Adding a “strongly” stationary subset of $\mathcal{P}_\kappa\lambda$ whose stationarity can be destroyed by a p.o. preserving cardinals*

Sakaé Fuchino (渕野 昌, 中部大学 (Chubu Univ.))

fuchino@isc.chubu.ac.jp

October 13, 2005 RIMS, Kyoto Japan

* The results are obtained in a joint work with **Greg Piper** (Kobe Univ.)

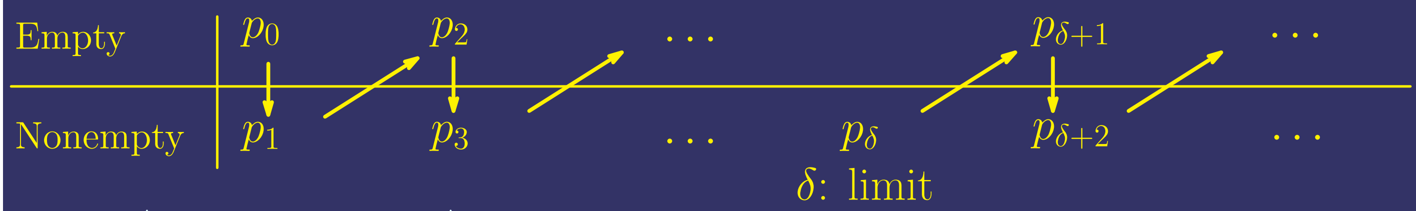
Reference

[1] S.F., Greg Piper, **Destructibility of stationary subsets of $\mathcal{P}_\kappa\lambda$**, to appear in Mathematical Logic Quarterly Vol.51 (6), (November 2005) 560-569.

[2] S.F., **A stronger version of stationarity preserved under $<\kappa$-strategically closed forcing**, 中部大学工学部紀要, to appear.

κ-strategically closed posets

A poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is said to be **κ-strategically closed** if Player Nonempty has a winning strategy in the following game $\mathcal{G}^{\mathrm{II}}_{\kappa}(\mathbb{P})$:



where $\langle p_\alpha : \alpha < \eta \leq \kappa \rangle$ should be a descending chain in $\mathbb{P}$ with respect to $\leq_{\mathbb{P}}$.

Nonempty wins $\Leftrightarrow$ for all $\alpha < \kappa$ p_α can be played.

$\mathbb{P}$ is **$<\kappa$-strategically closed** $\Leftrightarrow$ $\mathbb{P}$ is λ-strategically closed for all $\lambda < \kappa$.

$\mathbb{P}$ has a $<\kappa$-closed dense subset $\Rightarrow$ $\mathbb{P}$ is κ-strategically closed $\Rightarrow$ $\mathbb{P}$ is $<\kappa$-strategically closed.

Preservation of stationarity

定理 1 (Folklore?) (a) Suppose that κ is a regular cardinal. If $S \subseteq \kappa$ is stationary and $\mathbb{P}$ is κ-strategically closed *then* $\Vdash_{\mathbb{P}}$ "S is stationary".

(b) If $S \subseteq \mathcal{P}_{\omega_1}\lambda$ for a regular λ is stationary and $\mathbb{P}$ is ω_1-strategically closed, *then* $\Vdash_{\mathbb{P}}$ "S is stationary".

定理 2 (... and T. Usuba 200?) Suppse that $\kappa < \lambda$ are regular cardinals and κ is $\lambda^{<\kappa}$-supercompact. Then there are stationary $S \subseteq \mathcal{P}_\kappa\lambda$ and κ-strategically closed κ^+-c.c. $\mathbb{P}$ s.t. $\Vdash_{\mathbb{P}}$ "S is not stationary".

Internally approachable filter over $\mathcal{P}_\kappa\lambda$

$M \prec \mathcal{H}(\theta)$ is (strongly) internally approachable ((s.)i.a. for short) $\Leftrightarrow$ there is an increasing sequence $\langle M_\alpha : \alpha < \delta\rangle$ of elementary submodels of M s.t. (1) $M = \bigcup_{\alpha<\delta} M_\alpha$ and (2) $\forall\gamma < \delta\ \langle M_\alpha : \alpha < \gamma\rangle \in M_\gamma$;
$\Big($ (3) $\forall\gamma < \delta\ \mathcal{P}(\gamma) \in M_\gamma$ $\Big)$.

For $a \in \mathcal{H}(\theta)$, let

$$S^{\mathrm{IA},\theta,a}_\kappa\lambda = \{\lambda \cap M : M \prec \mathcal{H}(\theta),\ M \text{ is i.a.},\ \kappa,\lambda,a \in M,\ |M| < \kappa\}.$$

$\mathcal{F}^{\mathrm{IA}}_\kappa\lambda$ = the filter over $\mathcal{P}_\kappa\lambda$ generated by

$$\{S^{\mathrm{IA},\theta,a}_\kappa\lambda : \theta \text{ is sufficiently large}, a \in \mathcal{H}(\theta)\}$$

$S^{\mathrm{SIA},\theta,a}_\kappa\lambda$, $\mathcal{F}^{\mathrm{SIA}}_\kappa\lambda$: $S^{\mathrm{IA},\theta,a}_\kappa\lambda$, $\mathcal{F}^{\mathrm{IA}}_\kappa\lambda$ with "i.a." replaced by "s.i.a".

