${}^{\kappa}\kappa$ in light of the Tukey ordering

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Abstract

For a regular κ, we show that, under $\kappa^{<\kappa} = \kappa$ or $\mathfrak{b}(\kappa) > \kappa^+$ as well as in any κ-c.c. extension of a model of $\kappa^{<\kappa} = \kappa$ or $\mathfrak{b}(\kappa) > \kappa^+$, there is no subset $\mathcal{F}$ of ${}^{\kappa}\kappa$ of cardinality κ^+ such that $\mathcal{F}$ has less than κ many elements below every $g \in {}^{\kappa}\kappa$ with respect to the partial ordering $\leq$ on ${}^{\kappa}\kappa$ by coordinatewise comparison. By Lemma 3.5 in Karato [7], the non-existence of such $\mathcal{F}$ is equivalent to $\langle \mathcal{P}_\kappa \kappa^+, \subseteq \rangle \not\leq \langle {}^{\kappa}\kappa, \leq \rangle$ in the Tukey ordering. Todorčević pointed out that this condition is actually equivalent to what is called Galvin's proposition for κ in Abraham and Shelah [1]. Thus our arguments provide an alternative proof of Galvin's proposition. By this equivalence, a result in Abraham and Shelah [1] reads e.g. that $\langle \mathcal{P}_{\omega_n} \omega_m, \subseteq \rangle \leq \langle {}^{\omega_n}\omega_n, \leq \rangle$ is consistent for any $1 \leq n < m < \omega$. We also show that $\langle \mathcal{P}_\kappa \lambda, \subseteq \rangle \not\leq \langle {}^{\kappa}\kappa, \leq \rangle$ holds if λ has a certain large cardinal property.

1 Introduction

The *Tukey ordering* $\leq$ on the class of (upward) directed sets is defined as follows ([9]):
For directed sets $D = \langle D, \leq_D \rangle$ and $E = \langle E, \leq_E \rangle$,

$$\text{(1.1)} \qquad D \leq E \ \Leftrightarrow\ \exists f : E \to D \ \forall d \in D \ \exists e \in E \ (f[E \uparrow e] \subseteq D \uparrow d).$$

Here, for a directed set $D = \langle D, \leq_D \rangle$ (or for a partial ordering more generally), $X \subseteq D$ and $d \in D$, we denote:

$$X \uparrow d = \{x \in X \,:\, d \leq_D x\} \quad \text{and} \quad X \downarrow d = \{x \in X \,:\, d \geq_D x\}.$$

f as above is called a *convergent function.* If D is downward complete (i.e. every subset of D has its infimum with respect to $\leq_D$) then the convergent function f in (1.1) may be taken to be order preserving as well: simply replace f by the mapping $e \mapsto \bigwedge\!\!\!\bigwedge\{f(e') \,:\, e' \in E \uparrow e\}$.

For cardinals κ, λ with $\kappa \leq \lambda$, let

$${}^{\kappa}\kappa = \{f \,:\, f : \kappa \to \kappa\} \quad \text{and} \quad \mathcal{P}_\kappa \lambda = \{x \in \mathcal{P}(\lambda) \,:\, |\,x\,| < \kappa\}.$$

For f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{rng}(f)$, $\mathrm{rng}(g) \subseteq \mathrm{On}$, let

(1.2) $\quad f \leq g \;\Leftrightarrow\; f(x) \leq g(x)$ for all $x \in \operatorname{dom}(f)$.

For a directed set $D = \langle D, \leq_D \rangle$, let

$$\operatorname{add}(D)\ (= \mathfrak{b}(D)) \ = \min\{|\,X\,| \,:\, X \subseteq D \text{ is unbounded in } D\} \text{ and}$$
$$\operatorname{cof}(D)\ (= \mathfrak{d}(D)) \ = \min\{|\,X\,| \,:\, X \subseteq D \text{ is cofinal in } D\}.$$

Note that

(1.3) $\quad$ if $D \leq E$ then we have $\operatorname{add}(D) \geq \operatorname{add}(E)$ and $\operatorname{cof}(D) \leq \operatorname{cof}(E)$.

Karato [7] observed the following:

Lemma 1.1 (Lemma 3.5 in [7]) *Suppose that κ is a regular cardinal and $\kappa \leq \lambda$. For a directed set $D = \langle D, \leq_D \rangle$ with $\operatorname{add}(D) \geq \kappa$ the following are equivalent:*

(a) *There is an $X \in [D]^\lambda$ such that $|\,X \downarrow d\,| < \kappa$ for all $d \in D$;*

(b) *$\langle \mathcal{P}_\kappa \lambda, \subseteq \rangle \leq D$;*

(c) *There is an order preserving function $f : D \to \mathcal{P}_\kappa \lambda$ such that $f[D]$ is cofinal in $\mathcal{P}_\kappa \lambda$ (with respect to $\subseteq$).*

Of course, each of (a) $\sim$ (c) implies that $\operatorname{add}(D) \leq \kappa$ so that the lemma above is only relevant for directed sets D with $\operatorname{add}(D) = \kappa$.

By Lemma 1.1,

(1.4) $\quad \langle \mathcal{P}_\kappa \lambda, \subseteq \rangle \leq \langle \mathcal{P}_\kappa \lambda', \subseteq \rangle$

for regular κ and λ, λ' with $\kappa \leq \lambda \leq \lambda'$.

