \omega_1\}$ is well-founded. Hence any uncountable chain in S is cell ordered. But since $S \subseteq \mathcal{P}(\omega)$, this is impossible. By (a), it follows that there is an uncountable pairwise incomparable subset $T \subseteq S$.

(d): Let $X = U \times V$ and $X^2 = K_0 \,\dot\cup\, K_1$ be an open coloring defined by

$$K_0 = \{\{\langle u_0, v_0\rangle, \langle u_1, v_1\rangle\} : u_0 < u_1 \text{ and } v_0 < v_1\}.$$

Hence $\{\langle u_0, v_0\rangle, \langle u_1, v_1\rangle\}$ is an element of K_0 if and only if $\{\langle u_0, v_0\rangle, \langle u_1, v_1\rangle\}$ is a strictly increasing function. We show that the case (2) in ($\mathrm{OCA}_{\aleph_0}$) is impossible for this open coloring. Assume in contrary that there are $Y_n \subseteq X$ for $n \in \omega$ such that $X = \bigcup_{n\in\omega} Y_n$ and Y_n is 1-homogeneous for all $n \in \omega$. Let

$$S_n = \{u \in U : \text{there are } v, v' \in V \text{ such that } v \neq v' \text{ and } \langle u, v\rangle, \langle u, v'\rangle \in Y_n\}$$

for $n \in \omega$.

Claim 20.1 *S_n is countable for all $n \in \omega$.*

$\vdash$ For each $u \in S_n$, let $a_u = \inf\{v \in V : \langle u, v\rangle \in Y_n\}$, $b_u = \sup\{u \in V : \langle u, v\rangle \in Y_n\}$ and I_u be the open interval with end points a_u and b_u. Then I_u, $u \in S_n$ are pairwise disjoint nonempty open intervals. It follows that $\{I_u : u \in S_n\}$ and hence S_n is countable.

$\dashv$ (Claim 20.1)

Since U is uncountable, there is $u^* \in U \setminus (\bigcup_{n\in\omega} S_n)$. Since V and hence $\{u^*\} \times V$ is uncountable, we can find $n^* \in \omega$ and v, $v' \in V$ such that $v \neq v'$ and $\langle u^*, v\rangle, \langle u^*, v'\rangle \in Y_{n^*}$. But then $u^* \in S_{n^*}$. This is a contradiction.

By $\mathrm{OCA}_{\aleph_1}$ there is an uncountable $Y \subseteq X$ such that $Y^2 \subseteq K_0$. By definition of K_0, Y is a strictly increasing function on the uncountable $U' = \{u \in U : \langle u, v\rangle \in Y \text{ for a } v \in V\}$ in V.

□ (Theorem 20)

Theorem 21 *If the conclusion of Theorem 18* (a) *holds then* $\mathbf{b} \geq \aleph_2$*. In particular* (OCA^-) *implies* $\mathbf{b} \geq \aleph_2$.

For the proof of the theorem we use the following lemma which is also of its own interest.

Lemma 22 *Let κ be a cardinal number and $(f_\alpha)_{\alpha<\kappa}$ a sequence in ${}^\omega\omega$ such that*

(0) *f_α is increasing for all $\alpha < \kappa$;*

(1) *$f_\alpha \leq^* f_\beta$ and $f_\alpha \neq^* f_\beta$ for all $\alpha < \beta < \kappa$ and*

(2) *$\{f_\alpha : \alpha < \kappa\}$ is unbounded with respect to $\leq^*$.*

Then for each $I \subseteq [\kappa]^\kappa$ there are α, $\beta \in I$ such that $\alpha < \beta$ and $f_\alpha \leq f_\beta$.

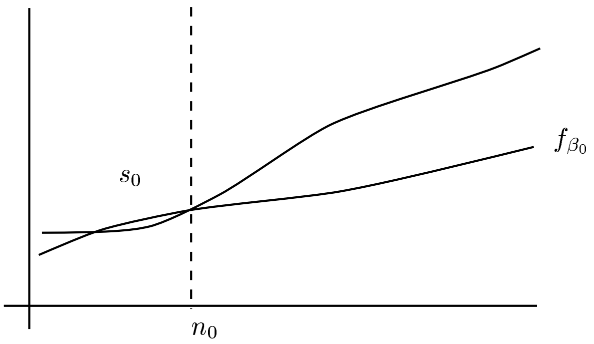
Proof For the increasing enumeration $\langle \xi_\alpha : \alpha < \kappa \rangle$ of I, the sequence $\langle f_{\xi_\alpha} : \alpha < \kappa \rangle$ satisfies also the conditions $(0), (1), (2)$. Hence it is enough to prove the lemma for the case $I = \kappa$. Let

$$S = \{ s \in {}^{\omega >}\omega : s \subseteq f_\alpha \text{ for an } \alpha < \kappa \}.$$

For $s \in S$ let $\alpha_s = \min\{\alpha < \kappa : s \subseteq f_\alpha\}$ and $\beta_0 = \sup\{\alpha_s : s \in S\}$. Clearly we have $\mathrm{cf}(\kappa) \geq \mathsf{b}$. Since S is countable, it follows that $\beta_0 < \kappa$. Note that $f_{\alpha_s} \leq^* f_{\beta_0}$ for all $s \in S$. Let $\varphi : \kappa \setminus \beta_0 \to \omega$ be such that

$$f_{\beta_0} \restriction (\omega \setminus \varphi(\alpha)) \leq f_\alpha \restriction (\omega \setminus \varphi(\alpha)) \qquad \text{for all } \alpha \in \kappa \setminus \beta_0.$$

Then there are $n_0 \in \omega$ and $X_0 \subseteq \kappa$ such that X_0 is cofinal in κ and $\varphi[X_0] = \{n_0\}$. $\{f_\alpha : \alpha \in X_0\}$ is still unbounded in ${}^\omega\omega$. We may modify X_0 if necessary so that there is $s_0 \in {}^{n_0}\omega$ such that $s_0 \subseteq f_\alpha$ for all $\alpha \in X_0$.



Claim 22.1 *There is an $n \in \omega$ such that for all $\alpha < \kappa$ and for all $m \in \omega$ there is a $\beta \in X_0$ such that $\alpha \leq \beta$ and $f_\beta(n) \geq m$.*

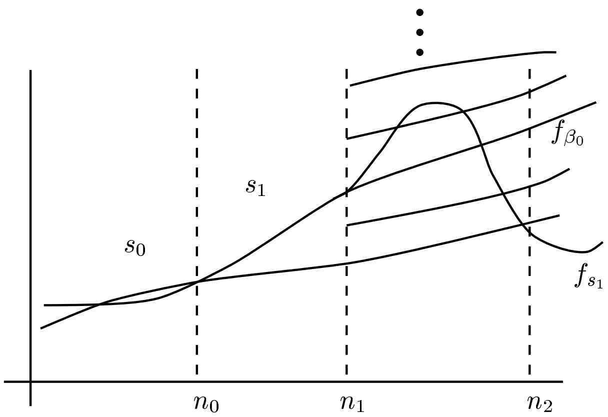
$\vdash$ Otherwise there are $\alpha_n < \kappa$ and $m_n \in \omega$ for each $n \in \omega$ such that

$$\forall \beta \in X_0 \, [\, \alpha_n \leq \beta \;\rightarrow\; f_\beta(n) < m_n \,].$$

Let $\gamma = \sup_{n \in \omega} \alpha_n$. Then $\gamma < \kappa$. Let $f \in {}^\omega\omega$ be defined by $f(n) = m_n$ for each $n \in \omega$. For all $\alpha < X_0$ such that $\gamma \leq \alpha$ we have then $f_\alpha \leq f$. But this is a contradiction to the fact that $\{f_\alpha : \alpha \in X_0\}$ is unbounded in ${}^\omega\omega$. $\dashv$ (Claim 22.1)

Let $n_1 \in \omega$ be minimal with

$$\forall \alpha < \kappa \, \forall m \in \omega \exists \beta \in X_0 [\, \alpha \leq \beta \wedge f_\alpha(n_1) \geq m \,].$$



Then $n_0 \leq n_1$. By minimality of n_1, there are an $s \in {}^{n_1}\omega$ and $\beta_1 < \kappa$ such that $\beta_0 \leq \beta_1$ and $f_\alpha \restriction n_0 \leq s$ for all $\alpha \in X_0$ with $\beta_1 \leq \alpha$. Since $S' = \{t \in {}^{n_1}\omega \,:\, t \leq s\}$ is finite there are an $s_1 \in S'$ and $X_1 \subseteq X_0$ such that $s_1 \subseteq f_\alpha$ for all $\alpha \in X_1$ and $\{f_\alpha(n_1) \,:\, \alpha \in X_1\}$ in ω is unbounded.

Let $n_2 \in \omega$ be such that $n_1 \leq n_2$ and

$$f_{\alpha_{s_1}} \restriction (\omega \setminus n_2) \leq f_{\beta_0} \restriction (\omega \setminus n_2).$$

Now let $\alpha^* \in X_1$ be such that $f_{\alpha^*}(n_1) \geq f\alpha_{s_1}(n_2)$. Then we have $f_{\alpha_{s_1}} \leq f_{\alpha^*}$: let $n \in \omega$. If $n < n_1$, then $f_{\alpha_{s_1}}(n) = s_1(n) = f_{\alpha^*}(n)$. If $n \in n_2 \setminus n_1$, then $f_{\alpha_{s_1}}(n) \leq f_{\alpha_{s_1}}(n_2) \leq f_{\alpha^*}(n_1) \leq f_{\alpha^*}(n)$. If $n \in \omega \setminus n_2$, then $f_{\alpha_{s_1}}(n) \leq f_{\beta_0}(n) \leq f_{\alpha^*}(n)$. □ (Lemma 22)

Proof of Theorem 21: Otherwise there is a sequence $(f_\alpha)_{\alpha<\omega_1}$ in ${}^\omega\omega$ such that $(0), (1), (2)$ in Lemma 22 hold. For each $\alpha < \omega_1$ let $x_\alpha = \{\langle n, m\rangle \in \omega \times \omega \,:\, f_\alpha(n) \leq m\}$. $(x_\alpha)_{\alpha<\omega_1}$ is then a tower. Let $X = \{x_\alpha \,:\, \alpha < \omega_1\}$. We show that X has neither uncountable chain nor uncountable antichain with respect to $\subseteq$. This is a contradiction to the conclusion of Theorem 18 (a). Let $I \subseteq \omega_1$ be uncountable. Then there are α, $\beta \in I$ such that $\alpha < \beta$ and $x_\alpha \not\supseteq x_\beta$: otherwise $\langle x_\alpha \,:\, \alpha \in I\rangle$ would be strictly increasing with respect to $\subseteq$. But this is impossible since $\omega \times \omega$ is countable. Hence $\{x_\alpha \,:\, \alpha \in I\}$ has no chain. On the other hand, by Lemma 22, there are α, $\beta \in I$ such that $\alpha < \beta$ and $f_\alpha \leq f_\beta$. Hence $x_\alpha \supseteq x_\beta$ and $\{x_\alpha \,:\, \alpha \in I\}$ is neither an antichain. □ (Theorem 21)

We show now that $\mathbf{b} = \aleph_2$ under (OCA). For the proof, we need the following two theorems. For arbitrary $f_0, \ldots, f_{k-1} \in {}^\omega\omega$, let $\Gamma(f_0, \ldots, f_{k-1})$ be the minimal $m \in \omega$ such that:

$$f_i \leq^* f_j \;\Leftrightarrow\; f_i \restriction (\omega \setminus m) \leq f_j \restriction (\omega \setminus m)$$

for all i, $j < k$.

