

Sakaé Fuchino · Stefan Geschke · Lajos Soukup

## How to drive our families mad

Received: date / Revised version: date – © Springer-Verlag 2010

**Abstract.** Given a family  $\mathcal{F}$  of pairwise almost disjoint (ad) sets on a countable set  $S$ , we study families  $\tilde{\mathcal{F}}$  of maximal almost disjoint (mad) sets extending  $\mathcal{F}$ .

We define  $\mathfrak{a}^+(\mathcal{F})$  to be the minimal possible cardinality of  $\tilde{\mathcal{F}} \setminus \mathcal{F}$  for such  $\tilde{\mathcal{F}}$  and  $\mathfrak{a}^+(\kappa) = \max\{\mathfrak{a}^+(\mathcal{F}) : |\mathcal{F}| \leq \kappa\}$ . We show that all infinite cardinals less than or equal to the continuum  $\mathfrak{c}$  can be represented as  $\mathfrak{a}^+(\mathcal{F})$  for some ad  $\mathcal{F}$  (Theorem 10) and that the inequalities  $\aleph_1 = \mathfrak{a} < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$  (Corollary 1) and  $\mathfrak{a} = \mathfrak{a}^+(\aleph_1) < \mathfrak{c}$  (Theorem 9) are both consistent.

We also give several constructions of mad families with some additional properties.

---

### 1. Introduction

Given a family  $\mathcal{F}$  of pairwise almost disjoint countable sets, we can ask how the maximal almost disjoint (mad) families extending  $\mathcal{F}$  look like. In this

---

Sakaé Fuchino: Graduate School of System Informatics, Kobe University, Kobe, Japan. [fuchino@diamond.kobe-u.ac.jp](mailto:fuchino@diamond.kobe-u.ac.jp)

Stefan Geschke: Hausdorff Center for Mathematics, Universität Bonn, Bonn, Germany. [stefan.geschke@hcm.uni-bonn.de](mailto:stefan.geschke@hcm.uni-bonn.de)

Lajos Soukup: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary. [soukup@math.cs.kitami-it.ac.jp](mailto:soukup@math.cs.kitami-it.ac.jp)

The first author was partially supported by Chubu University grant 16IS55A and Grant-in-Aid for Scientific Research (C) 19540152, Japan Society for the Promotion of Science. The second author was supported by Center of Excellence grant from the European Union. The third author was supported by Bolyai Grant. The research of this paper was began when the first and second authors visited Alfréd Rényi Institute in Budapest in 2002. The research was then resumed when the first author visited Centre de Recerca Matemàtica in Barcelona and the third author Barcelona University at the same time in 2006. The first and third authors would like to thank Joan Bagaria and Juan-Carlos Martínez for the arrangement of the visit and their hospitality during the stay in Barcelona. The authors also would like to thank Kenneth Kunen for allowing them to include his unpublished results and Andreas Blass as well as the referee of the paper for reading carefully the manuscript and giving many valuable suggestions.

*Key words:* cardinal invariants – almost disjoint number – Cohen model – destructible maximal almost disjoint family

note and [5], we address some instances of this question and other related problems.

Let us begin with the definition of some notions and notation about almost disjointness we shall use here. Two countable sets  $A, B$  are said to be *almost disjoint* (*ad* for short) if  $A \cap B$  is finite. A family  $\mathcal{F}$  of countable sets is said to be *pairwise almost disjoint* (*ad* for short) if any two distinct  $A, B \in \mathcal{F}$  are *ad*.

If  $\mathcal{X} \subseteq [S]^{\aleph_0}$  and  $S = \bigcup \mathcal{X}$ ,  $\mathcal{F} \subseteq \mathcal{X}$  is said to be *mad in  $\mathcal{X}$*  if  $\mathcal{F}$  is *ad* and there is no *ad*  $\mathcal{F}'$  such that  $\mathcal{F} \subsetneq \mathcal{F}' \subseteq \mathcal{X}$ . Thus an *ad* family  $\mathcal{F}$  is *mad in  $\mathcal{X}$*  if and only if there is no  $X \in \mathcal{X}$  which is *ad* from every  $Y \in \mathcal{F}$ . If  $\mathcal{F}$  is *mad in  $[S]^{\aleph_0}$*  for  $S = \bigcup \mathcal{F}$ , we say simply that  $\mathcal{F}$  is a *mad family* (on  $S$ ).  $S$  is called the *underlying set* of  $\mathcal{F}$ .

Let

$$(1.1) \quad \mathfrak{a}(\mathcal{X}) = \min\{|\mathcal{F}| : |\mathcal{F}| \geq \aleph_0 \text{ and } \mathcal{F} \text{ is mad in } \mathcal{X}\}.$$

Clearly, the cardinal invariant  $\mathfrak{a}$  known as the almost disjoint number ([2]) can be characterized as:

*Example 1.*  $\mathfrak{a} = \mathfrak{a}([S]^{\aleph_0})$  for any countable  $S$ .

In this paper we concentrate on the case where the underlying set  $S = \bigcup \mathcal{X}$  (or  $S = \bigcup \mathcal{F}$ ) is countable. In [5] and the forthcoming continuation of this paper, we will deal with the cases where  $S$  may be also uncountable.

As the countable  $S = \bigcup \mathcal{X}$ , we often use  $\omega$  or  $T = {}^{\omega}2$  where  $T$  is considered as a tree growing downwards. That is, for  $b, b' \in T$ , we write  $b' \leq_T b$  if  $b \subseteq b'$ . Each  $f \in {}^{\omega}2$  induces the (maximal) branch

$$(1.2) \quad B(f) = \{f \upharpoonright n : n \in \omega\} \subseteq T$$

in  $T$ .

In Section 2, we consider several cardinal invariants of the form  $\mathfrak{a}(\mathcal{X})$  for some  $\mathcal{X} \subseteq [T]^{\aleph_0}$ .

For  $\mathcal{X} \subseteq [S]^{\aleph_0}$  with  $S = \bigcup \mathcal{X}$ , let

$$(1.3) \quad \mathcal{X}^\perp = \{Y \in [S]^{\aleph_0} : \forall X \in \mathcal{X} \mid |X \cap Y| < \aleph_0\}.$$

If  $Y \in \mathcal{X}^\perp$  we shall say that  $Y$  is *almost disjoint* (*ad*) *to  $\mathcal{X}$* .