Also, since $\operatorname{add}(\langle {}^\kappa\kappa, \leq \rangle) = \kappa$ for a regular κ[2], it is easy to see by Lemma 1.1 that

(1.5) $\quad \langle \mathcal{P}_\kappa \kappa, \subseteq \rangle \leq \langle {}^\kappa\kappa, \leq \rangle$.

With these facts in background, Karato, the second author, asked if

(1.6) $\quad \langle \mathcal{P}_\kappa \kappa^+, \subseteq \rangle \leq \langle {}^\kappa\kappa, \leq \rangle$.

By Lemma 1.1, (1.6) is equivalent to the question whether the following holds:

[2] Here, the partial ordering $\leq$ on ${}^\kappa\kappa$ is defined by (1.2).

$(\star\star)_\kappa$: *There is a family $\mathcal{F} \subseteq {}^\kappa\kappa$ such that $|\mathcal{F}| = \kappa^+$ and $|\mathcal{F} \downarrow g| < \kappa$ for all $g \in {}^\kappa\kappa$ with respect to the ordering $\leq$ on ${}^\kappa\kappa$.*

More generally for a regular cardinal κ and $\lambda \geq \kappa$

(1.7) $\langle \mathcal{P}_\kappa\lambda, \subseteq \rangle \leq \langle {}^\kappa\kappa, \leq \rangle$.

is equivalent to

$(\star\star)_{\kappa,\lambda}$: *There is a family $\mathcal{F} \subseteq {}^\kappa\kappa$ such that $|\mathcal{F}| = \lambda$ and $|\mathcal{F} \downarrow g| < \kappa$ for all $g \in {}^\kappa\kappa$ with respect to the ordering $\leq$ on ${}^\kappa\kappa$.*

In Sections 2, 3 we show that $\neg(\star\star)_\kappa$ holds under $\kappa^{<\kappa} = \kappa$ as well as in any c.c.c. extension of a model of $\neg(\star\star)_\kappa$.

In Section 5 we show that $\neg(\star\star)_\kappa$ is equivalent to what is called Galvin's proposition in Abraham and Shelah [1]. The models constructed in [1] provide a consistency proof of $(\star\star)_\kappa$ and more for many κ. So e.g. for any $1 \leq n < m < \omega$, we have the consistency of $\langle \mathcal{P}_{\omega_n}\omega_m, \subseteq \rangle \leq \langle {}^{\omega_n}\omega_n, \leq \rangle$.

In Section 6, we show that $\neg(\star\star)_{\kappa,\lambda}$ holds for λ with certain large cardinal properties.

The results in this note give an answer to the question the second author asked during the set-theory meeting held at RIMS Kyoto, on October 10$\sim$12, 2005.

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2 $\neg(\star\star)_\kappa$ under $\kappa^{<\kappa} = \kappa$

In this and following sections we always assume that κ is regular.

For $f, g \in {}^\kappa\kappa$, let

$$f \leq^* g \;\Leftrightarrow\; \text{there is some } \xi < \kappa \text{ such that } f(\alpha) \leq g(\alpha) \text{ for all } \alpha \in \kappa \setminus \xi.$$

In [2], $\mathrm{add}(\langle {}^{\kappa}\kappa, \leq^*\rangle)$ and $\mathrm{cof}(\langle {}^{\kappa}\kappa, \leq^*\rangle)$ are denoted by $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ respectively. In particular, $\mathfrak{b}(\omega)$ and $\mathfrak{d}(\omega)$ coincide with the usual bounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$.

Lemma 2.1 *Suppose that $\mathcal{F} \subseteq {}^{\kappa}\kappa$ is of cardinality $\geq \kappa^+$ and λ is a cardinal with $\mathrm{cof}(\lambda) > \kappa$ and $\lambda \leq |\mathcal{F}|$. If $\mathcal{F}$ is bounded with respect to $\leq^*$ then there is a $g \in {}^{\kappa}\kappa$ such that $\mathcal{F} \downarrow g = \{f \in \mathcal{F} : f \leq g\}$ has the cardinality $\geq \lambda$.*

Proof. Note that we have $\lambda \geq \kappa^+$. Let $g_0 \in {}^{\kappa}\kappa$ be such that $f \leq^* g_0$ for all $f \in \mathcal{F}$. For each $f \in \mathcal{F}$, let $\alpha_f < \kappa$ be such that

$$f(\alpha) \leq g_0(\alpha) \text{ for all } \alpha \in \kappa \setminus \alpha_f.$$

Since $|\mathcal{F}| \geq \lambda \geq \kappa^+$, there is an $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality $\geq \lambda$ and $\alpha^* < \kappa$ such that $\alpha_f = \alpha^*$ for all $f \in \mathcal{F}'$. Now for each $f \in \mathcal{F}'$ let

$$\beta_f = \sup\{f(\gamma) : \gamma < \alpha^*\}.$$

Since κ is regular, we have $\beta_f < \kappa$ for all $f \in \mathcal{F}'$. Hence there is $\mathcal{F}'' \subseteq \mathcal{F}'$ of cardinality $\geq \lambda$ and a $\beta^* < \kappa$ such that $\beta_f = \beta^*$ for all $f \in \mathcal{F}''$. Let

$$g = \{\langle \gamma, \beta^* \rangle : \gamma < \alpha^*\} \cup g_0 \restriction (\kappa \setminus \alpha^*).$$

Then $\mathcal{F} \downarrow g \supseteq \mathcal{F}'' \downarrow g = \mathcal{F}''$. It follows that

$$|\mathcal{F} \downarrow g| \geq |\mathcal{F}''| \geq \lambda. \qquad \square \text{ (Lemma 2.1)}$$

Theorem 2.2 $\neg(\star\star)_\omega$.