Theorem 23 *There is no (κ, ω)-gap for any $\kappa < \mathsf{b}$.*

Proof Toward a contradiction, let us assume that there is a (κ, ω)-gap $\langle f_\alpha, g_k\rangle_{\alpha<\kappa,k<\omega}$ for some $\kappa < \mathsf{b}$. Without loss of generality, we may assume that $g_k \geq g_{k+1}$ for all $k < \omega$. For each $\alpha < \kappa$, let $h_\alpha \in {}^\omega\omega$ be defined by:

$$h_\alpha(k) = \Gamma(f_\alpha, g_0, \ldots, g_k)$$

for $k \in \omega$.

Since $\kappa < \mathsf{b}$, there is an $h \in {}^\omega\omega$ such that $h_\alpha \leq^* h$ for all $\alpha < \kappa$. Let $h' \in {}^\omega\omega$ be defined by: $h'(k) = \max(\{h'(n) \,:\, n < k\} \cup \{h(k)\}) + 1$. Then h' is strictly increasing, $h_\alpha \leq^* h'$ and $h_\alpha \neq^* h'$ for all $\alpha < \kappa$. Let $g \in {}^\omega\omega$ be now defined by:

$$g(n) = \begin{cases} 0, & \text{if } n < h'(0); \\ g_k(n) \ (= \min\{g_l(n) \,:\, l \leq k\}), & \text{if } h'(k) \leq n < h'(k+1) \text{ for some } k \in \omega. \end{cases}$$

We show that $f_\alpha \leq^* g \leq^* g_k$ holds for all $\alpha < \kappa$ and $k < \omega$. This is a contradiction to the fact that $\langle f_\alpha, g_k\rangle_{\alpha<\kappa,k<\omega}$ is a gap.

$g \leq^* g_k$: for all $n \in h'(l+1) \setminus h'(l)$ such that $k \leq l$, we have $g(n) = g_l(n) \leq g_k(n)$ by definition of g. Hence $\{n \in \omega : g(n) > g_k(n)\} \subseteq h'(k)$.

$f_\alpha \leq^* g$: let $m \in \omega$ be such that $h_\alpha \restriction (\omega \setminus m) \leq h' \restriction (\omega \setminus m)$. By definition of g, we have

$$f_\alpha(n) = f_\alpha \restriction (\omega \setminus h'(k))(n) \leq g_k \restriction (\omega \setminus h'(k))(n) = g_k \restriction (h'(k+1) \setminus h'(k))(n) = g(n)$$

for $k \geq m$ and $n \in h'(k+1) \setminus h'(k)$. Hence $\{n \in \omega : f_\alpha(n) > g(n)\} \subseteq h'(m)$. □ (Theorem 23)

Theorem 24 *For $\kappa < \mathsf{b}$ such that $\mathrm{cf}(\kappa) > \omega$, there is a (κ, λ)-gap for some $\lambda \in \mathrm{On}$ such that $\mathrm{cf}(\lambda) \geq \omega_1$.*

Proof Let $\langle f_\alpha : \alpha < \kappa \rangle$ be inductively constructed as follows: let $f_0 \in {}^\omega\omega$ be arbitrary. If f_ξ for all $\xi < \alpha$ have been constructed, then let $f \in {}^\omega\omega$ be such that $f_\xi \leq^* f$ for all $\xi < \alpha$ (we can find such f since $\alpha < \kappa < \mathsf{b}$). Let $f_\alpha \in {}^\omega\omega$ be defined by:

$$f_\alpha(n) = \max\{f(m) : m \leq n\} + 1.$$

Then f_α is increasing and, for all $\alpha < \kappa$, there is an $n \in \omega$ such that

$$f_\alpha \restriction (\omega \setminus n) < f_{\alpha+1} \restriction (\omega \setminus n).$$

Now let $\langle g_\beta : \beta < (2^{\aleph_0})^+ \rangle$ be constructed inductively so that:

(1) $f_\alpha \leq^* g_0$ for all $\alpha < \kappa$ (we can find such g_0 since $\kappa < \mathsf{b}$);

(2) If g_ξ for all $\xi < \beta$ have been constructed, then let $g_\beta = g$ for a $g \in {}^\omega\omega$ such that $f_\alpha \leq^* g \leq^* g_\xi$ and $g \neq^* g_\xi$ for all $\alpha < \kappa$ and $\xi < \beta$, if there is such g. Otherwise let $g_\beta = \emptyset$.

Let $\lambda = \sup\{\beta < (2^{\aleph_0})^+ : g_\beta \neq \emptyset\}$.

Claim 24.1 *λ is not a successor.*

$\vdash$ For $\beta < \lambda$, let $g \in {}^\omega\omega$ be defined by: $g(n) = g_\beta(n) \dot{-} 1$. Then $g \leq^* g_\beta$ and $g \neq^* g_\beta$. $f_\alpha \leq^* g$ for all $\alpha < \omega_1$: for $\alpha < \kappa$, let $n \in \omega$ such that

$$f_\alpha \restriction (\omega \setminus n) < f_{\alpha+1} \restriction (\omega \setminus n) \leq g_\beta \restriction (\omega \setminus n).$$

Then we have $f_\alpha \restriction (\omega \setminus n) \leq f_{\alpha+1} \restriction (\omega \setminus n) - 1 \leq g \restriction (\omega \setminus n)$. Hence $g_{\beta+1} \neq$ and $\beta + 1 < \lambda$.
$\dashv$ (Claim 24.1)

Thus $\langle f_\alpha, g_\beta \rangle_{\alpha<\kappa, \beta<\lambda}$ is a gap. By Theorem 23 it follows that $\mathrm{cf}(\lambda) \geq \omega_1$.
□ (Theorem 24)

Theorem 25 (OCA) *There is no (κ, λ)-gap for any κ, $\lambda \in \mathrm{On}$ such that $\mathrm{cf}(\kappa) \geq \omega_2$ and $\mathrm{cf}(\lambda) \geq \omega_1$.*

Proof Suppose that $\langle g_\alpha, f_\beta\rangle_{\alpha<\kappa,\beta<\lambda}$ is a (κ,λ)-gap for some κ and λ such that $\mathrm{cf}(\kappa)\geq\omega_2$ and $\mathrm{cf}(\lambda)\geq\omega_1$. Then we show that we can construct a counter-example to OCA. Without loss of generality, we may assume that κ and λ are regular cardinal such that $\kappa\geq\omega_2$ and $\lambda\geq\omega_1$.

For each $\alpha<\kappa$ let n_α be such that $\{\beta<\lambda\,:\,\Gamma(f_\alpha,g_\beta)=n_\alpha\}$ is cofinal in λ. Then there is an $n^*<\omega$ such that $\{\alpha<\kappa\,:\,n_\alpha=n^*\}$ is cofinal in κ. Without loss of generality, we may assume that $\{\alpha<\kappa\,:\,n_\alpha=n^*\}=\kappa$: otherwise we may replace $\langle g_\alpha\,:\,\alpha<\kappa\rangle$ by $\langle g_{\xi_\alpha}\,:\,\alpha<\kappa\rangle$, where $\langle\xi_\alpha\,:\,\alpha<\kappa\rangle$ is a strictly increasing enumeration of $\{\alpha<\kappa\,:\,n_\alpha=n^*\}$.

For each $\alpha<\kappa$ let $B_\alpha=\{\beta<\lambda\,:\,\Gamma(f_\alpha,g_\beta)=n^*\}$. Now let

$$X=\{\langle f_\alpha,g_\beta\rangle\,:\,\alpha<\kappa,\,\beta\in B_\alpha\}$$

and $[X]^2=K_0\,\dot\cup\,K_1$ be a partition given by:

$$K_0=\{\{\langle f,g\rangle,\langle f',g'\rangle\}\in[X]^2\,:\,\max\{\Gamma(f,g'),\Gamma(f',g)\}>n^*\}.$$

Clearly K_0 is symmetric.

Claim 25.1 *K_0 is open in $[X]^2$.*

$\vdash$ For $\{\langle f,g\rangle,\langle f',g'\rangle\,:\in\}K_0$, let $n^\dagger=\max\{\Gamma(f,g'),\Gamma(f',g)\}$ and

$$\begin{aligned}O=\{((f_0,g_0),(f_1,g_1))\in({}^\omega\omega)^2\times({}^\omega\omega)^2\,:\,&f_0\restriction n^\dagger=f\restriction n^\dagger,\ g_0\restriction n^\dagger=g\restriction n^\dagger,\\ &f_1\restriction n^\dagger=f'\restriction n^\dagger\text{ and }g_1\restriction n^\dagger=g'\restriction n^\dagger\}.\end{aligned}$$

O is then open in $({}^\omega\omega)^2\times({}^\omega\omega)^2$ and $\{\langle f,g\rangle,\langle f',g'\rangle\}\in O\cap X^2\subseteq K_0$.

$\dashv$ (Claim 25.1)

We show that none of (a) and (b) in the definition of OCA(X) holds for this partition.

Toward a contradiction, let us first assume that there is an uncountable $Y\subseteq X$ such that $[Y]^2\subseteq K_0$.

Claim 25.2 *$g\neq g'$ for all distinct $\langle f,g\rangle$, $\langle f',g'\rangle\in Y$.*

Proof Otherwise there are f, f', g, g' such that $\langle f,g\rangle$, $\langle f',g'\rangle\in Y$, $f\neq f'$ and $g=g'$. Then $\Gamma(f,g')=\Gamma(f,g)=n^*$ and $\Gamma(f',g)=\Gamma(f',g')=n^*$. But this is a contradiction to $[Y]^2\subseteq K_0$. $\dashv$ (Claim 25.2)

Claim 25.3 *There is a strictly increasing sequence $\langle\alpha_\xi\,:\,\xi<\omega_1\rangle$ in κ and a strictly increasing sequence $\langle\beta_\xi\,:\,\xi<\omega_1\rangle$ in λ such that $\langle f_{\alpha_\xi},g_{\beta_\xi}\rangle\in Y$ for all $\xi<\omega_1$.*

$\vdash$ Let $I=\{\alpha<\kappa\,:\,\langle f_\alpha,g_\beta\rangle\in Y$ for some $\beta<\lambda\}$. By Claim 25.2, the mapping $\varphi:I\to\lambda;,\,\alpha\mapsto\min\{\beta<\lambda\,:\,\langle f_\alpha,g_\beta\rangle\in Y\}$ is injective. Since I is uncountable, there is an $I'\subseteq I$ such that $otp(I')=\omega_1$ and $otp(\varphi''I')=\omega_1$[6]. Let α_ξ, $\xi<\omega_1$ be a strictly

[6] Let $I_0\subseteq I$ be such that $otp(I_0)=\omega_1$.
Then $|\,\varphi''I_0\,|=\aleph_1$. Let $\beta^*<\lambda$ be such that $\beta^*=\min\{\beta<\lambda\,:\,|\,\varphi''I_0\cap\beta\,|=\aleph_1\}$. Let $I'=\varphi^{-1\prime\prime}(\varphi''I_0\cap\beta^*)$.

increasing sequence in I' such that $\alpha_\xi > \sup\{\alpha \in I' : \varphi(\alpha) \leq \varphi(\alpha_\eta) \text{ for an } \eta < \xi\}$ for all $\xi < \omega_1$. The sequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ and $\langle \varphi(\alpha_\xi) : \xi < \omega_1 \rangle$ are then as desired.

$\dashv$ (Claim 25.3)

Let $\langle \alpha_\xi : \xi < \omega_1 \rangle$ and $\langle \beta_\xi : \xi < \omega_1 \rangle$ be as in Claim 25.3. Let $\alpha^* = \sup\{\alpha_\xi : \xi < \omega_1\}$. Since $\kappa = \mathrm{cf}\,\kappa > \omega_1$, we have $\alpha^* < \kappa$. Then, there is an $n^{**} \in \omega$ such that $n^* \leq n^{**}$ and uncountable $I_0 \subseteq \omega_1$, such that:

$(*)$ $\Gamma(f_{\alpha_\xi}, f_{\alpha^*})$, $\Gamma(f_{\alpha^*}, g_{\alpha_\xi}) \leq n^{**}$ for all $\xi \in I_0$.