For an *ad* family  $\mathcal{F}$ , let

$$(1.4) \quad \mathfrak{a}^+(\mathcal{F}) = \mathfrak{a}(\mathcal{F}^\perp).$$

For a cardinal  $\kappa$ , let

$$(1.5) \quad \mathfrak{a}^+(\kappa) = \sup\{\mathfrak{a}^+(\mathcal{F}) : \mathcal{F} \text{ is an ad family on } \omega \text{ of cardinality } \leq \kappa\}.$$

Clearly,  $\mathfrak{a}^+(\omega) = \mathfrak{a}$  and  $\mathfrak{a}^+(\kappa) \leq \mathfrak{a}^+(\lambda) \leq \mathfrak{c}$  for any  $\kappa \leq \lambda \leq \mathfrak{c}$ . In Section 3 we give several constructions of *ad* families  $\mathcal{F}$  for which  $\mathcal{F}^\perp$  has some particular property. Using these constructions, we show in Section 4 that  $\mathfrak{a}^+(\mathfrak{c}) = \mathfrak{c}$  (actually we have  $\mathfrak{a}^+(\mathfrak{d}) = \mathfrak{c}$ , see Theorem 7) and the consistency of the inequalities  $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$  (see Corollary 1). We also show the consistency of  $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$  (Theorem 9).

For undefined notions connected to the forcing, the reader may consult [7] or [8]. We mostly follow the notation and conventions set in [7] and/or [8]. In particular, the forcing is denoted in such a way that stronger conditions are smaller. We assume that  $\mathbb{P}$ -names are constructed just as in [8] for a poset  $\mathbb{P}$  but different from [8] we use symbols with tilde below them like  $\tilde{a}$ ,  $\tilde{b}$  etc. to denote the  $\mathbb{P}$ -names corresponding to the sets  $a$ ,  $b$  etc. in the generic extension.  $V$  denotes the ground model (in which we live). The canonical  $\mathbb{P}$ -names of elements  $a$ ,  $b$  etc. of  $V$  are denoted by the same symbols with hat like  $\hat{a}$ ,  $\hat{b}$  etc. For a poset  $\mathbb{P}$  (in  $V$ ) we use  $V^{\mathbb{P}}$  to denote a “generic” generic extension  $V[G]$  of  $V$  by some  $(V, \mathbb{P})$ -generic filter  $G$ . Thus  $V^{\mathbb{P}} \models \dots$  is synonymous to  $\Vdash_{\mathbb{P}} \dots$  or  $V \models \Vdash_{\mathbb{P}} \dots$  and a phrase like: “Let  $W = V^{\mathbb{P}}$ ” is to be interpreted as saying: “Let  $W$  be a generic extension of  $V$  by some/any  $(V, \mathbb{P})$ -generic filter”.

For the notation connected to the set theory of reals see [1] and [2]. With  $\mathfrak{c}$  we denote the size of the continuum  $2^{\aleph_0}$ .  $\mathcal{M}$  and  $\mathcal{N}$  are the ideals of meager sets and null sets (e.g. over the Cantor space  ${}^\omega 2$ ) respectively. For  $I = \mathcal{M}, \mathcal{N}$  etc.,  $\text{cov}(I)$  and  $\text{non}(I)$  are *covering number* and *uniformity* of  $I$ .

For an infinite cardinal  $\kappa$  let  $\mathcal{C}_\kappa = \text{Fn}(\kappa, 2)$  or, more generally  $\mathcal{C}_X = \text{Fn}(X, 2)$  for any set  $X$ .  $\mathcal{C}_\kappa$  is the Cohen forcing for adding  $\kappa$  many Cohen reals.  $\mathcal{R}_\kappa$  denotes the random forcing for adding  $\kappa$  many random reals.  $\mathcal{R}_\kappa$  is the poset consisting of Borel sets of positive measure in  ${}^\kappa 2$ , which corresponds to the homogeneous measure algebra of Maharam type  $\kappa$ .

For a poset  $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ ,  $X \subseteq \mathbb{P}$  and  $p \in \mathbb{P}$ , let

$$X \downarrow p = \{q \in X : q \leq_{\mathbb{P}} p\}.$$

## 2. Mad families and almost disjoint numbers

One of the advantages of using  $T = {}^{\omega} 2$  as the countable underlying set is that we can define some natural subfamilies of  $[T]^{\aleph_0}$  such as  $\mathcal{O}_T, \mathcal{A}_T, \mathcal{B}_T$  etc. below.

For  $X \subseteq T$ , let

$$(2.1) \quad [X] = \{f \in {}^\omega 2 : B(f) \subseteq X\}, \text{ and}$$

$$(2.2) \quad [X] = \{f \in {}^\omega 2 : |B(f) \cap X| = \aleph_0\}.$$

Clearly, we have  $[X] \subseteq [X]$ . For  $X \subseteq T$ , let  $X^\uparrow$  be the upward closure of  $X$ , that is:

$$(2.3) \quad X^\uparrow = \{t \upharpoonright n : t \in X, n \leq \ell(t)\}.$$

Then we have  $[X] \subseteq [X^\uparrow]$  for any  $X \subseteq T$ .

**Definition 1 (Off-binary sets, [9]).** *Let*

$$\mathcal{O}_T = \{X \in [T]^{\aleph_0} : [X] = \emptyset\}.$$

T. Leathrum [9] called elements of  $\mathcal{O}_T$  off-binary sets. Note that  $[X] = \emptyset$  if and only if there is no branch in  $T$  with infinite intersection with  $X$ .

**Definition 2 (Antichains).** *Let*

$$\mathcal{A}_T = \{X \in [T]^{\aleph_0} : X \text{ is an antichain in } T\}.$$

Clearly, we have  $\mathcal{A}_T \subseteq \mathcal{O}_T$ .

Using the notation above, the cardinal invariants  $\mathfrak{o}$  and  $\bar{\mathfrak{o}}$  introduced by Leathrum [9] can be characterized as:

$$(2.4) \quad \mathfrak{o} = \mathfrak{a}(\mathcal{O}_T),$$

$$(2.5) \quad \bar{\mathfrak{o}} = \mathfrak{a}(\mathcal{A}_T)$$

(see [9]). Leathrum also showed  $\mathfrak{a} \leq \mathfrak{o} \leq \bar{\mathfrak{o}}$ . J. Brendle [3] proved  $\text{non}(\mathcal{M}) \leq \mathfrak{o}$ .