Proof. Suppose that $\mathcal{F} \subseteq {}^{\omega}\omega$ with $|\mathcal{F}| \geq \omega_1$. Let $\mathbb{P}$ be a c.c.c. poset forcing $\mathfrak{b} > \left(2^{\aleph_0}\right)^V$. Then

$$\Vdash_{\mathbb{P}} \text{“} \mathcal{F} \text{ is bounded with respect to } \leq^* \text{”}.$$

By Lemma 2.1, there is a $\mathbb{P}$-name $\underset{\sim}{g}$ of a function from ω to ω such that

$$\Vdash_{\mathbb{P}} \text{“} \mathcal{F} \downarrow \underset{\sim}{g} \text{ is uncountable”}.$$

Let $\langle p_i : i \in \omega\rangle$, $\langle f_i : i \in \omega\rangle$ and $\langle n_i : i \in \omega\rangle$ be such that

(2.1) $\quad \langle p_i : i \in \omega\rangle$ is a decreasing sequence in $\mathbb{P}$;

(2.2) $\quad \langle f_i : i \in \omega \rangle$ is a sequence of distinct elements of $\mathcal{F}$;

(2.3) $\quad p_i \Vdash_{\mathbb{P}}$ " $\underset{\sim}{g}(i) = n_i$ " and

(2.4) $\quad p_i \Vdash_{\mathbb{P}}$ " $f_i \in \mathcal{F} \downarrow \underset{\sim}{g}$ " for all $i \in \omega$.

Let $g : \omega \to \omega$ be defined by $g(i) = n_i$ for all $i \in \omega$. Then we have

$$\mathcal{F} \downarrow g \supseteq \{f_i : i \in \omega\}.$$

In particular, $|\mathcal{F} \downarrow g| \geq \aleph_0$. Since $\mathcal{F}$ was arbitrary it follows that $\neg(\star\star)_\omega$. ❏ (Theorem 2.2)

By Lemma 7 in [2], if $\kappa^{<\kappa} = \kappa$ then there is a κ-closed κ^+-c.c. poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ " $\mathfrak{b}(\kappa) > (2^\kappa)^V$ ". Using such a $\mathbb{P}$, we can argue similarly to the proof of Lemma 2.2 to show that $\neg(\star\star)_\kappa$ holds. This idea proves the following Theorem 2.3.

The second author obtained Lemma 2.1, and Theorem 2.2 with a slightly different proof. The first and third authors found then independently the above mentioned forcing proof of the following Theorem 2.3 which extends Theorem 2.2. Finally the third and fourth author found a proof of the theorem without the forcing which is presented below.

Theorem 2.3 *Suppose that $\mathcal{F} \subseteq {}^\kappa\kappa$ and $|\mathcal{F}| > \kappa^{<\kappa}$. Then there is a $g \in {}^\kappa\kappa$ such that $|\mathcal{F} \downarrow g| \geq \kappa$.*

Proof. For $\varphi \in {}^{\kappa>}\kappa$ let

$$\mathcal{F}_\varphi = \{f \in \mathcal{F} : f \restriction \operatorname{dom}\varphi \leq \varphi\}.$$

By the assumption on the cardinality of $\mathcal{F}$, $\mathcal{F} \setminus \bigcup\{\mathcal{F}_\varphi : \varphi \in {}^{\kappa>}\kappa, |\mathcal{F}_\varphi| < \kappa\}$ is nonempty. Fix a

$$g^* \in \mathcal{F} \setminus \bigcup\{\mathcal{F}_\varphi : \varphi \in {}^{\kappa>}\kappa, |\mathcal{F}_\varphi| < \kappa\}.$$

Note that

Claim 2.3.1 *For any $\varphi \in {}^{\kappa>}\kappa$, if $g^* \restriction \operatorname{dom}(\varphi) \leq \varphi$ then $|\mathcal{F}_\varphi| \geq \kappa$.*

For $\alpha < \kappa$ let $\varphi_\alpha \in {}^{\kappa>}\kappa$ and $g_\alpha \in \mathcal{F}$ be taken inductively such that

(2.5) $\quad g_0 = g^*$ and $\varphi_0 = \emptyset$;

(2.6) $\quad \varphi_{\alpha+1} = \varphi_\alpha \cup \{\langle \alpha, \sup\{g_\beta(\alpha) : \beta \leq \alpha\}\rangle\}$;

(2.7) $\quad g_\alpha \in \mathcal{F}_{\varphi_\alpha} \setminus \{g_\beta : \beta < \alpha\}$;

(2.8) $\quad \varphi_\gamma = \bigcup\{\varphi_\alpha : \alpha < \gamma\}$ if $\gamma < \kappa$ is a limit.

Note that (2.7) is always possible because of Claim 2.3.1.

Let $g = \bigcup\{\varphi_\alpha : \alpha < \kappa\}$. Then we have

$$\mathcal{F} \downarrow g \supseteq \{g_\alpha : \alpha < \kappa\}$$

by (2.6). By (2.7), it follows that $|\,\mathcal{F} \downarrow g\,| \geq |\,\{g_\alpha : \alpha < \kappa\}\,| = \kappa$. ❑ (Theorem 2.3)

Corollary 2.4 $(\kappa^{<\kappa} = \kappa)$ $\neg(\star\star)_\kappa$.