Since ${}^{n^{**}}\omega$ is countable, We can find uncountable $I_1 \subseteq I_0$ and $s \in {}^{n^{**}}\omega$ such that

$(**)$ $f_{\alpha_\xi} \restriction n^{**} = s$ for all $\xi \in I_1$.

By $Y^2 \subseteq K_0$ the following assertion leads to a desired contradiction:

Claim 25.4 *Let ξ, $\eta \in I_1$ be such that $\xi \neq \eta$. Then $\Gamma(f_{\alpha_\xi}, g_{\beta_\eta}) \leq n^*$.*

$\vdash$ For $n \in \omega \setminus n^{**}$, we have $f_{\alpha_\xi}(n) \leq f_{\alpha^*}(n) \leq g_{\beta_\eta}(n)$ by $(*)$.
For $n \in n^{**} \setminus n^*$, $f_{\alpha_\xi}(n) = s(n) = f_{\alpha_\eta}(n) \leq g_{\beta_\eta}(n)$ by $(**)$ and $\beta_\eta \in B_{\alpha_\eta}$.

$\dashv$ (Claim 25.4)

Assume now that there are $Y_n \subseteq X$, $n < \omega$ such that $X = \bigcup_{n<\omega} Y_n$ and $[Y_n]^2 \subseteq K_1$ for all $n < \omega$. For $n < \omega$, let

$$I_n = \{\alpha < \kappa : \langle f_\alpha, g_\beta \rangle \in Y_n \text{ for some } \beta < \lambda\} \qquad \text{and}$$

$$J_n = \{\beta < \lambda : \langle f_\alpha, g_\beta \rangle \in Y_n \text{ for some } \alpha < \omega_1\}.$$

Claim 25.5 *There is an $n_0 \in \omega$ such that I_{n_0} is cofinal in κ and J_{n_0} is cofinal in λ.*

$\vdash$ Otherwise for each $n < \omega$, there would be an $\alpha_n < \kappa$ and a $\beta_n < \lambda$ such that

$$Y_n \subseteq (\alpha_n \times \lambda) \cup (\kappa \times \beta_n).$$

Let $\alpha^* = \sup_{n<\omega} \alpha_n$ and $\beta^* = \sup_{n<\omega} \beta$. Then $\alpha^* < \kappa$ and $\beta^* < \lambda$. Let $\beta^\dagger \in B_{\alpha^*}$ be such that $\beta^* \leq \beta^\dagger$. Then $\langle f_{\alpha^*}, g_{\beta^\dagger} \rangle \in X$. But by $\bigcup_{n<\omega} (\alpha_n \times \lambda) \cup (\kappa \times \beta_n) \not\ni \langle \alpha^*, \beta^\dagger \rangle$ is $\langle f_{\alpha^*}, g_{\beta^\dagger} \rangle \notin \bigcup_{n<\omega} Y_n$. This is a contradiction. $\dashv$ (Claim 25.5)

Let n_0 be as above in Claim 25.5. Let $\langle f^*, g^* \rangle$ be an element in in Y_{n_0}. For each further element $\langle f, g \rangle$ of Y_{n_0} we have $f \restriction (\omega \setminus n^*) \leq g^* \restriction (\omega \setminus n^*)$ by $[Y_{n_0}]^2 \subseteq K_1$. Hence we can define $h \in {}^\omega\omega$ by:

$$h(n) = \begin{cases} 0, & \text{if } n < n^*; \\ \max\{f(n) : (f, g) \in Y_{n_0} \text{ for some } g\}, & \text{otherwise.} \end{cases}$$

For all $\langle f, g \rangle \in Y_{n_0}$ we have then $f \leq^* h \leq^* g$. Since I_{n_0} is cofinal in κ and J_{n_0} is cofinal in λ, we have $f_\alpha \leq^* h \leq^* g_\beta$ for all $\alpha < \kappa$ and $\beta < \lambda$. But since $\langle f_\alpha, g_\beta \rangle_{\alpha<\kappa, \beta<\lambda}$ is a gap, this is a contradiction. ❑ (Theorem 25)

Theorem 26 (OCA) $\mathfrak{b} = \aleph_2$.

Proof By Theorem 21, Theorem 24 and Theorem 25. ❑ (Theorem 26)

7 Entangled sets

In a Cohen model (i.e. a model of ZFC obtained by adding Cohen reals to any ground model), all of the consequences of OCA and BA considered in previous sections do not hold. Hence we see that none of these consequences of OCA and BA are theorems in ZFC or even in ZFC + $\neg$CH. The notion of entangledness of subsets of $\mathbb{R}$ in the following sense can be used to see this: for cardinals κ, λ with $\lambda \leq \kappa$, a linear ordering $\langle I, \leq_I \rangle$ is said to be (κ, λ)-*entangled* if the following holds:

> For any $\lambda^* < \lambda$ and any system $\langle a_{\alpha,\beta} : \alpha < \kappa, \beta \leq \lambda^* \rangle$ of pairwise distinct elements of I, whenever $u \subseteq \lambda^*$, there are always some $\alpha' < \alpha'' < \kappa$ such that $u = \{\beta \in \lambda^* : a_{\alpha',\beta} <_I a_{\alpha'',\beta}\}$.

Proposition 27 (U. Avraham and S. Shelah [1]) *Suppose that G is $\mathrm{Fn}(\kappa \times \omega, 2)$-generic over V for some $\kappa \geq \aleph_1$. For each $\alpha < \kappa$ let $g_\alpha : \omega \to 2$ be defined in $V[G]$ by $g_\alpha(n) = \left(\bigcup G\right)(\alpha, n)$. Then $R = \{g_\alpha : \alpha < \kappa\}$ endowed with lexicographical ordering is $(\aleph_1, \aleph_0)$-entangled.*

Note that R above is embeddable into the Contor discontinuum and hence also embeddable into $\mathbb{R}$ (see also Lemma 30).

Proof of Proposition 27: Let $P = \mathrm{Fn}(\kappa \times \omega, 2)$ and let $\dot{g}_\alpha$, $\alpha < \kappa$ be canonical P-names of g_α, $\alpha < \kappa$ and let $\dot{R}$ be a P-name such that $\Vdash_P$ "$\dot{R} = \{\dot{g}_\alpha : \alpha < \kappa\}$". Let $\dot{<}$ be a P-name for the lexicographical ordering on $\dot{R}$.

Suppose that

$$p \Vdash_P \text{“}\langle \dot{a}_{\xi,i} : \xi < \omega_1, i < n \rangle \text{ is a pairwise distinct system of elements of } \dot{R}\text{”}$$

for some $n < \omega$, and $u \subseteq n$. For each $\xi < \omega_1$ let $p_\xi \in P$ be such that $p_\xi \leq p$ and there are $\alpha_{\xi,0}, \ldots, \alpha_{\xi,n-1} < \kappa$ such that

$$p_\xi \Vdash_P \text{“}\dot{a}_{\xi,0} = \dot{g}_{\alpha_{\xi,0}}, \ldots, \dot{a}_{\xi,n-1} = \dot{g}_{\alpha_{\xi,n-1}}\text{”}.$$

By Δ-system lemma and by thinning out the index set ω_1, we may assume that $\langle \mathrm{dom}(p_\xi) : \xi < \omega_1 \rangle$ form a Δ-system with the root r and that $p_\xi \restriction r$ is the same for all $\xi < \omega_1$. Since p_ξ, $\xi < \omega_1$ are then pairwise compatible, $\langle \alpha_{\xi,i} : \xi < \omega_1, i < n \rangle$ is pairwise distinct. Hence we may further assume that, for $\xi < \xi' < \omega_1$,

$$\mathrm{dom}(p_\xi) \cap \{\alpha_{\xi',0}, \ldots, \alpha_{\xi',n-1}\} = \emptyset.$$

For each $\xi < \omega_1$, let

$$t_\xi = \{\langle i, k, l \rangle : \langle \langle \alpha_{\xi,i}, k \rangle, l \rangle \in p_\xi\}.$$

Since there are only countably many possibilities of t_ξ, there are $\xi^* < \xi^{**} < \omega_1$ such that $t_{\xi^*} = t_{\xi^{**}}$. Now it is easy to see that we can extend $p_{\xi^*} \cup p_{\xi^{**}}$ to a $q \in P$ such that

$$q \Vdash_P \text{“}u = \{i < n : \dot{a}_{\xi^*,i} \mathrel{\dot{<}} \dot{a}_{\xi^{**},i}\}\text{”}.$$

❑ (Proposition 27)

Actually we can obtain an entangled set just by adding a single Cohen real:

Theorem 28 (Y. Yuasa [15], M. Bekkali [?]) *Let* $P = \mathrm{Fn}(\omega, 2)$. *Then*

$$\Vdash_P \text{“}\textit{there is an } (\aleph_1, \aleph_0)\textit{-entangled set of reals of cardinality } 2^{\aleph_0}\text{”}. \qquad ❑$$

Proposition 29 (1) *Suppose that* $\langle I, \leq \rangle$ *is an uncountable* $(\aleph_1, 3)$*-entangled subordering of* $\langle \mathbb{R}, \leq \rangle$. *If* I_0 *and* I_1 *are disjoint uncountable subsets of* I *then there is no order preserving injection from* I_0 *to* I_1. *Hence, if there exists an uncountable* $(\aleph_1, 3)$*-entangled subordering of* $\langle \mathbb{R}, \leq \rangle$, *then the assertions of all of the following propositions do not hold: Theorem 18* (b), *Theorem 18* (a), *Theorem 18* (d), (†) *in Corollary 12. In particular, we have* $\neg(\mathrm{OCA}^-)$ *and* $\neg(\mathrm{BA})$ *under the existence of such a subordering of* $\langle \mathbb{R}, \leq \rangle$.

(2) (S. Shelah) *For any linear ordering* $\langle I, \leq \rangle$ *and the interval algebra* B *over* $\langle I, \leq \rangle$, *the following are equivalent:*

(a) $\langle I, \leq \rangle$ *is* $(\kappa, \aleph_0)$*-entangled.*

(b) *For any* $X \in [B]^\kappa$ *there are two distinct* a, $b \in X$ *such that* $a < b$.

In particular, if there is an uncountable $(\kappa, \aleph_0)$*-entangled linear ordering, then the assertion in Theorem 18* (c) *does not hold.*

Proof (1): Suppose that I_0, $I_1 \in [I]^{\aleph_1}$ are such that $I_0 \cap I_1 = \emptyset$ and $f : I_0 \to I_1$ is 1-1. We show that f is not increasing or decreasing.

Let $I_0 = \{x_\alpha : \alpha < \omega_1\}$. For $\alpha < \omega_1$ and $i < 2$ let

$$x_{\alpha,i} = \begin{cases} x_\alpha, & \text{if } i = 0; \\ f(x_\alpha), & \textit{otherwise.} \end{cases}$$

Then by $(\aleph_1, 3)$-entangledness, there are $\alpha' < \alpha'' < \omega_1$ such that $x_{\alpha',0} < x_{\alpha'',0}$ and $x_{\alpha',1} > x_{\alpha'',1}$. But this means $x_{\alpha'} < x_{\alpha''}$ and $f(x_{\alpha'}) > f(x_{\alpha''})$. Hence f is not increasing.

Similarly there are $\alpha^* < \alpha^{**} < \omega_1$ such that $x_{\alpha^*,0} < x_{\alpha^{**},0}$ and $x_{\alpha^*,1} < x_{\alpha^{**},1}$. Hence f is not decreasing.