**Definition 3 (Sets without infinite antichains).** *Let*

$$\mathcal{B}_T = \{X \in [T]^{\aleph_0} : X \text{ does not contain any infinite antichain}\}.$$

Note that  $\mathcal{B}_T = \mathcal{A}_T^\perp$ . Elements of  $\mathcal{B}_T$  are those infinite subsets of  $T$  which can be covered by finitely many branches:

**Lemma 1 (K. Kunen).** *Let  $X \in [T]^{\aleph_0}$ . Then  $X \in \mathcal{B}_T$  if and only if  $X$  is covered by finitely many branches in  $T$ .*

*Proof.* If  $X$  is covered by finitely many branches in  $T$  then  $X$  clearly does not contain any infinite antichain since otherwise one of the finitely many branches would contain an infinite antichain.

Suppose now that  $X$  cannot be covered by finitely many branches. By induction on  $n$ , we choose  $t_n \in 2^n$  such that  $t_0 = \emptyset$ ,  $t_{n+1} = t_n \hat{\ } i$  for some  $i \in 2$  and

$$(2.6) \quad X_{n+1} = X \downarrow t_{n+1} \text{ can not be covered by finitely many branches.}$$

This is possible since  $X_0 = X$  and  $X_n \subseteq (X_n \downarrow (t_n \hat{\ } 0)) \cup (X_n \downarrow (t_n \hat{\ } 1)) \cup \{t_n\}$ .

By (2.6), the branch  $B = \{t_n : n < \omega\}$  does not cover  $X_n$  for each  $n \in \omega$ . So we can pick  $s_n \in X_n \setminus B$ . Let  $S = \{s_n : n \in \omega\}$ .  $S$  is an infinite subset of  $X$  since  $\ell(s_n) \geq n$  for all  $n \in \omega$ . If  $C$  is a branch in  $T$  different from  $B$  then  $t_n \notin C$  for some  $n \in \omega$  and so  $s_m \notin C$  for all  $m \geq n$ . Hence  $S \cap C$  is finite. Moreover  $S \cap B = \emptyset$ . So we have  $\lceil S \rceil = \emptyset$ . Thus  $S$  should contain an infinite antichain by König's Lemma.  $\square$

**Theorem 1 (K. Kunen).**  $\mathfrak{a}(\mathcal{B}_T) = \mathfrak{c}$ .

*Proof.* Suppose that  $\mathcal{F} \subseteq \mathcal{B}_T$  is an ad family of cardinality  $< \mathfrak{c}$ . We show that  $\mathcal{F}$  is not mad. For each  $X \in \mathcal{F}$  there is  $b_X \in [{}^\omega 2]^{< \aleph_0}$  such that  $X \subseteq \bigcup_{f \in b_X} B(f)$  by Lemma 1. Since  $\mathcal{S} = \bigcup \{b_X : X \in \mathcal{F}\}$  has cardinality  $\leq |\mathcal{F}| \cdot \aleph_0 < \mathfrak{c}$ , there is  $f^* \in {}^\omega 2 \setminus \mathcal{S}$ . We have  $B(f^*) \in \mathcal{B}_T$  and  $B(f^*)$  is ad to  $\mathcal{F}$ .  $\square$

Let us say  $X \subseteq T$  is *nowhere dense* if  $\lceil X \rceil$  is nowhere dense in the Cantor space  ${}^\omega 2$ . It can be easily shown that  $X$  is nowhere dense if and only if

$$(2.7) \quad \forall t \in T \exists t' \leq_T t \forall t'' \leq_T t' (t'' \notin X).$$

Note that, if  $X \subseteq T$  is not nowhere dense, then  $X$  is dense below some  $t \in T$  (in terms of forcing). Also note that from (2.7) it follows that the property of being nowhere dense is absolute.

**Definition 4 (Nowhere dense sets).** *Let*

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : X \text{ is nowhere dense}\}.$$

Note that, for  $X \in [T]^{\aleph_0}$  with  $X = \{t_n : n \in \omega\}$ , we have

$$\lceil X \rceil = \bigcap_{n \in \omega} \bigcup_{m > n} [T \downarrow t_m].$$

In particular  $\lceil X \rceil$  is a  $G_\delta$  subset of  ${}^\omega 2$ . Hence by Baire Category Theorem we have

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : \lceil X \rceil \text{ is a meager subset of } {}^\omega 2\}.$$

**Lemma 2.** *If  $X \in [T]^{\aleph_0}$  then there is  $X' \in [X]^{\aleph_0}$  such that  $X' \in \mathcal{ND}_T$ .*

*Proof.* If  $\lceil X \rceil = \emptyset$  then  $X \in \mathcal{ND}_T$ . Thus we can put  $X' = X$ . Otherwise let  $f \in \lceil X \rceil$  and let  $X' = X \cap B(f)$ .  $\square$

**Theorem 2.**  $\text{cov}(\mathcal{M}), \mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$ .

*Proof.* For the inequality  $\text{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathcal{ND}_T)$ , suppose that  $\mathcal{F} \subseteq \mathcal{ND}_T$  is an ad family of cardinality  $< \text{cov}(\mathcal{M})$ . Then  $\bigcup\{\lceil X \rceil : X \in \mathcal{F}\} \neq {}^\omega 2$ . Let  $f \in {}^\omega 2 \setminus \bigcup\{\lceil X \rceil : X \in \mathcal{F}\}$ . Then  $B(f) \in \mathcal{ND}_T$  and  $B(f)$  is ad from all  $X \in \mathcal{F}$ .

To show  $\mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$  suppose that  $\mathcal{F} \subseteq \mathcal{ND}_T$  is an ad family of cardinality  $< \mathfrak{a}$ . Then  $\mathcal{F}$  is not a mad family in  $[T]^{\aleph_0}$ . Hence there is some  $X \in [T]^{\aleph_0}$  ad to  $\mathcal{F}$ . By Lemma 2, there is  $X' \subseteq X$  such that  $X' \in \mathcal{ND}_T$ . Since  $X'$  is also ad to  $\mathcal{F}$ , it follows that  $\mathcal{F}$  is not mad in  $\mathcal{ND}_T$ .  $\square$

Let  $\sigma$  be the measure on Borel sets of the Cantor space  ${}^\omega 2$  defined as the product measure of the probability measure on 2. For  $X \subseteq T$ , let  $\mu(X) = \sigma(\lceil X \rceil)$ .