Proof. By Theorem 2.3. ❑ (Lemma 2.4)

The next lemma is a direct consequence of Lemma 2.1:

Lemma 2.5 *Suppose that $\mathcal{F} \subseteq {}^\kappa\kappa$ and $\lambda = |\,\mathcal{F}\,|$. If either* $\mathrm{cof}(\lambda) > \kappa$ *and* $\lambda < \mathfrak{b}(\kappa)$ *or* $\mathfrak{d}(\kappa) < \mathrm{cf}(\lambda)$ *then there is a $g \in {}^\kappa\kappa$ such that $|\,\mathcal{F} \downarrow g\,| = \lambda$.*

Proof. By Lemma 2.1, it is enough to show that there is an $\mathcal{F}' \in [\mathcal{F}]^\lambda$ which is bounded with respect to $\leq^*$.

If $\lambda < \mathfrak{b}(\kappa)$, then $\mathcal{F}$ is bounded with respect to $\leq^*$ by definition of $\mathfrak{b}(\kappa)$.

If $\mathfrak{d}(\kappa) < \mathrm{cf}(\lambda)$ then let X be cofinal in ${}^\kappa\kappa$ (with respect to $\leq^*$) with

(2.9) $\quad |\,X\,| < \mathrm{cf}(\lambda)$.

For each $f \in \mathcal{F}$ there is $h_f \in X$ such that $f \leq^* h_f$. By (2.9) there is $\mathcal{F}' \in [\mathcal{F}]^\lambda$ such that h_f, $f \in \mathcal{F}'$ is constant. ❑ (Lemma 2.5)

Corollary 2.6 (a) *If $(\star\star)_\kappa$ then $\mathfrak{b}(\kappa) = \kappa^+$.*

(b) *If $(\star\star)_{\kappa,\lambda}$ for some $\lambda \geq \kappa$, then $\lambda \leq \mathfrak{d}(\kappa)$.* ❑

3 Preservation of $\neg(\star\star)_\kappa$ in generic extensions

Proposition 3.1 *Suppose that $\neg(\star\star)_\kappa$ holds and $\mathbb{P}$ is a κ-c.c. poset. Then we have* $\Vdash_{\mathbb{P}}$ *"$\neg(\star\star)_\kappa$ holds".*

Proof. Suppose that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathcal{F}} \in [{}^{\kappa}\kappa]^{\kappa^+}$" for a $\mathbb{P}$-name $\underset{\sim}{\mathcal{F}}$.

Let $\langle \underset{\sim}{f}_{\xi} : \xi < \kappa^+\rangle$ be a sequence of $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \text{"}\langle \underset{\sim}{f}_{\xi} : \xi < \kappa^+\rangle \text{ is an injective sequence of elements of } \underset{\sim}{\mathcal{F}}\text{"}.$$

For each $\xi < \kappa^+$, let $f^*_{\xi} \in {}^{\kappa}\kappa$ be such that

$$\Vdash_{\mathbb{P}} \text{"}\underset{\sim}{f}_{\xi}(\alpha) \le f^*_{\xi}(\alpha) \text{ for all } \alpha < \kappa\text{"}.$$

This is possible by the κ-c.c. of $\mathbb{P}$.

If there is an $X \in [\kappa^+]^{\kappa^+}$ and $f \in {}^{\kappa}\kappa$ such that $f^*_{\xi} = f$ for all $\xi \in X$ then

$$\Vdash_{\mathbb{P}} \text{"}\forall \xi \in X\ (\underset{\sim}{f}_{\xi} \le f)\text{"}$$

and thus $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathcal{F}}$ is not a witness for $(\star\star)_{\kappa}$".

Otherwise there is a $Y \in [\kappa^+]^{\kappa^+}$ such that f^*_{ξ}, $\xi \in Y$ are pairwise distinct. By $\neg(\star\star)_{\kappa}$ (in the ground model) there is a $Z \in [Y]^{\kappa}$ and $g \in {}^{\kappa}\kappa$ such that $f^*_{\xi} \le g$ for all $\xi \in Z$. But then

$$\Vdash_{\mathbb{P}} \text{"}\forall \xi \in Z\ (\underset{\sim}{f}_{\xi} \le g)\text{"}.$$

Hence again we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathcal{F}}$ is not a witness for $(\star\star)_{\kappa}$".

Since the $\mathbb{P}$-name $\underset{\sim}{\mathcal{F}}$ for a subset of ${}^{\kappa}\kappa$ of cardinality of κ^+ was arbitrary, it follows that $\Vdash_{\mathbb{P}}$ "$\neg(\star\star)_{\kappa}$". ❑ (Proposition 3.1)

From Proposition 3.1 and Theorem 2.3, it follows that $\neg(\star\star)_{\omega_1}$ holds e.g. in a Cohen or random model (obtained by starting from a model M of CH and then by adding more than $\aleph_2$ Cohen or random reals to M). Similarly, if we start from a model of GCH with a measurable cardinal λ and force with the measure algebra of Maharam type λ, we obtain a model of real valued measurability in which $\neg(\star\star)_{\kappa}$ holds for every regular uncountable κ. The last property also holds in a standard model of Martin's axiom + $\neg$CH provided that we start from a model of GCH. Note that we have e.g. $\omega_1^{<\omega_1} > \omega_1$ in these models.