(2): (a) $\Rightarrow$ (b): Let B be the interval algebra of an $(\kappa, \aleph_0)$-entangled linear ordering $\langle I, \leq \rangle$. Let b_α, $\alpha < \kappa$ be pairwise distinct elements of B. Say

$$b_\alpha = [x_0^\alpha, x_1^\alpha) \,\dot\cup\, [x_2^\alpha, x_3^\alpha) \,\dot\cup \cdots \dot\cup\, [x_{2n_\alpha-2}^\alpha, x_{2n_\alpha-1}^\alpha)$$

where $n_\alpha \in \omega$; x_0^α, x_1^α,..., $x_{2n_\alpha-1}^\alpha \in I \cup \{-\infty, \infty\}$, and

$$x_0^\alpha < x_1^\alpha < \cdots < x_{2n_\alpha-2}^\alpha < x_{2n_\alpha-1}^\alpha.$$

Here we mean with the half-open interval $[-\infty, x)$ the set $\{y \in I : y < x\}$. By thinning out the sequence $\langle b_\alpha : \alpha \in \kappa \rangle$, we may assume that n_α, $\alpha < \kappa$ are all the same. Say $n_\alpha = n$ for $\alpha < \kappa$. We may further assume that $\langle \{x_0^\alpha, \ldots, x_{2n-1}^\alpha\} : \alpha < \kappa \rangle$ forms a Δ-system with the root r and there is an $s \subseteq 2n$ such that $r = \{x_i^\alpha : i \in s\}$ for all $\alpha < \kappa$.

Let $m = 2n - |s|$. Applying (κ, m)-entangledness of I to the system $\{x_i^\alpha : \alpha < \kappa,\ i \in 2n \setminus s\}$, we can find $\alpha' < \alpha'' < \kappa$ such that

$$\begin{aligned} &x_0^{\alpha'} \leq x_0^{\alpha''}, x_2^{\alpha'} \leq x_2^{\alpha''}, \ldots, x_{2n-2}^{\alpha'} \leq x_{2n-2}^{\alpha''},\\ &x_1^{\alpha'} \geq x_1^{\alpha''}, x_3^{\alpha'} \geq x_3^{\alpha''}, \ldots, x_{2n-1}^{\alpha'} \geq x_{2n-1}^{\alpha''}. \end{aligned}$$

But this means $b_{\alpha''} > b_{\alpha'}$.

(b) $\Rightarrow$ (a): Suppose that, for any $X \in [B]^\kappa$, there are two distinct $a, b \in X$ such that $a < b$. If $|I| < \kappa$, then I is $(\kappa, \aleph_0)$-entangled vacuously. So let us assume that $|I| \geq \kappa$.

Claim 29.1 *There are only $< \kappa$ many nonempty open intervals in I with ≤ 2 elements.*

$\vdash$ Otherwise we could find κ many pairwise disjoint nonempty open intervals say a_α, $\alpha < \kappa$. For each $\alpha < \kappa$ let $\tilde{a}_\alpha$ be the half open interval with the same end-points as a_α. Then $X = \{\tilde{a}_\alpha : \alpha < \kappa\}$ contradicts the assumption (b). $\dashv$ (Claim 29.1)

Claim 29.2 *I has a dense subset of size $< \kappa$.*

$\vdash$ Suppose otherwise. Then by the previous Claim we can find inductively x_ξ^i for $i < 4$ and $\xi < \kappa$ such that, for all $\xi < \kappa$, we have $x_\xi^0 < x_\xi^1 < x_\xi^2 < x_\xi^3$ and

$$[x_\xi^0, x_\xi^3] \cap \overline{\{x_\eta^i : i < 4,\ \eta < \xi\}} = \emptyset$$

where $[x, y]$ denotes the closed interval with end points x, y.

For $\xi < \kappa$ let

$$a_\xi = [-\infty, x_\xi^0) \mathbin{\dot\cup} [x_\xi^1, x_\xi^2) \mathbin{\dot\cup} [x_\xi^3, \infty).$$

By construction, for any $\xi < \xi' < \kappa$, either $[x_{\xi'}^0, x_{\xi'}^3] \subseteq [x_\xi^i, x_\xi^{i+1}]$ for some $i < 3$ or, $[x_\xi^0, x_\xi^3]$ and $[x_{\xi'}^0, x_{\xi'}^3]$ are disjoint. In both cases we can show easily that a_ξ and $a_{\xi'}$ are incomparable. Hence $X = \{a_\xi : \xi < \kappa\}$ contradicts the assumption of (2). $\dashv$ (Claim 29.2)

Now, to prove that (a) holds, let $\langle x_{\alpha,i} : \alpha < \kappa,\ i < n\rangle$ be a system of pairwise disjoint elements of I and $u \subseteq n$. By thinning out the index set κ and reindexing $i < n$ (and, if necessary, changing the set u depending on the reindexing), we may assume that $x_{\alpha,0} < x_{\alpha,1} < \cdots < x_{\alpha,n-1}$ for all $\alpha < \kappa$. By Claim above and by thinning out the index set κ further, we may assume that there are $y_0, \ldots, y_n \in I \cup \{-\infty, \infty\}$ such that

$$y_0 < x_{\alpha,0} < y_1 < x_{\alpha,1} < \cdots < y_{n-1} < x_{\alpha,n-1} < y_n$$

for all $\alpha < \kappa$. Let

$$a_\alpha = a_{\alpha,0} \mathbin{\dot\cup} a_{\alpha,1} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} a_{\alpha,n-1}$$

where

$$a_{\alpha,i} = \begin{cases} [y_i, x_{\alpha,i}), & \text{if } i \in u\\ [x_{\alpha,i}, y_{i+1}), & \text{otherwise.} \end{cases}$$

By assumption of (b), there are two distinct α, $\alpha' < \kappa$ such that $a_\alpha \leq a_{\alpha'}$. By definition of a_α and $a_{\alpha'}$, it follows that $\{i < n : x_{\alpha,i} < x_{\alpha',i}\} = u$. $\square$ (Proposition 29)

Lemma 30 *Every* $(\aleph_1,\aleph_0)$*-entangled set* $\langle I,\leq_I\rangle$ *has a countable dense subset and hence embeddable into* $\langle\mathbb{R},\leq\rangle$.

Proof By Claim 29.2. □ (Lemma 30)

By definition of entangledness we see immediately:

Fact 31 $\langle I,\leq\rangle$ *is* $(\kappa,3)$*-entangled if and only if for any disjoint* I^0, $I^1\in[I]^\kappa$ *there is no monotonous* $f: I^0\rightarrow I^1$. □

Hence Sierpinski's Theorem (Theorem 15) also reads:

There is a $2^{\aleph_0}$*-dense* $(2^{\aleph_0},3)$*-entangled* $K\in[\mathbb{R}]^{2^{\aleph_0}}$.

8 OCA does not imply BA

In this section, we consider a result by I. Farah in [5] which asserts that OCA does not imply BA. We shall begin with some well-known facts about c.c.c. forcing:

Lemma 32 *Suppose that* P *is a c.c.c. p.o. and* p_α, $\alpha<\omega_1$ *be a sequence of elements of* P. *Then there is* $\alpha^*<\omega_1$ *such that for all* $\alpha^*\alpha<\omega_1$,

$$p_\alpha \Vdash_P \text{“}\{\beta<\omega_1\,:\,p_\beta\in\dot{G}\}\ \textit{is uncountable}\text{”}.$$

Proof Otherwise there are ω_1 many $\alpha<\omega_1$ such that $p_\alpha\Vdash_P$ “$\{\beta<\omega_1\,:\,p_\beta\in\dot{G}\}$ is countable”. Hence by c.c.c. of P there is an $\eta_\alpha<\omega_1$ such that $p_\alpha\Vdash_P$ “$\{\beta<\omega_1\,:\,p_\beta\in\dot{G}\}\subseteq\eta_\alpha$”. Thus we can find an increasing sequence $\langle\alpha_\xi\,:\,\xi<\omega_1\rangle$ of ordinals $<\omega_1$ such that $p_{\alpha_\xi}\Vdash_P$ “$p_{\alpha_\eta}\notin\dot{G}$” for all $\xi<\eta<\omega_1$. But then $\{p_{\alpha_\xi}\,:\,\xi<\omega_1\}$ is an uncountable pairwise incompatible family. This is a contradiction to the c.c.c. of P. □ (Lemma 32)

Lemma 33 *For any c.c.c. Boolean algebras* A, B *the following are equivalent:*

(a) $A\oplus B$ *does not have the c.c.c.;*

(b) $a\Vdash_{A^+}$ “B *does not have the c.c.c.*” *for some* $a\in A^+$*;*

(c) $b\Vdash_{B^+}$ “A *does not have the c.c.c.*” *for some* $b\in B^+$.

Proof By symmetry, it is enough to show the equivalence of (a) and (b).

For (a) $\Rightarrow$ (b), suppose that $\{\langle a_\alpha,b_\alpha\rangle\,:\,\alpha<\omega_1\}$ is pairwise incompatible in $A\oplus B$. Then by Lemma 32, there is $a\in A^+$ such that $a\Vdash_{A^+}$ “$\{\alpha<\omega_1\,:\,a_\alpha\in\dot{G}\}$ is uncountable”. Then

$$a\Vdash_{A^+} \text{“}\{b_\alpha\,:\,a_\alpha\in\dot{G}\}\text{ is an uncountable family of pairwise incompatible elements of }B\text{”}.$$

Hence $a\Vdash_{A^+}$ “B does not have the c.c.c.”.

For (b) $\Rightarrow$ (a), suppose that $\dot{b}_\alpha$, $\alpha<\omega_1$ are A^+-names of elements of B such that

$$\Vdash_{A^+} \text{“}\dot{b}_\alpha\in B,\ \alpha<\omega_1\text{ are pairwise incompatible”}.$$

For each α, let $a_\alpha \in A^+$ and $b_\alpha \in B^+$ be such that

$$a_\alpha \Vdash_{A^+} \text{“}\,\dot{b}_\alpha = b_\alpha\,\text{”}.$$

Then $\{\langle a_\alpha, b_\alpha\rangle : \alpha < \omega_1\}$ is an uncountable family of pairwise incompatible elements of $A \oplus B$. ❑ (Lemma 33)

Lemma 34 *Suppose that $\langle x_\alpha : \alpha < \omega_1\rangle$ is an injective sequence of reals. Then there is $\alpha^* < \omega_1$ such that $\{x_\alpha : \alpha^* < \alpha < \omega_1\}$ is $\aleph_1$-dense.*

Proof Let $\langle I_n : n \in \omega\rangle$ be an enumeration of rational intervals in $\mathbb{R}$. For each $n \in \omega$, let

$$\alpha_n = \begin{cases} 0, & \text{if } I_n \cap \{x_\alpha : \alpha < \omega_1\} \text{ is uncountable} \\ \min\{\alpha < \omega_1 : I_n \cap \{x_\beta : \alpha < \beta < \omega_1\} = \emptyset\}, & \text{otherwise.} \end{cases}$$

Let $\alpha' = \sup_{n\in\omega} \alpha_n$ and $S = \{x_\alpha : \alpha' < \alpha < \omega_1\}$. Let $\alpha^* > \alpha'$ be large enough so that $\{x_\alpha : \alpha^* < \alpha < \omega_1\}$ does not contain any $p \in S$ such that either there is a rational $r < p$ such that $(r, p) \cap S = \emptyset$ or there is a rational $p < r'$ such that $(p, r) \cap S = \emptyset$. Then this α^* is as desired. ❑ (Lemma 34)

For a tree T, let us denote with P_T the partial ordering obtained by putting T “upside down” i.e. $P_T = \langle T, (\le_T)^{-1}\rangle$.