**Definition 5 (Null sets).** *Let*

$$\mathcal{N}_T = \{X \in [T]^{\aleph_0} : \mu(X) = 0\}.$$

**Theorem 3.**  $\text{cov}(\mathcal{N}), \mathfrak{a} \leq \mathfrak{a}(\mathcal{N}_T)$ .

*Proof.* Similarly to the proof of Theorem 2.  $\square$

**Definition 6 (Nowhere dense null sets).** *Let*

$$\mathcal{NDN}_T = \mathcal{ND}_T \cap \mathcal{N}_T.$$

**Lemma 3.**  $\mathfrak{a}(\mathcal{ND}_T) \leq \mathfrak{a}(\mathcal{NDN}_T)$  and  $\mathfrak{a}(\mathcal{N}_T) \leq \mathfrak{a}(\mathcal{NDN}_T)$ .

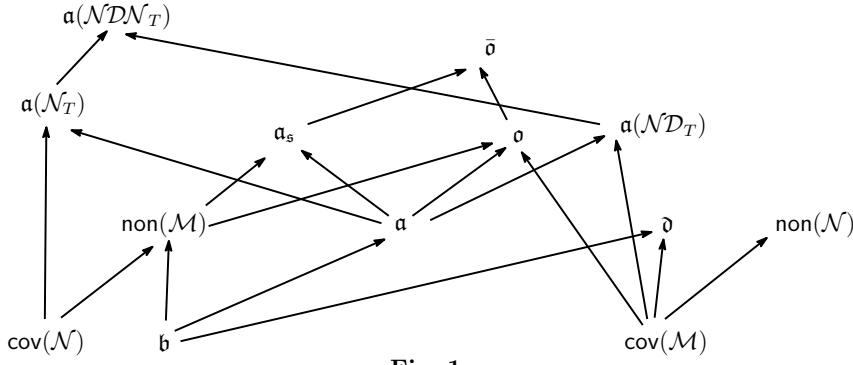


Fig. 1.

*Proof.* For the first inequality, suppose that  $\mathcal{F}$  is a mad family in  $\mathcal{N}\mathcal{D}\mathcal{N}_T$ . Then  $\mathcal{F}$  is an ad family in  $\mathcal{N}\mathcal{D}_T$ . It is also mad in  $\mathcal{N}\mathcal{D}_T$ . Suppose not. Then there is an  $X \in \mathcal{N}\mathcal{D}_T$  ad to  $\mathcal{F}$ . Let  $X' \in [X]^{\aleph_0}$  be as in the measure analog of Lemma 2. Then  $X' \in \mathcal{N}\mathcal{D}\mathcal{N}_T$ . Hence  $\mathcal{F}$  is not mad in  $\mathcal{N}\mathcal{D}\mathcal{N}_T$ . This is a contradiction. The second inequality can be also proved similarly.  $\square$

The diagram Fig.1 summarizes the inequalities obtained in this section integrated into the cardinal diagram given in Brendle [4]. “ $\kappa \rightarrow \lambda$ ” in the diagram means that “ $\kappa \leq \lambda$  is provable in ZFC”. There are still some open questions concerning the (in)completeness of this diagram. In particular:

**Problem 1.** (a) Are the inequalities between  $\mathfrak{a}(\mathcal{N}_T)$ ,  $\mathfrak{a}(\mathcal{N}\mathcal{D}_T)$ ,  $\mathfrak{a}(\mathcal{N}\mathcal{D}\mathcal{N}_T)$  consistently strict and complete?

(b) Are  $\mathfrak{a}(\mathcal{N}\mathcal{D}_T)$  etc. independent from  $\mathfrak{o}$ ,  $\bar{\mathfrak{o}}$ ,  $\mathfrak{a}_s$  ?

### 3. Ad families $\mathcal{F}$ for which $\mathcal{F}^\perp$ is contained in a certain subfamily of $[T]^{\aleph_0}$

In this section we give several constructions of ad families with the property that the sets ad to them in a given generic extension are necessarily in a certain subfamily of  $[T]^{\aleph_0}$ . The constructions in this section are used in the proof of some results in the next sections.

**Theorem 4.** (CH) *There exists an ad family  $\mathcal{F} \subseteq \mathcal{A}_T$  of size  $\aleph_1$  such that for any cardinal  $\kappa$  we have*

$$(3.1) \quad V^{C_\kappa} \models \mathcal{F}^\perp \subseteq \mathcal{N}\mathcal{D}_T.$$

*Proof.* Let

$$(3.2) \quad \mathcal{S} = \{ \langle p, \tilde{B}, t \rangle : p \in \mathcal{C}_\omega, \tilde{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of a subset of } T, \\ t \in T \text{ and } p \Vdash_{\mathcal{C}_\omega} \text{“} \tilde{B} \text{ is dense below } t \text{”} \}.$$

Note that this set is of cardinality  $\aleph_1$  by CH. Let  $\langle \langle p_\alpha, \tilde{B}_\alpha, t_\alpha \rangle : \alpha < \omega_1 \setminus \omega \rangle$  be an enumeration of  $\mathcal{S}$ .

By induction on  $\alpha < \omega_1$ , we construct  $A_\alpha \subseteq T$ ,  $\alpha < \omega_1$  such that

- (3.3)  $A_\alpha \in \mathcal{A}_T$  for all  $\alpha < \omega_1$ ,
- (3.4)  $A_n$ ,  $n \in \omega$  is a partition of  $T \setminus \{\emptyset\}$  (note that  $\emptyset$  is the root of the tree  $T$ ),
- (3.5)  $|A_\beta \cap A_\alpha| < \aleph_0$  for all  $\beta < \alpha < \omega_1$ , and
- (3.6) if  $\alpha \in \omega_1 \setminus \omega$ , for each  $q \leq_{\mathcal{C}_\omega} p_\alpha$  and  $n \in \omega$ , there are  $r \leq_{\mathcal{C}_\omega} q$  and  $t \in A_\alpha \downarrow t_\alpha$  such that  $|t| \geq n$  and  $r \Vdash_{\mathcal{C}_\omega} "t \in \tilde{B}_\alpha"$  (in particular,  $p_\alpha \Vdash_{\mathcal{C}_\omega} "|A_\alpha \cap \tilde{B}_\alpha| = \aleph_0"$ ).