4 ${}^{\kappa}\kappa$ with almost dominance

We might also consider the following weakening of $(\star\star)_{\kappa}$:

$(\star\star\star)_\kappa$: *There is a family $\mathcal{F} \subseteq {}^\kappa\kappa$ such that $|\mathcal{F}| = \kappa^+$ and $|\mathcal{F} \downarrow g| \leq \kappa$ for all $g \in {}^\kappa\kappa$ with respect to the ordering $\leq$ on ${}^\kappa\kappa$.*

This assertion can be characterized as a condition on the size of the bounding number $\mathfrak{b}(\kappa)$ of $\langle {}^\kappa\kappa, \leq^* \rangle$:

Proposition 4.1 *The following are equivalent:*

(a) $(\star\star\star)_\kappa$ *;*

(b) $\langle \mathcal{P}_{\kappa^+}\kappa^+, \subseteq \rangle \leq \langle {}^\kappa\kappa, \leq^* \rangle$ *;*

(c) $\mathfrak{b}(\kappa) = \kappa^+$.

This follows from the next characterization of $\mathfrak{b}(\kappa)$:

Lemma 4.2 $\mathfrak{b}(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq {}^\kappa\kappa,\ |\mathcal{F}| > \kappa,\ \forall f \in {}^\kappa\kappa\ |\mathcal{F} \downarrow f| < |\mathcal{F}|\}$.

Proof. Let $\mathfrak{b}'$ be the right side of the equation. We show first the inequality $\mathfrak{b}(\kappa) \leq \mathfrak{b}'$. Suppose that $\kappa < \lambda < \mathfrak{b}(\kappa)$ and λ is regular. For any $\mathcal{F} \subseteq {}^\kappa\kappa$, if $|\mathcal{F}| = \lambda$ then $\mathcal{F}$ is bounded (with respect to $\leq^*$). By Lemma 2.1, there is an $f \in {}^\kappa\kappa$ such that $|\mathcal{F} \downarrow f| = \lambda = |\mathcal{F}|$. Hence $\lambda < \mathfrak{b}'$. This shows $\mathfrak{b}(\kappa) \leq \mathfrak{b}'$.

Next, we show $\mathfrak{b}(\kappa) \geq \mathfrak{b}'$. Let $\langle f_\alpha : \alpha < \mathfrak{b}(\kappa) \rangle$ be an increasing sequence (with respect to $\leq^*$) such that $\{f_\alpha : \alpha < \mathfrak{b}(\kappa)\}$ does not have any upper bound (with respect to $\leq^*$). Then $\{\alpha < \mathfrak{b}(\kappa) : f_\alpha \leq f\}$ is bounded in $\mathfrak{b}(\kappa)$ for any $f \in {}^\kappa\kappa$. This shows that $\mathfrak{b}' \leq \mathfrak{b}(\kappa)$. ❏ (Lemma 4.2)

Proof of Proposition 4.1: (a) $\Rightarrow$ (c): Suppose $\mathfrak{b}(\kappa) > \kappa^+$. For any $\mathcal{F} \subseteq {}^\kappa\kappa$ of cardinality $> \kappa$, let $\mathcal{F}' \subseteq \mathcal{F}$ be of cardinality κ^+. $\mathcal{F}'$ is bounded with respect to $\leq^*$. Hence by Lemma 2.1 there is an $f \in {}^\kappa\kappa$ such that $|\mathcal{F}' \downarrow f| = \kappa^+$. Thus $(\star\star\star)_\kappa$ does not hold.

(c) $\Rightarrow$ (b): Recall that $\mathfrak{b}(\kappa) = \mathrm{add}(\langle {}^\kappa\kappa, \leq^* \rangle)$. If $\mathfrak{b}(\kappa) = \kappa^+$ there is an increasing sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\langle {}^\kappa\kappa, \leq^* \rangle$ such that $X = \{f_\alpha : \alpha < \kappa^+\}$ is unbounded. Clearly X satisfies the condition of Lemma 1.1, (a) for κ there replaced with κ^+ and $\lambda = \kappa^+$. It follows that $\langle \mathcal{P}_{\kappa^+}\kappa^+, \subseteq \rangle \leq \langle {}^\kappa\kappa, \leq^* \rangle$.

(b) $\Rightarrow$ (c): Assume that $\langle \mathcal{P}_{\kappa^+}\kappa^+, \subseteq \rangle \leq \langle {}^\kappa\kappa, \leq^* \rangle$. By (1.3), it follows that $\mathfrak{b}(\kappa) = \mathrm{add}(\langle {}^\kappa\kappa, \leq^* \rangle) \leq \mathrm{add}(\langle \mathcal{P}_{\kappa^+}\kappa^+, \subseteq \rangle) = \kappa^+$.

(c) $\Rightarrow$ (a): Suppose $\mathfrak{b}(\kappa) = \kappa^+$. Then, by Lemma 4.2, there is $\mathcal{F} \subseteq {}^\kappa\kappa$ of cardinality κ^+ such that $|\mathcal{F} \downarrow f| \leq \kappa$. Thus $(\star\star\star)_\kappa$ holds. ❏ (Lemma 4.1)

5 Galvin's proposition

After we had written up the previous sections, David Asperó told us that Stevo Todorčević pointed out that the consistency of $(\star\star)_\kappa$ for may κ follows from the results in [1]. The main point of Todorčević's remark actually amounts to the equivalence of $\neg(\star\star)_\kappa$ with the following assertion cited in [1] as "Galvin's proposition":

$(G)_\kappa$: For any family $\mathcal{C}$ of club subsets of κ with $|\,\mathcal{C}\,| \geq \kappa^+$ there is an $\mathcal{C}' \in [\mathcal{C}]^\kappa$ such that $\bigcap \mathcal{C}'$ contains a club subset of κ.