補題 3. $\mathcal{F}^{\mathrm{IA}}_\kappa\lambda$ ($\mathcal{F}^{\mathrm{SIA}}_\kappa\lambda$) is a normal filter. In particular, for $X \subseteq \mathcal{P}_\kappa\lambda$, X is $\mathcal{F}^{\mathrm{IA}}_\kappa\lambda$-stationary ($\Leftrightarrow$ $X \in \big((\mathcal{F}^{\mathrm{IA}}_\kappa\lambda)^*\big)^+$) $\Rightarrow$ X is stationary.

Stationary sets of $\mathcal{P}_\kappa\lambda$ whose stationarity is preserved in generic extensions by a strategically closed poset

定理 4. (S.F. and G.Piper [1]) Suppose that $S \subseteq \mathcal{P}_\kappa\lambda$ is $\mathcal{F}^{\mathrm{IA}}_\kappa\lambda$-stationary. Then, for any κ-strategically closed poset $\mathbb{P}$, $\Vdash_{\mathbb{P}}$ "S is $\mathcal{F}^{\mathrm{IA}}_\kappa\lambda$-stationary".

定理 5. (S.F. [2]) Suppose that κ is inaccessible and so that $S \subseteq \{x \in \mathcal{P}_\kappa\lambda : x \cap \kappa$ is not a regular cardinal$\}$ is $\mathcal{F}^{\mathrm{SIA}}_\kappa\lambda$-stationary. Then, for any $<\kappa$-strategically closed poset $\mathbb{P}$, we have $\Vdash_{\mathbb{P}}$ "S is $\mathcal{F}^{\mathrm{SIA}}_\kappa\lambda$-stationary"

定理 6. (S.F. and G.Piper [1]) "$(<)\kappa$-strategically closed" in 定理 5,6 cannot be replaced e.g. by "$\mathbb{P}$ does not add any new element of $\mathcal{P}_\kappa\lambda$ + κ^+-c.c."

定理 6. in detail

定理 6. (S.F. and G.Piper [1])
Suppose that κ is a regular cardinal with $\kappa^{<\kappa} = \kappa$ and $\kappa \leq \lambda$.
Then there are a $<\kappa$-closed κ^+-c.c. poset $\mathbb{P}_0$, a strongly κ-strategically closed κ^+-c.c. poset $\mathbb{Q}_0$ and a $\mathbb{P}_0$-name $\underset{\sim}{S}$ s.t. $\mathbb{P}_0 \lessdot \mathbb{Q}_0$ and

$$\Vdash_{\mathbb{P}_0} \text{“} \underset{\sim}{S} \text{ is a } \mathcal{F}^{IA}_{\kappa}\lambda\text{-stationary subset of } \mathcal{P}_{\kappa}\lambda \text{”}$$

but $\Vdash_{\mathbb{Q}_0}$ “ $\underset{\sim}{S}$ is not stationary”.
Further, if κ is inaccessible, we have

$$\Vdash_{\mathbb{P}_0} \text{“} \underset{\sim}{S} \cap \{x \in \mathcal{P}_{\kappa}\lambda : x \cap \kappa \text{ is a singular ordinal}\} \text{ is a } \mathcal{F}^{SIA}_{\kappa}\lambda\text{-stationary subset of } \mathcal{P}_{\kappa}\lambda \text{”}.$$

Construction of $\mathbb{P}_0$ and $\mathbb{Q}_0$

Let θ be sufficiently large.

$p \in \mathbb{P}_0 \Leftrightarrow p = \langle M^p, s^p \rangle$; $\kappa, \lambda \in M^p \prec \mathcal{H}(\theta)$ or $M^p = \emptyset$; $|M^p| < \kappa$;
$s^p \subseteq \mathcal{P}(\lambda \cap M^p) \cap M^p$.

For $p, p' \in \mathbb{P}_0$, $p' \leq_{\mathbb{P}_0} p \;\Leftrightarrow\; M^p \subseteq M^{p'} \;\wedge\; s^p \subseteq s^{p'} \;\wedge\; s^{p'} \cap \mathcal{P}(\lambda \cap M^p) = s^p$.

Let $S_G = \bigcup\{s^p \,:\, p \in G\}$ for a $(V, \mathbb{P}_0)$-generic filter G and $\underset{\sim}{S}$ a $\mathbb{P}_0$-name of S_G.

$\langle p, q \rangle \in \mathbb{Q}_0 \Leftrightarrow p \in \mathbb{P}_0$; $q = \langle M^q, s^q \rangle$; $\kappa, \lambda \in M^q \prec \mathcal{H}(\theta)$ or $M^q = \emptyset$; $|M^q| < \kappa$;
$s^q \subseteq \mathcal{P}(\lambda \cap M^q)$; $M^q \subseteq M^p$; $b^q \in s^q \cap M^q$ for $b^q = \bigcup s^q$;
$\overline{s^q \cap M^q} = s^q$ (in particular, $\overline{s^q} = s^q$); $s^p \cap s^q = \emptyset$, where,
for $s \subseteq \mathcal{P}(x)$, $\overline{s}$ = the closure of s w.r.t. union of increasing chain of length $< \kappa$.

For $\langle p, q \rangle, \langle p', q' \rangle \in \mathbb{Q}_0$, $\langle p', q' \rangle \leq_{\mathbb{Q}_0} \langle p, q \rangle \Leftrightarrow p' \leq_{\mathbb{P}_0} p$, $M^q \subseteq M^{q'}$, $s^q \subseteq s^{q'}$ and $s^{q'} \cap \mathcal{P}(\lambda \cap M^q) = s^q$.

$\mathbb{P}_0 \hookrightarrow \mathbb{Q}_0$; $p \mapsto \langle p, \langle \emptyset, \emptyset \rangle \rangle$