We show first that  $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$  for  $A_\alpha$ 's as above satisfies (3.1). Since every subset of  $T$  in  $V^{\mathcal{C}_\omega}$  is contained in  $V^{\mathcal{C}_\omega}$  for some countable  $X \subseteq \omega$ , it is enough to show (3.1) for  $\kappa = \omega$ . Assume for contradiction that, for some  $\mathcal{C}_\omega$ -name  $\tilde{B}^*$  of subset of  $T$  and  $p \in \mathcal{C}_\omega$ , we have  $p \Vdash_{\mathcal{C}_\omega} "\tilde{B}^* \in \mathcal{F}^\perp \setminus \mathcal{N}\mathcal{D}_T"$ . Then there are  $t^* \in T$  and  $p^* \leq_{\mathcal{C}_\omega} p$  such that

$$(3.7) \quad p^* \Vdash_{\mathcal{C}_\omega} "\tilde{B}^* \text{ is dense below } t^* \text{ and } |\tilde{B}^* \cap A_\alpha| < \aleph_0 \text{ for all } \alpha < \omega_1"$$

We may assume that  $\tilde{B}^*$  is a nice  $\mathcal{C}_\omega$ -name. Let  $\alpha < \omega_1 \setminus \omega$  be such that  $\langle p_\alpha, \tilde{B}_\alpha, t_\alpha \rangle = \langle p^*, \tilde{B}^*, t^* \rangle$ . Then  $p^* \Vdash_{\mathcal{C}_\omega} "|A_\alpha \cap \tilde{B}^*| = \aleph_0"$  by (3.6). This is a contradiction.

To see that the construction of  $A_\alpha$ ,  $\alpha < \omega_1$  is possible, assume that  $\langle A_\beta : \beta < \alpha \rangle$  satisfying (3.3), (3.4), (3.5) and (3.6) has been constructed for  $\alpha \in \omega_1 \setminus \omega$ .

For  $q \leq_{\mathcal{C}_\omega} p_\alpha$  let

$$I(\tilde{B}_\alpha, q) = \{t \in T : t \leq_T t_\alpha \wedge \exists r \leq_{\mathcal{C}_\omega} q (r \Vdash_{\mathcal{C}_\omega} "\hat{t} \in \tilde{B}_\alpha")\}.$$

Note that  $I(\tilde{B}_\alpha, q)$  is dense below  $t_\alpha$  by the definition (3.2) of  $\langle p_\alpha, \tilde{B}_\alpha, t_\alpha \rangle \in \mathcal{S}$ .

Fix an enumeration  $\{\langle q_i, n_i \rangle : i < \omega\}$  of  $(\mathcal{C}_\omega \downarrow p_\alpha) \times \omega$  and an enumeration  $\{\beta_i : i < \omega\}$  of  $\alpha$ .

By induction on  $m \in \omega$  we choose  $u_m \in T$  and  $r_m \in \mathcal{C}_\omega$  according to the following (3.8) – (3.12) and let

$$A_\alpha = \{u_m : m < \omega\}.$$

In the  $m$ 'th step of the construction, let  $u_m \in T$  and  $r_m \in \mathcal{C}_\omega$  be such that

- (3.8)  $\{u_i : i \leq m\}$  is an antichain in  $T \downarrow t_\alpha$  which is not maximal below  $t_\alpha$ ;
- (3.9)  $u_m \in I(\tilde{B}_\alpha, q_m) \setminus \bigcup \{A_{\beta_i} : i < m\}$ ;
- (3.10)  $|u_m| \geq n_m$ ;
- (3.11)  $r_m \leq_{\mathcal{C}_\omega} q_m$ ; and
- (3.12)  $r_m \Vdash_{\mathcal{C}_\omega} "\hat{u}_m \in \tilde{B}_\alpha"$ .

This can be carried out. Indeed, at the  $m$ 'th step if  $\{u_i : i < m\}$  has been chosen so that it is a non-maximal antichain below  $t_\alpha$ , then we can find

$u'_m \in T \downarrow t_\alpha$  distinct from all  $u_i, i < m$  such that  $\{u_i : i < m\} \cup \{u'_m\}$  is still a non-maximal antichain below  $t_\alpha$ . We can also choose  $u'_m$  so that  $|u'_m| \geq n_m$ . Since  $\{A_{\beta_i} : i < m\}$  are antichains we can find  $u''_m \leq_T u'_m$  such that there is no  $t \leq_T u''_m$  with  $t \in \cup\{A_{\beta_i} : i < m\}$ . Since  $I(\tilde{B}_\alpha, q_m)$  is dense below  $t_\alpha$  we can find  $u_m \in I(\tilde{B}_\alpha, q_m)$  such that  $u_m \leq_T u''_m$ . By the definition of  $I(\tilde{B}_\alpha, q_m)$  there is an  $r_m \leq_{\mathcal{C}_\omega} q_m$  such that  $r_m \Vdash_{\mathcal{C}_\omega} \hat{u}_m \in \tilde{B}_\alpha$ .

It is easy to see that  $A_\alpha$  defined as above satisfies (3.3), (3.5) and (3.6):  $A_\alpha \in \mathcal{A}_T$  by (3.8),  $|A_\beta \cap A_\alpha| < \aleph_0$  for all  $\beta < \alpha$  by (3.9). To show that  $A_\alpha$  also satisfies (3.6), suppose that  $q \leq_{\mathcal{C}_\omega} p_\alpha$  and  $n \in \omega$ . Let  $m \in \omega$  be such that  $\langle q, n \rangle = \langle q_m, n_m \rangle$ . Then we have  $r_m \leq_{\mathcal{C}_\omega} q$  by (3.11),  $u_m \in A_\alpha$  by definition of  $A_\alpha$ ,  $|u_m| \geq n$  by (3.10) and  $r_m \Vdash_{\mathcal{C}_\omega} \hat{u}_m \in \tilde{B}_\alpha$  by (3.12).

□

**Problem 2.** Is CH really necessary for the conclusion of Theorem 4?

In connection with the problem above, we can actually obtain a slightly stronger conclusion than that of Theorem 4 if our ground model is a generic extension of some inner model by adding uncountably many Cohen reals. Note that CH need not hold in such a model.