For $\kappa < \lambda$ we can also consider the following generalization of $(G)_\kappa$:

$(G)_{\kappa,\lambda}$: For any family $\mathcal{C}$ of club subsets of κ with $|\,\mathcal{C}\,| \geq \lambda$ there is an $\mathcal{C}' \in [\mathcal{C}]^\kappa$ such that $\bigcap \mathcal{C}'$ contains a club subset of κ.

Thus $(G)_\kappa$ is just $(G)_{\kappa,\kappa^+}$.

Theorem 5.1 *For any regular uncountable κ and $\lambda > \kappa$, the following are equivalent:*

(a) $\langle \mathcal{P}_\kappa\lambda, \subseteq\rangle \leq \langle {}^\kappa\kappa, \leq\rangle$;

(b) $(\star\star)_{\kappa,\lambda}$;

(c) $\neg(G)_{\kappa,\lambda}$.

It follows that the proof of Theorem 2.3 (and Corollary 2.4) just provides an alternative proof of Galvin's proposition.

Theorem 5.1 follows from the next Lemma 5.2.

Let κ be an uncountable regular cardinal. Let $\mathcal{C}_\kappa = \{C\, :\, C \subseteq \kappa,\ C \text{ is a club}\}$. Let $\Phi_\kappa : \mathcal{C}_\kappa \to {}^\kappa\kappa$ and $\Psi_\kappa : {}^\kappa\kappa \to \mathcal{C}_\kappa$ be mappings defined by

(5.1) $\quad \Phi_\kappa(C)(\xi) = \min\left(C \setminus (\xi+1)\right)$ for $C \in \mathcal{C}_\kappa$ and $\xi < \kappa$, and

(5.2) $\quad \Psi_\kappa(f) = \{\xi < \kappa\, :\, \forall \eta < \xi\, (f(\eta) \leq \xi)\}$ for $f \in {}^\kappa\kappa$.

Lemma 5.2 (1) *Φ_κ is an order preserving map from $\langle \mathcal{C}_\kappa, \supseteq\rangle$ to $\langle {}^\kappa\kappa, \leq\rangle$; Ψ_κ is an order preserving map from $\langle {}^\kappa\kappa, \leq\rangle$ to $\langle \mathcal{C}_\kappa, \supseteq\rangle$.*

(2) *For any $C \in \mathcal{C}_\kappa$, $\Psi_\kappa(\Phi_\kappa(C)) = C$; For any $f \in {}^\kappa\kappa$, $\Phi_\kappa(\Psi_\kappa(f)) \geq f$.*

(3) $\Phi_\kappa[\mathcal{C}_\kappa]$ *is cofinal in* $\langle {}^\kappa\kappa, \leq\rangle$*;* $\Psi_\kappa\left[{}^\kappa\kappa\right]$ *is cofinal in* $\langle \mathcal{C}_\kappa, \supseteq\rangle$*. In particular,* Φ_κ *and* Ψ_κ *are convergent functions.*

(4) $\langle {}^\kappa\kappa, \leq\rangle$ *and* $\langle \mathcal{C}_\kappa, \supseteq\rangle$ *are cofinally similar (i.e. equivalent with respect to the Tukey ordering).*

Proof. (1) is clear. (3) follows from (2) and (4) from (3). So it is enough to show (2):

Suppose first that $C \in \mathcal{C}_\kappa$. Then we have

$$\begin{aligned} \xi \in \Psi_\kappa(\Phi_\kappa(C)) \;&\Leftrightarrow\; \forall \eta < \xi\ (\Phi_\kappa(C)(\eta) \leq \xi) \\ &\Leftrightarrow\; \forall \eta < \xi\ (\min(C \setminus (\eta+1)) \leq \xi) \\ &\Leftrightarrow\; \xi \in C. \end{aligned}$$

Thus, we have $C = \Psi_\kappa(\Phi_\kappa(C))$.

Suppose now $f \in {}^\kappa\kappa$. Then, for $\xi < \kappa$, we have

$$\begin{aligned} \Phi_\kappa(\Psi_\kappa(f))(\xi) &= \min(\Psi_\kappa(f) \setminus (\xi+1)) \\ &= \min\{\eta \in \kappa \setminus (\xi+1) \,:\, \forall \zeta < \eta\ f(\zeta) \leq \eta\} \\ &\geq f(\xi). \end{aligned}$$

❏ (Lemma 5.2)

Theorem 5.1 follows now from Lemma 5.2 and Lemma 1.1.

For many uncountable regular κ, Abraham and Shelah gave in [1] a model of set-theory in which the negation of Galvin's proposition for κ (i.e. $\neg(G)_\kappa$ in our notation) holds. More precisely:

Theorem 5.3 (Theorem 1.1 in Abraham and Shelah [1]) *Assume* GCH*. Then for a regular cardinal* κ *and* $\lambda > \kappa$ *with* $cf(\lambda) \geq \kappa^+$ *there is a p.o.-set* $\mathbb{P}$ *which does not add any* κ *sequences and preserves all cardinals such that*

$$\Vdash_{\mathbb{P}} \text{“}2^{\kappa^+} = \lambda \text{ and } \neg(G)_{\kappa^+,\lambda} \text{ holds”}. \qquad ❏$$

By Theorem 5.1, it follows e.g.:

Corollary 5.4 *For any natural numbers* $1 \leq n < m < \omega$*, the assertion* $(\star\star)_{\omega_n,\omega_m}$ *is consistent with* ZFC*.* ❏