Lemma 35 (OCA) *If T is a Suslin tree then $\Vdash_{P_T}$ “OCA”.*

Proof Let $p \in P_T$, $\dot{X}$ a P_T-name of a subset of $\mathbb{R}$ and $\dot{c}$ a P_T-name for a Borel code for an open coloring of X. Suppose that there is $q' \le p$ such that

$$q' \Vdash_{P_T} \text{“}\,\dot{X} \text{ is not } \sigma\text{-1-homogeneous with respect to the open coloring coded by } c\,\text{”}.$$

Since P_T does not add any new real, there is $c \in \mathbb{R}$ and $q'' \le q$ such that $q'' \Vdash_{P_T}$ “$\dot{c} = c$”. Let $X = \{x \in \mathbb{R} : \exists r \le q''\ [r \Vdash_{P_T} \text{“}\,x \in \dot{X}\,\text{”}]\}$. Then X is not σ-1-homogeneous with respect to the open coloring coded by c. Hence there is $Y \in [X]^{\aleph_1}$ such that Y is 0-homogeneous. By Lemma 32, we have $q \Vdash_{P_T}$ “$\dot{X} \cap Y$ is uncountable” for some $q \le q''$. Clearly we also have $q \Vdash_{P_T}$ “$\dot{X} \cap Y$ is 0-homogeneous”. ❑ (Lemma 35)

Lemma 36 *For a Suslin tree T we have $\Vdash_{P_T}$ “$\mathbf{t} = \aleph_1$” and $\Vdash_{P_T}$ “$\neg$(BA)”.*

Proof We may assume that the nodes of T are elements of $[\omega]^{\aleph_0}$, $(\le_T)^{-1}$ coincides with $\subseteq^*$ and each level of T consists of pairwise almost disjoint elements of $[\omega]^{\aleph_0}$.

Let B be an ω_1-branch of T added generically by P_T. Then B is a tower in $([\omega]^{\aleph_0}, \subseteq^*)$: Otherwise there is $x \in [\omega]^{\aleph_0}$ such that $x \subseteq^* b$ for all $b \in B$. Since no new reals are added by P_T, x is in the ground model. But then B is definable from T and x and hence also in the ground model. This is a contradiction to the assumption that T is Suslin.

Now let $B = \{x_\alpha : \alpha < \omega_1\}$. Then, by Lemma 34, there is an $\alpha^* < \omega_2$ such that $B' = \{t(x_\alpha) : \alpha^* < \alpha < \omega_1\}$ is an ω_1-dense set where $t(x_\alpha) = \sum_{i\in x_\alpha} 3^{-i}$.

Let X be an $\aleph_1$-dense set in the ground model. Then there is no order preserving mapping from X to B' : If there were such an order preserving mapping from X to B' then the mapping is coded by some Borel code d which is a ground model set. Hence using X and d we could define an uncountable subset B'' of B' such an uncountable subset B'' of B' is in the ground model. But then we could reconstruct B from B''. Hence we would have $B \in V$ which is a contradiction as before. □ (Lemma 36)

Lemma 37 *Suppose that* CH *holds and T is a Suslin tree. Let $\langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\omega_2}$ and P_{ω_2} be the FS-iteration of c.c.c. p.o.s as in the proof of Theorem 8. Then we have* $\Vdash_{P_\alpha}$ "T remains a suslin tree" *for all $\alpha \leq \omega_2$.*

Proof We show the lemma by induction on $\alpha \leq \omega_2$. For successor step, it is enough to show the following:

(∗) If $V \models$ CH and

$V \models Q$ is the c.c.c. p.o. as defined in Theorem 7 for an open coloring $[X]^2 = K_0 \,\dot\cup\, K_1$ such that X is not σ-1-homogeneous.

Then $\Vdash_Q$ "T is a suslin tree".

By Lemma 33 it is enough to show that $\Vdash_{P_T}$ "Q is c.c.c."

Since T is Suslin P_T does not add any real, hence we have:

$$\Vdash_{P_T} \text{“}[X]^2 = K_0 \,\dot\cup\, K_1 \text{ is an open coloring such that } X \text{ is not } \sigma\text{-1-homogeneous”}.$$

Since $|\,T\,| = \aleph_1$, we also have $\Vdash_{P_T}$ "CH". Hence the proof of Theorem 7 goes through in the generic extension with P_T to show that $\Vdash_{P_T}$ "Q has the c.c.c.".

For limit steps, again by Lemma 33 it is enough to show that $\Vdash_{P_T}$ "P_α satisfies the c.c.c.". But this follows immediately from

$$\Vdash_{P_T} \text{“}\langle P_\beta, \dot{Q}_\beta \rangle_{\beta<\alpha} \text{ is a FS-iteration of c.c.c. p.o.s”}.$$

Hence by the theorem on the preservation of c.c.c. under FS-iteration (applied in V^{P_T}), it follows that $\Vdash_{P_T}$ "P_α satisfies the c.c.c.". □ (Lemma 37)

Putting together the lemmas above, we obtain the following:

Theorem 38 (I. Farah, [5]) OCA *does not imply* $\mathbf{t} = \aleph_2$. OCA *does not imply* BA.

Proof Starting from $V = L$, let T be a Suslin tree (which exists in L). Let $\langle P_\alpha, \dot{Q}_\alpha \rangle_{\alpha<\omega_2}$ and P_{ω_2} be as in Theorem 8. Then $\Vdash_{P_{\omega_2}}$ "OCA" by Theorem 8 and we have $\Vdash_{P_{\omega_2}}$ "T is a suslin tree" by Lemma 37. By Lemma 35, $\Vdash_{P_{\omega_2} * P_T}$ "OCA". On the other hand, by Lemma 36, we have $\Vdash_{P_{\omega_2} * P_T}$ "$\mathbf{t} = \aleph_1$ and $\neg$(BA)". □ (Lemma 38)

The argument in the proof of Lemma 37 for successor steps applies also to the p.o.s in Theorem 10. Hence we obtain:

Theorem 39 (I. Farah, [5]) *If* ZFC *is consistent then so is* ZFC *with the following assertions:*

$$\text{OCA}, \ \ \mathbf{t} = \aleph_1, \ \ (\dagger) \textit{ in Corollary 12}, \ \ \neg(\text{BA}). \qquad \square$$

Question 2 *Does* BA *imply* $\mathbf{t} > \aleph_1$ *?*

9 Weak versions of PFA which imply versions of OCA

It seems that we need some amount of consistency strength beyond that of ZFC alone to force general principles in some form of natural weakening of PFA which imply BA and/or versions of OCA.

One of such principles is the Bounded Proper Forcing Axiom (BPFA) by Goldstern and Shelah [6]:

(BPFA): *For any structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* $\aleph_1$, *if there is a proper p.o.* P *such that* $\Vdash_P$ "$\mathfrak{A} \cong \mathfrak{B}$" *then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are really isomorphic.*

It is proved in [6] that BPFA is equi-consistent with the existence of a reflecting cardinal. A cardinal κ is called *reflecting*, if κ is regular and for any $a \in \mathcal{H}(\kappa)$ and a formula $\varphi(x)$ in the language of ZFC, if $\mathcal{H}(\chi) \models \varphi[a]$ for some $\chi \geq \kappa$, then there is $\delta < \kappa$ such that $a \in \mathcal{H}(\delta)$ and $\mathcal{H}(\delta) \models \varphi[a]$. A reflecting cardinal sits quite low in the hierarchy of large cardinals:

Lemma 40 ([6])

(0) *If* κ *is reflecting then* κ *is hyperinaccessible.*

(1) *If* κ *is Mahlo, then* $V_\kappa \models$"there are 'stationary many' reflecting cardinals".

(2) *If* κ *is reflecting, then* $L \models$"κ is reflecting". $\square$

Proposition 41 BPFA *implies* BA *and* $\text{OCA}_{\aleph_1}$.

Proof For the implication of BA, suppose that X and Y are $\aleph_1$-dense sets. Consider the proper p.o. $(P * \dot{Q}) * \dot{R}$ where P is the σ-closed p.o. collapsing $|\,2^{\aleph_0}\,|$ to $\aleph_1$, $\Vdash_P$ "$\dot{Q}$ is the p.o. as in Theorem 10 for X and Y" and $\Vdash_{P * \dot{Q}}$ "$\dot{R}$ is a c.c.c. p.o. forcing $\mathbf{t} > \aleph_1$". Note that $\Vdash_P$ "CH" and hence $\Vdash_P$ "$\dot{Q}$ has the c.c.c." by Theorem 10. By Theorem 13 we have $\Vdash_{(P*\dot{Q})*\dot{R}}$ "$X \cong Y$". By BPFA, it follows that $X \cong Y$.

For the implication of $\text{OCA}_{\aleph_1}$, we use the following characterization of BPFA from [6]:

Lemma 42 ([6]) BPFA *is equivalent to the following assertion:*

> *For any proper* P, *if* $\mathcal{A} = \langle A_\alpha \,:\, \alpha < \omega_1 \rangle$ *is a family of maximal anti-chain in r.o.*(P) *of size* $\leq \omega_1$, *then there is an* $\mathcal{A}$*-generic filter over r.o.*(P). $\square$

Now suppose that $[X]^2 = K_0 \dot\cup K_1$ is an open coloring for some $X \in [\mathbb{R}]^{\aleph_1}$ which is not σ-1-homogeneous. Applying the version of BPFA as above to the proper p.o. $P * \dot{Q}'$ where P is again the standard σ-closed p.o. collapsing $|\,2^{\aleph_0}\,|$ to $\aleph_1$ and $\Vdash_P$ "$\dot{Q}'$ is as in Theorem 7 for the open coloring $[X]^2 = K_0 \dot\cup K_1$", we see that there is an uncountable 0-homogeneous set with respect to the open coloring. Note that we need $|\,X\,| = \aleph_1$ so that we can encode the condition, that the 0-homogeneous set to be obtained is uncountable, in maximal antichains of size $\leq \aleph_1$. □ (Proposition 42)

It is not clear if BPFA implies OCA. FA(Axiom A, $\leq 2^{\aleph_0}$) is one of the natural principles from which OCA immediately follows:

FA(Axiom A, $\leq 2^{\aleph_0}$): For any axiom A p.o. P of size $\leq 2^{\aleph_0}$, and for any family $\mathcal{D}$ of dense subsets of P of size $\leq \aleph_1$ there is a $\mathcal{D}$-generic set G over P.

Proposition 43 FA(Axiom A, $\leq 2^{\aleph_0}$) *implies* BA *and* OCA.

Proof An argument similar to the proof of Theorem 41 for the partial orderings defined there will do. Note the partial orderings in the proof of Theorem 41 are Axiom A p.o.s of size $\leq 2^{\aleph_0}$. Also note that, for the second part of the proof, we do not need the restriction that $|\,X\,| = \aleph_1$.

□ (Proposition 43)

In [2], it is stated that starting from a model with a weakly compact cardinal, we can construct a model of FA(Axiom A, $\leq 2^{\aleph_0}$)[7]. The reason for restriction to Axiom A can be seen in Lemma 49.

10 FA(Axiom A, $\leq 2^{\aleph_0}$)

Theorem 44 (Baumgartner) *Suppose that* ZFC + *"there exists a weakly compact cardinal" is consistent, then so is* ZFC + FA(Axiom A, $\leq 2^{\aleph_0}$).

To show this, we can just follow the usual consistency proof of PFA from a supercompact cardinal with some necessary modifications. The key to the proof is a local version of Laver function generically added to a weakly compact cardinal by fast function forcing. In the next subsection, we shall review some facts about fast function forcing needed in our context. The presentation in the next subsection is based on [7].

10.1 Fast Function Forcing

Suppose that κ is a (large) cardinal. Let $\mathrm{INACC}_{<\kappa} = \{\gamma < \kappa : \gamma \text{ is inaccessible}\}$ and

$$\mathbb{F}_\kappa = \{p : \mathrm{dom}(p) \subseteq \mathrm{INACC}_{<\kappa},\ \mathrm{rng}(p) \subseteq \kappa,\\ \forall\gamma \in \mathrm{dom}(p)\,[\,p''\gamma \subseteq \gamma \ \wedge\ |\,p\restriction\gamma\,| < \gamma\,]\}$$

[7]Actually, we can prove the equiconsistncy (see 10.3).

with reverse inclusion. For $\lambda < \kappa$, let

$$\mathbb{F}_{\lambda,\kappa} = \{p \in \mathbb{F}_\kappa \;:\; \mathrm{dom}(p) \cap \lambda = \emptyset\}$$

also endowed with reverse inclusion. If $f : \kappa \supseteq\!\!\rightarrow \kappa$ is a function generically added over V by $\mathbb{F}_\kappa$, then f is called a *fast function* over V (for $\mathbb{F}_\kappa$).