**Theorem 5.** *Suppose that  $W = V^{\mathcal{C}_{\omega_1}}$ . Then, in  $W$ , there is an ad family  $\mathcal{F} \subseteq \mathcal{A}_T$  of cardinality  $\aleph_1$  such that,*

$$(3.13) \text{ for any c.c.c. poset } \mathbb{P} \text{ with } \mathbb{P} \in V, \text{ we have } W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{N}\mathcal{D}_T.$$

*Proof.* Let  $A \in [T]^{\aleph_0} \cap V$  be an antichain and let  $\langle t_n^* : n \in \omega \rangle$  be a 1-1 enumeration of  $A$ .

Let  $G$  be a  $(V, \mathcal{C}_{\omega_1})$ -generic filter and  $W = V[G]$ . For  $p \in \mathcal{C}_{\omega_1}$ ,  $\alpha < \omega_1$  and  $k \in \omega$ , let

$$\begin{aligned} f_\alpha^p &= \{\langle n, i \rangle \in \omega \times \omega : \langle \omega\alpha + 3n, i \rangle \in p\}; \\ n_{\alpha, k}^p &= \begin{cases} n, & \text{if } [\omega\alpha, \omega\alpha + 3n + 1] \subseteq \text{dom}(p), \\ & p(\omega\alpha + 3n + 1) = 1 \text{ and} \\ & |\{m < n : p(\omega\alpha + 3m + 1) = 1\}| = k, \\ \text{undefined,} & \text{if there is no such } n \text{ as above;} \end{cases} \\ t_\alpha^p &= \begin{cases} \{\langle n, i \rangle \in \omega \times \omega : n < n_{\alpha, 0}^p, \langle \omega\alpha + 3n + 2, i \rangle \in p\}, \\ & \text{if } n_{\alpha, 0}^p \text{ is defined,} \\ \text{undefined,} & \text{otherwise} \end{cases} \\ \text{and} \\ t_{\alpha, k}^p &= \begin{cases} \{\langle n, i \rangle \in \omega \times \omega : n < n_{\alpha, k+1}^p, \langle \omega\alpha + 3n + 2, i \rangle \in p\}, \\ & \text{if } n_{\alpha, k+1}^p \text{ is defined,} \\ \text{undefined,} & \text{otherwise.} \end{cases} \end{aligned}$$

Let



$$\begin{aligned} f_\alpha^G &= \bigcup_{p \in G} f_\alpha^p, \\ t_\alpha^G &= t_\alpha^p \text{ for some } p \in G \text{ such that } t_\alpha^p \text{ is defined, and} \\ t_{\alpha,k}^G &= t_{\alpha,k}^p \text{ for some } p \in G \text{ such that } t_{\alpha,k}^p \text{ is defined.} \end{aligned}$$

For  $\alpha \in \omega_1$ , let

$$(3.14) \quad A_\alpha = \{t_\alpha^G \frown t_k^* \frown t_{\alpha,k}^G : k \in \omega\}.$$

Clearly each  $A_\alpha$  is an antichain in  $T$ .

$A_\alpha, \alpha < \omega_1$  are pairwise almost disjoint: Suppose that  $\alpha < \beta < \omega_1$ . Then there is  $k_0 < \omega$  such that  $t_{\alpha,k}^G \neq f_{\beta,k}^G$  for all  $k \in \omega \setminus k_0$ . It follows that  $A_\alpha \cap A_\beta \subseteq \{t_\alpha^G \frown t_k^* \frown t_{\alpha,k}^G : k < k_0\}$ .

We show that  $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$  satisfies (3.13).

Suppose that  $\mathbb{P}$  is a c.c.c. poset (in  $W$ ) and  $\mathbb{P} \in V$ . Let  $H$  be a  $(W, \mathbb{P})$ -generic filter. It is enough to show that, in  $W[H]$ , if  $X \in [T]^{\aleph_0}$  is not nowhere dense then  $X$  is not almost ad to  $\mathcal{F}$ .

By the c.c.c. of  $\mathcal{C}_{\omega_1} * \dot{\mathbb{P}} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$ , there is an  $\alpha^* \in \omega_1$  such that  $X \in V[G \upharpoonright \mathcal{C}_{\omega\alpha^*}][H]$ . Let  $t \in T$  be such that  $X$  is dense below  $t$ . Then

$$D = \{p \in \mathcal{C}_{\omega_1 \setminus \omega\alpha^*} : t_\alpha^p \supseteq t \text{ for some } \alpha \in \omega_1 \setminus \omega\alpha^*\}$$

is dense in  $\mathcal{C}_{\omega_1 \setminus \omega\alpha^*}$ .

For  $p \in D$  and  $\alpha \in \omega_1 \setminus \omega\alpha^*$  such that  $t_\alpha^p \supseteq t$ , letting  $\tilde{A}_\alpha$  a  $\mathcal{C}_{\omega_1 \setminus \omega\alpha^*}$ -name of  $A_\alpha$ , we have  $p \Vdash_{\mathcal{C}_{\omega_1 \setminus \omega\alpha^*}} \text{“} |\tilde{A}_\alpha \cap X \downarrow t| = \aleph_0 \text{”}$  by (3.14) and since  $X$  is dense below  $t$ .

By genericity, it follows that, in  $W[G]$ , there is  $\alpha < \omega_1$  such that  $|A_\alpha \cap X| = \aleph_0$ .  $\square$

A measure version of Theorem 5 also holds:

**Theorem 6.** *Let  $W = V^{\mathcal{C}_{\omega_1}}$ . Then, in  $W$ , there is an ad family  $\mathcal{F}$  in  $\mathcal{N}_T$  of cardinality  $\aleph_1$  such that for any c.c.c. poset  $\mathbb{P}$  with  $\mathbb{P} \in V$ , we have  $W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{O}_T$ .*

For the proof of Theorem 6 we note first the following:

**Lemma 4.** *Suppose that  $X \subseteq T$  is such that  $X = \{t_k : k \in \omega\}$  for some enumeration  $t_k, k \in \omega$  of  $X$  with  $\ell(t_k) \geq k$  for all  $k \in \omega$ . Then  $X \in \mathcal{N}_T$ .*

*Proof.* For all  $n \in \omega$ , we have  $[X] \subseteq \bigcup_{k \in \omega \setminus n} [T \downarrow t_k]$ . Hence

$$\mu(X) = \sigma([X]) \leq \sum_{k \in \omega \setminus n} \sigma([T \downarrow t_k]) \leq \sum_{k \in \omega \setminus n} 2^k = 2^{-n}.$$

It follows that  $\mu(X) = 0$ .  $\square$

*Proof (of Theorem 6).* Let  $G$  be a  $(V, \mathcal{C}_{\omega_1})$ -generic filter and  $W = V[G]$ . In  $W$ , let

$$f_\alpha^G = \{\langle n, i \rangle : \langle \omega\alpha + n, i \rangle \in p \text{ for some } p \in G\}$$

for  $\alpha < \omega_1$  and let  $g_\alpha^G \in {}^\omega\omega$  be the increasing enumeration of  $(f_\alpha^G)^{-1}[\{1\}]$ .