Starting from a model of GCH, Abraham and Shelah also constructed in [1] a model of $\neg(G)_{\omega_1,\lambda}$ for $\lambda > \omega_1$ such that the failure of Galvin's proposition is absolute for further extensions:

Theorem 5.5 (Theorem 2.2 in Abraham and Shelah [1]) *Assume that $V \models$ ZFC+ GCH and λ is a cardinal $\geq \omega_2$ in V. Then there is a generic extension W of V preserving cardinals such that $W \models \neg(G)_{\omega_1,\lambda}$ (and more see [1]) and for any further generic extension W' of W we have $W' \models \neg(G)_{\omega_1,\lambda}$ provided that ω_1 and "$|\lambda| > \omega_1$" are preserved.*

In [1] the model was constructed starting from $V = L$ but by virtue of Shelah's Club Guessing Lemma now available, a model of GCH (actually less than that) is enough to start with.

This result combined with the usual construction of a model of MA implies the following:

Corollary 5.6 *For any $n \geq 1$ $(\star\star)_{\omega_1,\omega_n}$ is consistent with* $\neg$CH + MA. ❑

Corollary 5.6 together with the remark at the end of Section 3 prove the next corollary:

Corollary 5.7 $(\star\star)_{\omega_1}$ *is independent from* ZFC + $\neg$CH + MA. ❑

We can also start from a model of GCH with a measurable cardinal κ and make e.g. $\neg(G)_{\omega_1,\omega_2}$ indestructible first and then force with the measure algebra with Maharam type κ to obtain a model of real-valued measurability (and $\neg(G)_{\omega_1,\omega_2}$). This together with the remark at the end of Section 3 proves the following:

Corollary 5.8 $(\star\star)_{\omega_1}$ *is independent from* ZFC + *"$2^{\aleph_0}$ is real-valued measurable".* ❑

The following shows that $\omega_1^{<\omega_1} = \omega_1$ in the ground model for the construction of $\mathfrak{b}(\omega_1) > \omega_2$ in [2] is necessary.

Corollary 5.9 *Let W be a model as in* Theorem 5.5 *for a $\lambda \geq \omega_2$. Then we have $\mathfrak{b}(\omega_1) = \omega_2$ in any generic extension of W preserving ω_1 and "$\lambda > \omega_1$".*

Proof. By Corollary 2.6, (a). ❑ (Corollary 5.9)

The following questions seem to be still open:

Problem 5.10 *Is* MM *consistent with* $\neg(\star\star)_{\omega_1}$ *or does* MM *even imply* $\neg(\star\star)_{\omega_1}$ *?*

Problem 5.11 *Is* MA *consistent with* $(\star\star)_{\omega_1,2^{\aleph_0}}$ *?*

6 $\neg(\star\star)_{\kappa,\lambda}$ for λ with large cardinal properties

In this section we show that $\neg(\star\star)_{\kappa,\lambda}$ holds if there is an ideal over λ with certain precipitousness.

An ideal I over a cardinal λ is said to be δ-strategically closed if the player Nonempty has a winning strategy in the following infinitary game over the partial ordering $\mathbb{P}_I = \langle \mathbb{P}_I, \leq_{\mathbb{P}_I} \rangle$ where

(6.1) $\quad \mathbb{P}_I = (\mathcal{P}(\lambda)/I) \setminus \{\emptyset/I\}$

and $\leq_{\mathbb{P}_I}$ is defined by

(6.2) $\quad X/I \leq_{\mathbb{P}_I} Y/I \;\Leftrightarrow\; X \setminus Y \in I.$

The player Empty begins the game with his move $p_0 \in \mathbb{P}_I$. At $\alpha = 1+\xi+2n+1$'st move for a limit ordinal $\xi < \delta$ (or $\xi = 0$) and $n \in \omega$, the player Empty plays $p_\alpha \in \mathbb{P}_I$; Nonempty plays $p_\alpha \in \mathbb{P}_I$ at $\alpha = 1+\xi+2n$'th move; each p_α must be below the previous moves p_ξ, $\xi < \alpha$ (with respect to $\leq_{\mathbb{P}_I}$) so that p_ξ, $\xi < \alpha$ form a decreasing sequence in $\mathbb{P}_I$ for all $\alpha < \delta$. Nonempty wins if the game can be played through all the moves p_ξ, $\xi < \delta$. Note that p_ξ at all limit $\xi < \delta$ is Nonempty's move as far as the game is played that far.

See e.g. [5], [10] for more about this game. κ-strategically closed ideals in our terminology are called $\prec\kappa$-strategically closed ideals in [8].

An ideal I over λ is μ-complete if $\bigcup X \in I$ for all $X \in [I]^{<\mu}$. I is precipitous if any generic ultrapower constructed on the basis of any $(V, \mathbb{P}_I)$-generic filter is well-founded.

It follows from a characterization of precipitousness that λ-complete $(\omega+1)$-strategically closed ideals are precipitous. Note that by definition if $\mathbb{P}_I$ is δ-strategically closed and $\delta' \leq \delta$ then $\mathbb{P}_I$ is δ'-strategically closed.

Lemma 6.1 *Suppose that there is a κ-strategically closed κ^+-complete ideal I over κ^+. Then $\kappa^{<\kappa} = \kappa$.*

Proof. Let $\mathbb{P}_I = (\mathcal{P}(\kappa^+)/I) \setminus \{\emptyset/I\}$ and G be a $(V, \mathbb{P}_I)$-generic filter. Let $\tilde{G}$ be the the corresponding filter over κ^+ and M be (the Mostowski collapse of) the generic ultrapower associated with $\tilde{G}$ and $j : V \to M$ the be the canonical embedding of V into M. We have $crit(j) = \kappa^+$.