Lemma 45 $\mathbb{F}_{\lambda,\kappa}$ *is* $\leq \lambda$*-directedly closed.*

Proof Suppose that $\langle p_\alpha \;:\; \alpha < \delta\rangle$ is a decreasing sequence in $\mathbb{F}_{\lambda,\kappa}$ for some $\delta \leq \lambda$. Let $p^* = \bigcup_{\alpha<\delta} p_\delta$. It is enough to show that $p^* \in \mathbb{F}_{\lambda,\kappa}$. Suppose that $\gamma \in \mathrm{dom}(p^*)$. Then $\gamma \in \mathrm{dom}(p_\alpha)$ for some $\alpha < \delta$. Hence $\gamma \in \mathrm{INACC}_{<\kappa}$ and $\lambda \leq \gamma$. Also we have $p^{*\prime\prime}\gamma = \bigcup_{\alpha<\delta} p_\alpha{}''\gamma \subseteq \gamma$. If $\lambda = \gamma$ then $|\, p^*\restriction \gamma\,| \leq 1 < \gamma$. Otherwise $\delta \leq \lambda < \gamma$. Since γ is regular, it follows that $|\, p^*\restriction \gamma\,| = |\bigcup_{\alpha<\delta} p_\alpha\restriction \gamma\,| < \gamma$. □ (Lemma 45)

If f is a fast function and $\gamma \in \mathrm{dom}(f)$, then $f\restriction \gamma$ is a fast function over V for $\mathbb{F}_\gamma$. Moreover, $f\restriction \gamma \times f\restriction [\gamma, \kappa)$ corresponds to a generic set over $\mathbb{F}_\gamma \times \mathbb{F}_{\lambda,\kappa}$ for $\lambda = \min\left([\gamma, \kappa) \cap \mathrm{dom}(f)\right)$.

Theorem 46 $\mathbb{F}_\kappa$ *preserves all cardinals and cofinalities and does not affect on the continuum function. In particular* $\mathbb{F}_\kappa$ *preserves all inaccessibles.*

Proof Let f be a fast function over V for $\mathbb{P}_\kappa$ and suppose that γ is regular in V but has cofinality $\delta < \gamma$ in $V[f]$. Since $|\,\mathbb{F}_\kappa\,| = \kappa$, we may assume that $\gamma \leq \kappa$.

Case I: $f''\delta \subseteq \delta$. Then $f = f\restriction \delta \,\dot\cup\, f\restriction [\delta, \kappa)$ is $\mathbb{F}_\delta \times \mathbb{F}_{\delta,\kappa}$-generic. Hence $V[f]$ can be seen as $V[f''][f']$ where f'' is V-generic over $\mathbb{F}_{\delta,\kappa}$ and f' is $V[f'']$-generic over $\mathbb{F}_\delta$. Now, since $\mathbb{F}_{\delta,\kappa}$ is $\leq \delta$-directedly closed, it cannot add a cofinal mapping from δ to γ. In $V[f'']$, $|\,\mathbb{F}_\delta\,| = \delta$ and so $\mathbb{F}_\delta$ satisfies δ^+-c.c. Hence it cannot add a cofinal mapping from δ to γ.

Case II: $f(\beta) > \delta$ for some $\beta < \delta$. Then $f = f\restriction \beta \,\dot\cup\, f\restriction [f(\beta), \kappa)$ is $\mathbb{F}_\beta \times \mathbb{F}_{f(\beta),\kappa}$-generic. Hence we can argue similarly to Case I to drive a contradiction.

Similar argument also shows that $(|\,2^\delta\,|)^V = |\,2^\delta\,|^{V[f]}$. □ (Theorem 46)

Theorem 47 $\mathbb{F}_\kappa$ *preserves every weakly compact cardinal. Indeed if* κ *is weakly compact cardinal in* V *then for any transitive* N *with sufficiently strong absoluteness such that* $\mathbb{F}_\kappa \in N$, ${}^{\kappa>}N \subseteq N$ *and* $|\,N\,| = \kappa$, *and for any* $\alpha < \kappa^+$, *there is a weakly compact embedding* $\tilde{j} : N[f] \to M[\tilde{j}(f)]$ *such that* $j \subseteq \tilde{j}$ *and* $\tilde{j}(f)(\kappa) = \alpha$ *for some weakly compact embedding* $j : N \to M$ *in* V.

Proof Suppose that $f : \kappa \supseteq\!\!\rightarrow \kappa$ is a fast function for $\mathbb{F}_\kappa$ over V. Since κ remains inaccessible in $V[f]$ by Theorem 46, it is enough to show that κ has the tree property.

Suppose that T is a κ-tree in $V[f]$ and let $\alpha < \kappa^+$ be arbitrary. Let $\dot{T}$ be a $\mathbb{F}_\kappa$-name of T. In V, let $N \prec \mathcal{H}(\kappa^+)$ be transitive such that $\mathbb{F}_\kappa$, $\dot{T} \in N$, ${}^{\kappa>}N \subseteq N$ and $|\,N\,| = \kappa$. Since κ is weakly compact, there is a weakly compact embedding $j : N \to M$ for a transitive M with $\mathit{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. Let

$$M' = \{j(h)(\kappa) \;:\; h \in N\}.$$

Then

Claim 47.1 (1) $j''N \subseteq M' \subseteq M$.

(2) $M' \prec M$.

$\vdash$ (1): For $x \in N$, let $i_x : \kappa \to \{x\}$. By elementarity $M \models (\forall\alpha < j(\kappa))\,[\,j(i_x)(\alpha) = j(x)\,]$. Hence we have $M \models j(i_x)(\kappa) = j(x)$. It follows that $j(x) \in M'$.

(2): Let $h_1,\ldots, h_n \in N$ and suppose that

$$M \models \exists x\varphi[j(h_1)(\kappa),\ldots, j(h_n)(\kappa), x].$$

Let $C = \{\alpha < \kappa : \exists x\varphi[j(h_1)(\alpha),\ldots, j(h_n)(\alpha), x]\}$. Then by elementarity $N \models$"C is unbounded below κ". Let $g \in N$ be such that

$$N \models \mathrm{dom}(g) = C \ \wedge\ \forall\alpha \in C\,[\,\varphi(h_1(\alpha),\ldots, h_n(\alpha), g(\alpha))\,].$$

Since $M \models \kappa \in j(C)$, it follows by elementarity that

$$M \models \varphi[j(h_1)(\kappa),\ldots, j(h_n)(\kappa), j(g)(\kappa)].$$

$\dashv$ (Claim 47.1)

By replacing M with the Mostowski collapse of M' in Claim 47.1, we may assume that

$$M = \{j(h)(\kappa) : h \in N\}.$$

Claim 47.2 $V \models {}^{\kappa>}M \subseteq M$.

$\vdash$ Suppose that $\langle a_\alpha : \alpha < \beta\rangle \in {}^{\kappa>}M$ for some $\beta < \kappa$. For each $\alpha < \beta$, let $h_\alpha \in N$ be such that $j(h_\alpha)(\kappa) = a_\alpha$. Since ${}^{\kappa>}N \subseteq N$, $\langle h_\alpha : \alpha < \beta\rangle \in N$. We may assume that $\mathrm{dom}(h_\alpha) = \kappa$ for all $\alpha < \beta$. Let

$$h = \{\langle\delta, \langle h_\alpha(\delta) : \alpha < \beta\rangle\rangle : \delta < \kappa\}.$$

Then $h \in N$ and $M \models j(h)(\kappa) = \langle a_\alpha : \alpha < \beta\rangle$ by elementarity. $\dashv$ (Claim 47.2)

Let $p = \{\langle\kappa, \alpha\rangle\}$. Then $p \in j(\mathbb{F}_\kappa)$. Let $\lambda \in M$ be such that

$$M \models \lambda \text{ is the least inaccessible } > \max\{\alpha, \kappa\}.$$

Note that by $\max\{\alpha, \kappa\} < j(\kappa)$ and by elementarity, there *is* such a λ. Since $M \models$"$(\mathbb{F}_{\lambda,j(\kappa)})^M$ is $\leq\kappa$-directedly closed", it is $<\kappa$-directedly closed by Claim 47.2. Since there are only κ many dense subsets of $(\mathbb{F}_{\lambda,j(\kappa)})^M$ in M, there is a M-generic $f_{tail} \in V$ over $(\mathbb{F}_{\lambda,j(\kappa)})^M$. Let

$$\tilde{f} = f \mathbin{\dot\cup} p \mathbin{\dot\cup} f_{tail}.$$

Now, f is really generic. Hence f is $M[f_{tail}]$-generic (note that $M[f_{tail}] \in V$). It follows that $\tilde{f}$ is M-generic over $(\mathbb{F}_{j(\kappa)})^M$. So we can lift j to $\tilde{j} : N[f] \to M[\tilde{f}]$ by letting $j(\dot{a}^{N[f]}) = j(\dot{a})^{M[\tilde{f}]}$ for each $\mathbb{F}_\kappa$-name $\dot{a} \in N$. Then $\tilde{j}(f) = \tilde{f}$ and we have $\tilde{j}(f)(\kappa) = \tilde{f}(\kappa) = \alpha$. Note that $\tilde{j} \in V[f]$. Since T is a κ-tree in N it follows by elementarity that $\tilde{j}(T)$

is a $j(\kappa)$-tree in M. Let t^* be an element of κth level of $\tilde{j}(T)$ then $b = \{t \in T : j(t) < t^*\}$ is a κ-branch in T.

For arbitrary weakly compact $\lambda \neq \kappa$, we can show easily by factor argument and by what we proved above, that λ remains weakly compact in the extension by $\mathbb{F}_\kappa$.

□ (Theorem 47)

Theorem 48 (Theorem 2.4 in [7]) *Suppose that κ is weakly compact in V and f a fast function over V for $\mathbb{F}_\kappa$. In V, enumerate V_κ as $\vec{a} = \langle a_\alpha : \alpha < \kappa \rangle$ such that for any $a \in V_\kappa$, $\{\alpha < \kappa : a_\alpha = a\}$ is cofinal in κ. Let $\ell : \kappa \to V_\kappa$ be defined by:*

$$\ell(\gamma) = \begin{cases} (a_{f(\gamma)})^{V[f \restriction \gamma]}, & \textit{if } \gamma \in \mathrm{dom}(f) \textit{ and } a_{f(\gamma)} \textit{ is an } \mathbb{F}_\gamma\textit{-name}, \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Then for any $z \in (\mathcal{H}(\kappa^+))^{V[f]}$ there is a weakly compact embedding $j^* : N[f] \to M[j^*(f)]$ such that $j^*(\ell)(\kappa) = z$.

Proof Suppose that $z \in (\mathcal{H}(\kappa^+))^{V[f]}$. Since $|\,\mathbb{F}_\kappa\,| = \kappa$ and $\mathbb{F}_\kappa \in \mathcal{H}(\kappa^+)$, we can find a $\mathbb{F}_\kappa$-name $\dot{z}$ of z such that $\dot{z}$ is hereditary of size $\leq \kappa$. Since $\mathcal{H}(\kappa^+) \subseteq V_{\kappa^+}$, there is $\xi < \kappa^+$ such that $\dot{z} \in V_\xi$. Let N be transitive such that N is sufficiently absolute, $\dot{z}$, $\mathbb{F}_\kappa$, $\vec{a} \in N$, $|\,N\,| = \kappa$ and ${}^{\kappa>}N \subseteq N$. Let $j : N \to M$ be a weakly compact embedding for some transitive M with $crit(j) = \kappa$ and $j(\kappa) > \xi$. As in the proof of Theorem 47, we may assume that $M = \{j(h)(\kappa) : h \in N\}$ and hence ${}^{\kappa>}M \subseteq M$.