Further in  $W$ , we construct inductively  $A_\alpha \in \mathcal{N}_T$ ,  $\alpha < \omega_1$  as follows.

For  $n \in \omega$ , let  $A_n \in \mathcal{N}_T$  be such that  $\langle A_n : n \in \omega \rangle$  is a partition of  $T$  in  $V$ . We can easily find such  $A_n$ 's by Lemma 4.

For  $\omega \leq \alpha < \omega_1$ , suppose that pairwise almost disjoint  $A_\beta$ ,  $\beta < \alpha$  have been constructed. Let  $\langle B_\ell : \ell \in \omega \rangle$  be an enumeration of  $\{A_\beta : \beta < \alpha\}$  and, for each  $n \in \omega$ , let  $\langle b_{n,m} : m \in \omega \rangle$  be an enumeration of

$$(3.15) \quad C_n = T \setminus ({}^{n>}2 \cup \{B_\ell : \ell < n\}).$$

Let

$$(3.16) \quad A_\alpha = \{b_{n,g_\alpha^G(n)} : n \in \omega\}.$$

$A_\alpha \in \mathcal{N}_T$  by (3.15) and Lemma 4.  $A_\alpha$  is ad to  $\{A_\beta : \beta < \alpha\}$  by (3.15) and (3.16).

We show that  $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$  is as desired. Suppose that  $\mathbb{P}$  is c.c.c. (in  $W$ ) and  $\mathbb{P} \in V$ . Let  $H$  be a  $(W, \mathbb{P})$ -generic filter. It is enough to show that, in  $W[H]$ , if  $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$  then  $X$  is not ad to  $\mathcal{F}$ . So suppose that (in  $W[H]$ )  $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$  and  $f \in [X]$ . Let  $B = X \cap B(f)$ . By the c.c.c. of  $\mathcal{C}_{\omega_1} * \hat{\mathbb{P}} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$ , there is an  $\alpha^* \in \omega_1 \setminus \omega$  such that  $B \in V[(G \upharpoonright \mathcal{C}_{\omega_{\alpha^*}})][H]$ . If  $B \cap A_\alpha$  is infinite for some  $\alpha < \alpha^*$  then we are done. So assume that  $B$  is ad to all  $A_\alpha$ ,  $\alpha < \alpha^*$ . Then  $B \cap C_n$  is infinite for all  $n \in \omega$ . Since  $f_{\alpha^*}^G$  is a Cohen real generic over  $V[(G \upharpoonright \mathcal{C}_{\omega_{\alpha^*}})][H]$ , it follows that  $B \cap A_{\alpha^*}$  is infinite.  $\square$

#### 4. Almost disjoint numbers over ad families

In this section we turn to questions on the possible values of  $\mathfrak{a}^+(\cdot)$ .

**Theorem 7.** (K. Kunen)  $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$ .

*Proof.* Let  $\mathcal{F}$  be any mad family in  $\mathcal{A}_T$  of cardinality  $\bar{\mathfrak{o}}$ . By maximality of  $\mathcal{F}$  we have  $\mathcal{F}^\perp = \mathcal{B}_T$ . If  $\mathcal{G} \subseteq [T]^{\aleph_0}$  is disjoint from  $\mathcal{F}$  and  $\mathcal{F} \cup \mathcal{G}$  is mad then  $\mathcal{G}$  is mad in  $\mathcal{B}_T$  and hence  $|\mathcal{G}| = \mathfrak{c}$  by Theorem 1.  $\square$

**Theorem 8.**  $V^{\mathcal{C}_\kappa} \models \mathfrak{a}^+(\aleph_1) \geq \kappa$  for all regular  $\kappa$ .

*Proof.* If  $\kappa = \omega_1$  this is trivial. So suppose that  $\kappa > \omega_1$ . Let  $W = V^{\mathcal{C}_{\omega_1}}$ . Then  $V^{\mathcal{C}_\kappa} = W^{\mathcal{C}_{\kappa \setminus \omega_1}}$ . Let  $\mathcal{F}$  be as in the proof of Theorem 5. Suppose that  $\tilde{\mathcal{F}} \supseteq \mathcal{F}$  is mad on  $T$  in  $V^{\mathcal{C}_\kappa}$ . Then  $\tilde{\mathcal{F}} \subseteq (\mathcal{N}\mathcal{D}_T)^{V^{\mathcal{C}_\kappa}}$ . Since  $V^{\mathcal{C}_\kappa} \models \text{cov}(\mathcal{M}) \geq \kappa$ , it follows that  $|\tilde{\mathcal{F}}| \geq \kappa$  by Theorem 2.  $\square$

**Corollary 1.** *The inequality  $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$  is consistent.*

*Proof.* Start from a model  $V$  of CH. Since there is a  $\mathcal{C}_\kappa$ -indestructible mad family in  $V$  it follows that  $V^{\mathcal{C}_{\omega_2}} \models \mathfrak{a} = \aleph_1$  (see e.g. [8], Theorem 2.3). On the other hand we have  $V^{\mathcal{C}_{\omega_2}} \models \mathfrak{a}^+(\aleph_1) = \aleph_2 = \mathfrak{c}$  by Theorem 8.  $\square$

**Theorem 9.** *The inequality  $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$  is consistent.*

For the proof of the theorem we use the following forcing notions: for a family  $\mathcal{I} \subseteq \{A \in [\omega]^{\aleph_0} : |\omega \setminus A| = \aleph_0\}$  closed under union, let  $\mathbb{Q}_{\mathcal{I}} = \langle \mathbb{Q}_{\mathcal{I}}, \leq_{\mathbb{Q}_{\mathcal{I}}} \rangle$  be the poset defined by

$$\mathbb{Q}_{\mathcal{I}} = \mathcal{C}_{\omega} \times \mathcal{I};$$

For all  $\langle s, A \rangle, \langle s', A' \rangle \in \mathbb{Q}_{\mathcal{I}}$

$$(4.1) \quad \langle s', A' \rangle \leq_{\mathbb{Q}_{\mathcal{I}}} \langle s, A \rangle \Leftrightarrow s \subseteq s', A \subseteq A' \text{ and} \\ \forall n \in \text{dom}(s') \setminus \text{dom}(s) (n \in A \rightarrow s'(n) = 0).$$

Clearly  $\mathbb{Q}_{\mathcal{I}}$  is  $\sigma$ -centered.