Suppose toward a contradiction that $\kappa^{<\kappa} > \kappa$. Let $\lambda = \kappa^{<\kappa}$ and let $\langle \varphi_\alpha : \alpha < \lambda \rangle$ be an injective enumeration of ${}^{\kappa >}\kappa$. Let

$$\langle \tilde{\varphi}_\alpha : \alpha < j(\lambda) \rangle = j(\langle \varphi_\alpha : \alpha < \lambda \rangle).$$

Since $\kappa < crit(j)$ we have $\tilde{\varphi}_{j(\alpha)} = \varphi_\alpha$ for all $\alpha < \lambda$. We have $\kappa^+ \leq \lambda$ by assumption. Hence $\kappa^+ < j(\kappa^+) \leq j(\lambda)$. By the κ-strategical closedness of I we have $\tilde{\varphi}_{\kappa^+} \in V$. But since $\kappa^+ \notin j''V$ and since

$$M \models \langle \tilde{\varphi}_\alpha : \alpha < j(\lambda) \rangle \text{ is injective}$$

by elementarity,

$$M \models \tilde{\varphi}_{\kappa^+} \neq \tilde{\varphi}_{j(\alpha)} \ \text{ for all } \alpha < \lambda.$$

It follows that $\tilde{\varphi}_{\kappa^+} \neq \varphi_\alpha$ for all $\alpha < \lambda$ (in V). This is a contradiction.

❏ (Lemma 6.1)

Corollary 6.2 *Suppose that there is a κ-strategically closed κ^+-complete ideal I over κ^+. Then $\neg(\star\star)_\kappa$ holds.*

Proof. By Lemma 6.1 and Corollary 2.4. ❏ (Corollary 6.2)

The constellation of Corollary 6.2 can be created under a measurable cardinal: if a measurable cardinal λ is collapsed to κ^+ for a regular uncountable κ below λ by $\mathrm{Coll}(\kappa, <\lambda)$, then the ideal I generated from a regular ideal on λ in the ground model is κ-strategically closed and κ^+-complete (see [4] or [8]).

An ideal I over λ is said to be μ-saturated if $\mathbb{P}_I$ has the μ-c.c. where $\mathbb{P}_I$ is defined as in (6.1) and (6.2). It is known that λ^+-saturated ideal over λ is precipitous.

If λ is a measurable cardinal and we force with a κ-c.c. poset then in the generic extension the ideal generated from a maximal regular ideal over κ in the ground model is κ-saturated and λ-complete (see Theorem 17.1 in [6]).

Proposition 6.3 *Suppose that κ and λ are regular cardinals with $\omega_1 \leq \kappa < \lambda \leq 2^{\aleph_0}$. If there is a κ-saturated λ-complete ideal I on $\mathcal{P}(\lambda)$. Then $\neg(\star\star)_{\kappa,\lambda}$ holds.*

Furthermore, the following holds:

For every $\mathcal{F} \in [{}^\kappa\kappa]^\lambda$, there is a $g \in {}^\kappa\kappa$ such that $|\,\mathcal{F} \downarrow g\,| = \lambda$.

Proof. Let I be as in the proposition. I is precipitous by the remark above the proposition. Let G a $(V, \mathbb{P}_I)$-generic filter and let $j : V \to M$ be the corresponding generic ultrapower. We have $crit(j) = \lambda$.

Suppose that $\langle f_\alpha : \alpha < \lambda \rangle$ is an enumeration of a family $\mathcal{F} \in [{}^\kappa\kappa]^\lambda$. Let

$$\langle \tilde{f}_\alpha : \alpha < j(\lambda) \rangle = j(\langle f_\alpha : \alpha < \lambda \rangle).$$

For $\alpha < \lambda$, we have $\tilde{f}_\alpha = f_\alpha$ by α, $\kappa < crit(j)$. Let $\underset{\sim}{g}$ be a $\mathbb{P}_I$-name of $\tilde{f}_\lambda$. By the κ-c.c. of $\mathbb{P}_I$, there is a $g \in {}^\kappa\kappa$ (in V) such that

$$\Vdash_{\mathbb{P}_I} \text{“}\, \underset{\sim}{g} \leq g \,\text{”}.$$

Now back in $V[G]$, for any $\eta < \lambda$ we have

$$M \models \exists \nu < j(\lambda)\ (\eta < \nu \ \wedge\ \tilde{f}_\nu \leq g)$$

since $\tilde{f}_\lambda$ witnesses this. Hence by elementarity, we have

$$V \models \exists \nu < \lambda\ (\eta < \nu \ \wedge\ f_\nu \leq g).$$

Thus $|\,\mathcal{F} \downarrow g\,| = \lambda$. ❑ (Proposition 6.3)

Corollary 6.4 *Suppose that κ and λ are regular cardinals with $\omega_1 \leq \kappa < \lambda \leq 2^{\aleph_0}$. If λ is real-valued measurable then we have $\neg(\star\star)_{\kappa,\lambda}$.*

Proof. The null ideal over λ with respect to a real-valued measure over λ is ω_1-saturated. Hence by Proposition 6.3, $\neg(\star\star)_{\kappa,\lambda}$ follows. ❑ (Corollary 6.4)

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