Claim 48.1 *$\dot{z} \in M$.*

$\vdash$ It is enough to show that $trcl(\{\dot{z}\}) \in M$. Noting that $|\,trcl(\{\dot{z}\})\,| \leq \kappa$, in N, let $k : \kappa' \to trcl(\{\dot{z}\})$ be a bijection for some $\kappa' \leq \kappa$. Let

$$E = \{\langle \alpha, \beta \rangle \in (\kappa')^2 : k(\alpha) \in k(\beta)\}.$$

Then $\langle \kappa', E \rangle \in M$ since $\langle \kappa', E \rangle = \langle \kappa', j(E) \cap \kappa^2 \rangle$.

Now in M, let x be the transitive set such that $\langle \kappa', E \rangle \cong \langle x, \in \rangle$. Then $x = trcl(\{\dot{z}\})$.

$\dashv$ (Claim 48.1)

By $\xi < j(\kappa)$ we have $M \models$ "$\dot{z} \in V_{j(\kappa)}$". Hence by elementarity there is $\gamma^* < j(\kappa)$ such that $j(\vec{a})(\gamma^*) = \dot{z}$. Now as in the proof of Theorem 47, we can find a lifting

$$j^* : N[f] \to M[j^*(f)]$$

of j such that $j^*(f)(\kappa) = \gamma^*$. By definition of ℓ and by elementarity, we have

$$j^*(\ell)(\kappa) = (\dot{z})^{M[j^*(f) \restriction \kappa]} = (\dot{z})^{M[f]} = z.$$

□ (Theorem 48)

10.2 Proof of Theorem 44

Remembering the definition of Axiom A, it is easy to see the following:

Lemma 49 *Suppose that M is transitive and sufficiently absolute and ${}^{\omega}M \subseteq M$. If $P \in M$ is an Axiom A p.o. and $\langle \leq_n : n \in \omega \rangle \in M$ witnesses this, then it follows that* $M \models$"P is an Axiom A p.o.". ❑

Note that the corresponding lemma for proper p.o.s is not true in general.

Let κ be weakly compact and $\ell : \kappa \to V_\kappa$ be such that

(†) For any $x, y \in \mathcal{H}(\kappa^+)$ for many transitive and sufficiently absolute N with ℓ, κ, $x \in N$ and $|N| = \kappa$, there is a weakly compact embedding $j : N \to M$ with $crit(j) = \kappa$ for some transitive M such that $x, y \in M$ and $j(\ell)(\kappa) = x$.

We may assume that such ℓ exists by Theorem 48.

Let $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a CS-iteration defined inductively as follows: for αth step, if $\ell(\alpha) = \langle \dot{P}, \dot{\mathcal{D}} \rangle$ such that $\dot{P}$, $\dot{\mathcal{D}}$ are P_α-names and

$$\Vdash_{P_\alpha} \text{“}\dot{P} \text{ is an Axiom A forcing and } \dot{\mathcal{D}} \text{ is a } \gamma\text{-sequence of dense subsets of } \dot{P} \text{ for some } \gamma < \kappa\text{”},$$

then let $\dot{Q}_\alpha = \dot{P}$; otherwise, let $\dot{Q}_\gamma$ be (a P_α-name of) the trivial p.o.

We show that $\Vdash_{P_\kappa}$ "$\kappa = \aleph_2 \ \wedge \ 2^{\aleph_0} = \kappa \ \wedge \ \mathrm{FA}(\text{Axiom A}, \leq 2^{\aleph_0})$". Let G be a P_κ-generic set over V. Since P_κ is proper, $\aleph_1$ is preserved. Since $\ell(\alpha) \in V_\kappa$ for all $\alpha < \kappa$, we have $|P_\alpha| < \kappa$ for all $\alpha < \kappa$. Hence P_κ satisfies the κ-c.c. In particular, P_κ preserves all cardinals $\geq \kappa$. Since $|P_\kappa| = \kappa$, we have $2^{\aleph_0} \leq \kappa$. In the rest of the proof we show that $V[G]$ models:

(∗) If P is an Axiom A p.o. of cardinality $\leq \kappa$ and $\mathcal{D} = \{D_\alpha : \alpha < \gamma\}$ is a family of dense subsets of P for some $\gamma < \kappa$, then there is a $\mathcal{D}$-generic set G over P.

To see this is enough to show, consider a σ-closed p.o. P collapsing $\gamma < \kappa$ to cardinality $\aleph_1$. Then P is axiom A and of cardinality κ. Since the genericity of a filter on P asserted in (∗) is enough to find a mapping which actually collapse γ, it follows that $M[G] \models \kappa \leq \aleph_2$. Also it follows from (∗), that $M[G] \models 2^{\aleph_0} \geq \kappa$. Hence $M[G] \models 2^{\aleph_0} = \kappa = \aleph_2$, and $M[G] \models \mathrm{FA}(\text{Axiom A}, \leq 2^{\aleph_0})$.

To prove $M[G] \models (*)$, let $\langle P, \leq_P \rangle$ be an Axiom A p.o. in $V[G]$ of cardinality $\leq \kappa$ with the sequence $\langle \leq_{P,n} : n \in \omega \rangle$ of partial orderings on P witnessing that P satisfies Axiom A, and $\mathcal{D} = \{D_\alpha : \alpha < \gamma\}$ a family of dense subsets of P for some $\gamma < \kappa$. Without loss of generality we may assume that $P \subseteq \kappa$ and $P_\kappa \subseteq \kappa$. Let $\dot{P}$, $\dot{\leq}_P$, $\dot{\leq}_{P,n}$, $n < \omega$, $\dot{\mathcal{D}}$, $\dot{D}_\alpha$, $\alpha < \gamma$ be corresponding nice[8] P_κ-names. Then $\langle \dot{P}, \dot{\mathcal{D}} \rangle \in \mathcal{H}(\kappa^+)$ and $\dot{\leq}_{P,n} \in \mathcal{H}(\kappa^+)$ for all $n < \omega$.

By (†) we can find:

[8] In the sense of [9].

(1) a sufficiently absolute transitive N such that $\langle \dot{P}, \dot{\mathcal{D}}\rangle$, κ ℓ, $P_\kappa \in N$, $[\kappa]^{<\kappa} \in N$ and $|\,N\,| = \kappa$;

(2) a transitive M with $\langle \dot{P}, \dot{\mathcal{D}}\rangle \in M$ and $\dot{\leq}_{P,n} \in M$ for all $n \in \omega$, and an elementary embedding $j : N \rightarrow M$ such that $crit(j) = \kappa$ and $j(\ell)(\kappa) = \langle \dot{P}, \dot{D}\rangle$.

Claim 44.1 $\left({}^{\aleph_0}(\kappa^n)\right)^{V[G]} \subseteq M[G]$ *for all* $n \in \omega$.

$\vdash$ Suppose that $V[G] \models a \in {}^{\aleph_0}(\kappa^n)$ and let $\dot{a}$ be a nice P-name of a. Then for each $\langle n, \overline{\alpha}\rangle \in \aleph_0 \times \kappa^n$, there is a pairwise incompatible $B_{\langle n,\overline{\alpha}\rangle}$ such that

$$\dot{a} = \bigcup\{B_{\langle n,\overline{\alpha}\rangle} \times \{\langle n, \overline{\alpha}\rangle\} \,:\, \langle n, \overline{\alpha}\rangle \in \aleph_0 \times \kappa^n\}.$$

By $[\kappa]^{<\kappa} \subseteq N$ (and since N is a model of sufficiently large fragment of ZFC by (1)) we have $\dot{a} \in N$. Then $\dot{a} = j(\dot{a}) \cap (\kappa \times \aleph_0 \times \kappa^n)$ and hence $\dot{a} \in M$. It follows that $a = \dot{a}^{V[G]} \in M[G]$. $\dashv$ (Claim 44.1)

By Lemma 49 it follows from Claim 44.1 that

Claim 44.2 $M[G] \models\, \Vdash_{P_\kappa}$ "*$\dot{P}$ is an Axiom A p.o.*". □

By elementarity we have:

$$M \models j(P_\kappa) \text{ is the direct limit of a CS-iteration of length } j(\kappa).$$

Since $j \restriction \kappa$ is the identity, we have[9] $P_\kappa = j(P_\kappa)\restriction \kappa$. By Claim 44.2, $j(\ell)(\kappa) = \langle \dot{P}, \dot{\mathcal{D}}\rangle$ by assumption and

$$M[G] \models\, \Vdash_{j(P_\kappa)\restriction\,\kappa} \text{“}\,\dot{P} \text{ is an Axiom A p.o.''},$$

we have $j(P_\kappa) = P_\kappa * \dot{P} * \dot{R}$ for some $(P_\kappa * \dot{P})$-name $\dot{R}$ of a p.o.

Let H be $\dot{P}^{V[G]}$-generic set over $V[G]$ and K be $\dot{R}^{V[G][H]}$-generic set over $V[G][H]$. Let $j^* : N[G] \rightarrow M[G][H][K]$ be defined by

$$j^*(\dot{x}^{V[G]}) = j(\dot{x})^{V[G][H][K]}$$

for $\dot{x} \in V^{P_\kappa} \cap N$. To see that j^* is well-defined, suppose that $\dot{x}$, $\dot{y} \in V^{P_\kappa} \cap N$ are such that $p \Vdash_{P_\kappa}$ "$\dot{x} = \dot{y}$" for some $p \in G$. Then by absoluteness we have $N \models p \Vdash_{P_\kappa}$ "$\dot{x} = \dot{y}$". Hence by elementarity of j it follows that

$$M \models j(p) \Vdash_{j(P_\kappa)} \text{“}\,j(\dot{x}) = j(\dot{y})\,\text{''}.$$

Since $j(p) = p = p * 1_{\dot{P}} * 1_{\dot{R}}$, $j(p) \in G * H * K$. Hence $j(\dot{x})^{V[G][H][K]} = j(\dot{y})^{V[G][H][K]}$.

Similarly we can show the elementarity of j^* and $j \subseteq j^*$.

Now, since H is $V[G]$-generic over P $(= \dot{P}^{V[G]})$, $H \cap D_\alpha \neq \emptyset$ for all $\alpha < \gamma$. Let $\tilde{H}$ be a filter on $j^*(P)$ generated by $H = \{j(\alpha) \,:\, \alpha \in H\}$. Then $\tilde{H} \in M[G][H] \subseteq M[G][H][K]$

[9] Here we regard P_κ not as a subset of κ but as a CS-iterate in the usual sense.

and $\tilde{H} \cap j^*(D_\alpha) \supseteq H \cap D_\alpha \neq \emptyset$ for all $\alpha < \gamma$. Since $j^*(\mathcal{D}) = \{j^*(D_\alpha) : \alpha < \gamma\}$ by $\gamma < \kappa$, it follows that

$$M[G][H][K] \models \text{There exists a } j^*(\mathcal{D})\text{-generic set over } j^*(P).$$

Hence, by elementarity of j^*

$$N[G] \models \text{There exists a } \mathcal{D}\text{-generic set over } P.$$

So there do exist a $\mathcal{D}$-generic set over P in $V[G]$. ❑ (Theorem 44)

10.3 Equiconsistency

In this section we shall prove that the converse of Theorem 44 also holds:

Theorem 50 (Baumgartner) ZFC + FA(Axiom A, $\leq 2^{\aleph_0}$) *is equiconsistent with* ZFC + *"there is a weakly compact cardinal".*

Since one of the implications has been shown in Theorem 44, it is enough to prove the following:

Theorem 51 *Suppose that* FA(Axiom A, $\leq 2^{\aleph_0}$) *holds. Then* $L \models$"$(\aleph_2)^V$ is weakly compact"*.*

Theorem 51 follows immediately from Theorem 53 and Theorem 56 below. A tree T is said to be κ-special if T is the union of $\leq \kappa$ pairwise incomparable sets (i.e. anticains).

Lemma 52 *If A tree T is κ-special, then T has no branch of length $\geq \kappa^+$.*

Proof Let $T = \bigcup_{n\in\omega} T_n$ where elements of each T_n are pairwise incomparable. Let $g : T \to \omega$ be defined by $g(t) = \min\{n \in \omega : t \in T_n\}$ for each $t \in T$. For a branch b in T, the mapping $g\restriction b$ must be 1-1. Hence $|\, b \,| \leq \aleph_0$. ❑ (Lemma 52)

We say g as in the proof of Lemma 52 *specializes* T.

Theorem 53 (Jensen) *Assume that $\square_\kappa$ holds. Then there is a κ-special κ^+-Aronszajn tree.* ❑

For proof of Theorem 53, see e.g. [4]. Note that under GCH, this is already true for successors of regualr cardinals:

Theorem 54 (Specker) *Assume $\kappa^{<\kappa} = \kappa \geq \aleph_0$. Then there is a κ-special κ^+-Aronszajn tree.* ❑

For proof, see e.g. [11].

For a tree T, let $P(T)$ be the p.o.

$$P(T) = \{p \in [T]^{<\aleph_0} : \text{ elements of } p \text{ are pairwise incomparable}\}$$

with the reverse inclusion. $P^\omega(T)$ is the p.o.

$$P^\omega(T) = \{f \in {}^\omega P(T) \,:\, \{n \in \omega \,:\, f(n) \neq \emptyset\} \text{ is finite}\}$$

with the ordering:

$$f \leq g \quad \Leftrightarrow \quad f(n) \supseteq g(n) \text{ for all } n \in \omega$$

for $f, g \in P^\omega(T)$.

Proposition 55 *For a tree T of height ω_1, the following are equivalent:*
(i) *T has no uncountable branch;*
(ii) *$P(T)$ has the c.c.c.;*
(iii) *$P^\omega(T)$ has the c.c.c.*

Proof (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i): Suppose that T has an uncountable branch b. Then $\{\{t\} \,:\, t \in b\}$ is an uncountable pairwise incompatible subset of $P(T)$ and

$$\{\{\langle 0, \{t\}\rangle\} \cup \Big((\omega \setminus \{0\}) \times \{\emptyset\}\Big) \,:\, t \in b\}$$

is an uncountable pairwise incompatible subset of $P^\omega(T)$.

(i) $\Rightarrow$ (ii): Suppose that $\{p_\alpha \,:\, \alpha < \omega_1\} \subseteq P(T)$ is pairwise incompatible. Without loss of generality we may assume that $|\, p_\alpha \,| = n$ for all $\alpha < \omega_1$. Let $p_\alpha = \{t^0_\alpha, \ldots, t^{n-1}_\alpha\}$ for $\alpha < \omega_1$. By assumption, for any $\alpha < \beta < \omega_1$, there are $i, j < n$ such that t^i_α and t^j_β are comparable. By Δ-system lemma (and by discarding the root of a Δ-system $\subseteq \{p_\alpha \,:\, \alpha < \omega_1\}$) we may assume that p_α, $\alpha < \omega$ are pairwise disjoint.

Let U be a uniform ultrafilter over ω_1. For each $\alpha < \omega_1$, let $i_\alpha, j_\alpha < n$ be such that

$$A_\alpha = \{\beta < \omega_1 \,:\, t^{i_\alpha}_\alpha \text{ and } t^{j_\alpha}_\beta \text{ are comparable}\}$$

is an element of U. Since $t^{i_\alpha}_\alpha$ has only countably many predecessors in T and U is uniform, it follows that

$$A'_\alpha = \{\beta < \omega_1 \,:\, t^{i_\alpha}_\alpha <_T t^{j_\alpha}_\beta\}$$

is also an element of U.

Now, let $j^* < n$ and $B \subseteq \omega_1$ be such that B is uncountable and $j_\alpha = j^*$ for all $\alpha \in B$. The following claim implies that there is an uncountable branch in T:

Claim 55.1 *$t^{i_\alpha}_\alpha$, $\alpha \in B$ are pairwise comparable.*

$\vdash$ Suppose that $\alpha, \alpha' \in B$. $A'_\alpha \cap A'_{\alpha'} \neq \emptyset$ since $\beta \in A'_\alpha, \cap A'_{\alpha'} \in U$. But for $\beta \in A'_\alpha \cap A'_{\alpha'}$, we have $t^{i_\alpha}_\alpha$, $t^{i_{\alpha'}}_{\alpha'} \leq_T t^{j^*}_\beta$. $\dashv$ (Claim 55.1)

(iii) $\Rightarrow$ (i): Suppose that $\{f_\alpha \,:\, \alpha < \omega_1\} \subseteq P^\omega(T)$ is pairwise incompatible. By Δ-system lemma, we may assume that $\mathrm{supp}(f_\alpha) = \{n \in \omega \,:\, f_\alpha(n) \neq \emptyset\}$, $\alpha < \omega_1$ form a Δ-system with the root r. For $\alpha < \omega_1$, let f'_α be defined by

$$f'_\alpha(n) = \begin{cases} f_\alpha(n), & \text{if } n \in r \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $\mathrm{supp}(f'_\alpha) = r$ for all $\alpha < \omega_1$ and $\{f'_\alpha : \alpha < \omega_1\}$ is still pairwise incompatible. Hence by replacing f_α's with f'_α's, we may assume that $\mathrm{supp}(f_\alpha)$, $\alpha < \omega_1$ are all the same. Now by similar argument as in the proof of (ii) $\Rightarrow$ (i), we can find an uncountable family of pairwise comparable elements of T. $\square$ (Proposition 55)

Theorem 56 (Baumgartner) *Assume that* FA(Axiom A, $\leq 2^{\aleph_0}$) *holds. Then there are no $\aleph_2$-Aronszajn trees.*

Proof Assume on the contrary that FA(Axiom A, $\leq 2^{\aleph_0}$) holds and yet there is an $\aleph_2$-Aronszajn tree T. Without loss of generality we may assume that the underlying set of T is ω_2 and $T_\alpha = \{\xi : \omega_1\alpha \leq \xi < \omega_1(\alpha+1)\}$ for all $\aleph_1 \leq \alpha < \omega_2$. Let

$$P = \{f : \omega_1 \supseteq\!\!\rightarrow \omega_2 \, : \, |\, f \,| \leq \aleph_0\}$$

be the p.o. endowed with reverse inclusion. Then P is σ-closed and collapses $|\,(\aleph_2)^V\,|$ to $\aleph_1$.

Claim 56.1 *P does not add any branch in T of uncountable cofinality.*

$\vdash$ Suppose otherwise. Then there is a P-name $\dot{b}$ and $p_0 \in P$ such that

$$p_0 \Vdash_P \text{“ } \dot{b} \text{ is a branch in } T \text{ of uncountable cofinality.”}.$$

Clearly $p_0 \Vdash_P$ “ $\dot{b} \notin V$ ” by assumption on T. Hence we can choose for each $s \in {}^{\omega>}2$ a condition $p_s \in P$ and $t_s \in T$ such that

(0) $p_\emptyset = p_0$, $t_\emptyset =$ the smallest element of T;

(i) $p_s \Vdash_P$ “ $t_s \in \dot{b}$ ”;

(ii) $p_{s^\frown 0}, p_{s^\frown 1} \leq_P p_s$;

(iii) $t_s \leq_T t_{s^\frown 0}, t_{s^\frown 1}$; $t_{s^\frown 0}$ and $t_{s^\frown 1}$ are incomparable.

Having constructed such $p_s \in P$, $t_s \in T$ as above for all $s \in {}^{\omega>}2$, let $\alpha < \omega_2$ be such that $b_s \in T_{\leq\alpha}$ for each $s \in {}^{\omega>}2$. Now for each $f \in {}^{\omega}2$, let $p_f = \bigcup_{n\in\omega} p_{f\restriction n}$. Then $p_f \Vdash_P$ “ $\{t_{f\restriction n} : n \in \omega\} \subseteq \dot{b}$ ”. Let $q_f \leq_P p_f$ and $t_f \in T_\alpha$ be such that $q_f \Vdash_P$ “ $t_f \in \dot{b}$ ”. Then $t_{f\restriction n} \leq_T t_f$ for all $n \in \omega$. By (iii), t_f, $f \in {}^{\omega}2$ are pairwise incompatible. But this is a contradiction since $|\, T_\alpha \,| \leq \aleph_1 < 2^{\aleph_0}$. $\dashv$ (Claim 56.1)

Let G be V-generic over P and let c be a strictly increasing function from ω_1 to $(\omega_2)^V$ in $V[G]$ such that $c(0) = \omega_1$ and $c''\omega_1$ is cofinal in $(\omega_2)^V$.

In $V[G]$, let $T' = \bigcup_{\alpha\in\omega_1} T_{c(\alpha)}$. By Claim 56.1, T' is a tree of height ω_1 without cofinal branches. By Proposition 55, it follows that $P^\omega(T')$ is a c.c.c. p.o. Let H be a $V[G]$-generic set over $P^\omega(T')$. Then, in $V[G][H]$,

$$h^* = \{\langle n, \bigcup\{g(n) : g \in H\}\rangle : n \in \omega\}$$

gives a partition of T' into ω pairwise incomparable subsets. Let $g^* : T' \to \omega$ be defined by

$$g^*(t) = \min\{n \in \omega : t \in h^*(n)\}$$

for $t \in T$. Then g^* specializes T'.

Now, returning to V, let $\dot{Q}$ be a P-name for $\left(P^\omega(T')\right)^{V^P}$, $\dot{c}$ a $P * \dot{Q}$-name of c and $\dot{g}$ a $P * \dot{Q}$-name of g^*.

For α, $\xi < \omega_1$, let

$$D_{\alpha,\xi} = \{r \in P * \dot{Q} : r \Vdash_{P*\dot{Q}} \text{“}\dot{c}(\alpha) = \beta \ \wedge\ \dot{g}(\omega_1\beta + \xi) = n\text{”} \\ \text{for some } \omega_1 \leq \beta < \omega_2 \text{ and } n \in \omega\}.$$

Note that, by assumption on T, $\omega_1\beta + \xi$ is “the ξth element” of T_β. Clearly each $D_{\alpha,\xi}$ is dense in $P * \dot{Q}$. Let $\mathcal{D} = \{D_\alpha, \xi : \alpha, \xi < \omega_1\}$. Since $P * \dot{Q}$ is an Axiom A p.o. (as a two step iteration of a σ-closed p.o. and then a c.c.c. p.o), there is a $\mathcal{D}$-generic filter K over $P * \dot{Q}$. Let $d : \omega_1 \to \omega_2$ be defined by

$$d(\alpha) = \beta \quad \Leftrightarrow \quad p \Vdash_{P*\dot{Q}} \text{“}\dot{c}(\alpha) = \beta\text{”} \ \text{ for some } p \in K.$$

Then d is strictly increasing and so $T'' = \bigcup_{\alpha<\omega_1} T_{d(\alpha)}$ is a tree of height ω_1. Since $\sup\{d(\alpha) : \alpha \in \omega_1\} < \omega_2$, T'' has cofinal branches. But the function $g^{**} : T'' \to \omega$ defined by $g^{**}(t) = n$ if $p \Vdash_{P*\dot{Q}}$ “$\dot{g}(t) = n$” for some $p \in K$ specializes T''. By Lemma 52, this is a contradiction. □ (Theorem 56)

Now we are ready to prove Theorem 51. Suppose that FA(Axiom A, $\leq 2^{\aleph_0}$) holds. Then $\aleph_2$ has the tree property by Theorem 56. Hence $L \models$“$(\aleph_2)^V$ has the tree property”. Since $\aleph_2$ is regular, $L \models$“$(\aleph_2)^V$ is regular”. By Theorem 53 we also have: $L \models$“$(\aleph_2)^V$ is a limit”. Hence it follows that $L \models$“$(\aleph_2)^V$ is weakly compact” as desired. □ (Theorem 51)

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