For a  $(V, \mathbb{Q}_{\mathcal{I}})$ -generic  $G$ , let

$$f_G = \bigcup \{s : \langle s, A \rangle \in G \text{ for some } A \in \mathcal{I}\} \text{ and} \\ A_G = f_G^{-1} \{1\}.$$

Let  $\tilde{\mathcal{I}}$  be the ideal in  $[\omega]^{\aleph_0}$  generated from  $\mathcal{I}$  (i.e. the downward closure of  $\mathcal{I}$  with respect to  $\subseteq$ ). By the genericity of  $G$  and the definition of  $\leq_{\mathbb{Q}_{\mathcal{I}}}$  it is easy to see that  $A_G$  is infinite and

$$(4.2) \quad \text{for every } B \in ([\omega]^{\aleph_0})^V, A_G \text{ is almost disjoint from } B \Leftrightarrow B \in \tilde{\mathcal{I}}.$$

*Proof (of Theorem 9).* Working in a ground model  $V$  of  $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$ , let

$$\langle \mathbb{P}_{\alpha}, \mathbb{Q}_{\beta} : \alpha \leq \omega_2, \beta < \omega_2 \rangle$$

be the finite support iteration of c.c.c. posets defined as follows: for  $\beta < \omega_2$ , let  $\mathbb{Q}_{\beta}$  be the  $\mathbb{P}_{\beta}$ -name of the finite support (side-by-side) product of

$$(4.3) \quad \mathbb{Q}_{\tilde{\mathcal{F}}}, \tilde{\mathcal{F}} \in \Phi$$

where

$$\Phi = \{\tilde{\mathcal{F}} : \tilde{\mathcal{F}} \text{ is an ideal in } [\omega]^{\aleph_0} \\ \text{generated from an ad family in } [\omega]^{\aleph_0} \text{ of cardinality } \aleph_1\}$$

in  $V^{\mathbb{P}_{\beta}}$ . We have

$$V^{\mathbb{P}_{\beta}} \models \mathbb{Q}_{\beta} \text{ satisfies the c.c.c.}$$

since  $V^{\mathbb{P}_{\beta}} \models \mathbb{Q}_{\tilde{\mathcal{F}}}$  is  $\sigma$ -centered for all  $\tilde{\mathcal{F}} \in \Phi$ . By induction on  $\alpha \leq \omega_2$ , we can show that  $\mathbb{P}_{\alpha}$  satisfies the c.c.c. and  $|\mathbb{P}_{\alpha}| \leq 2^{\aleph_1} = \aleph_3$  for all  $\alpha \leq \omega_2$ . It follows that

$$(4.4) \quad V^{\mathbb{P}_{\omega_2}} \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3.$$

Thus the following claim finishes the proof:

$$\textit{Claim. } V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \mathfrak{a}^+(\aleph_1) = \aleph_2.$$

$\vdash$  Working in  $V^{\mathbb{P}_{\omega_2}}$ , suppose that  $\mathcal{F}$  is an ad family in  $[\omega]^{\aleph_0}$  of cardinality  $\aleph_1$ . By the c.c.c. of  $\mathbb{P}_{\omega_2}$ , there is some  $\alpha^* < \omega_2$  such that  $\mathcal{F} \in V^{\mathbb{P}_{\alpha^*}}$ . By (4.3) and (4.2), there are  $A_{\alpha}$ ,  $\alpha \in \omega_2 \setminus \alpha^*$  such that

(4.5) for every  $B \in ([\omega]^{\aleph_0})^{V^{\mathbb{P}^\alpha}}$ ,  $A_\alpha$  is ad from  $B \Leftrightarrow B \in$  the ideal generated from  $\mathcal{F} \cup \{A_\beta : \beta \in \alpha \setminus \alpha^*\}$ .

Since  $([\omega]^{\aleph_0})^{V^{\mathbb{P}^{\omega_2}}} = \bigcup_{\alpha < \omega_2} ([\omega]^{\aleph_0})^{V^{\mathbb{P}^\alpha}}$ , it follows that  $\mathcal{F} \cup \{A_\alpha : \alpha \in \omega_2 \setminus \alpha^*\}$  is a mad family in  $V^{\mathbb{P}^{\omega_2}}$ . This shows that  $V^{\mathbb{P}^{\omega_2}} \models \mathfrak{a}^+(\aleph_1) \leq \aleph_2$ .

We also have  $V^{\mathbb{P}^{\omega_2}} \models \mathfrak{a} \geq \aleph_2$ : for any ad family  $\mathcal{G} \subseteq ([\omega]^{\aleph_0})^{V^{\mathbb{P}^{\omega_2}}}$  of cardinality  $\leq \aleph_1$ , there is some  $\alpha^* < \omega_2$  such that  $\mathcal{G} \in V^{\mathbb{P}^{\alpha^*}}$ . But  $\mathbb{Q}_{\alpha^*}$  adds an infinite subset of  $\omega$  almost disjoint to every element of  $\mathcal{G}$ . Hence  $\mathcal{G}$  is not mad.  $\dashv$   $\square$

Clearly, the method of the proof of Theorem 9 cannot produce a model of  $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$ .

**Problem 3.** Is  $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$  consistent?

All infinite cardinals less than or equal to the continuum  $\mathfrak{c}$  can be represented as  $\mathfrak{a}^+(\mathcal{F})$  for some  $\mathcal{F}$ .

**Theorem 10.** *For any infinite  $\kappa \leq \mathfrak{c}$ , there is an ad family  $\mathcal{F} \subseteq [T]^{\aleph_0}$  of cardinality  $\mathfrak{c}$  such that  $\mathfrak{a}^+(\mathcal{F}) = \kappa$ .*

*Proof.* Let  $\mathcal{F}'$  be a mad family in  $\mathcal{A}_T$ . Then by Lemma 1, we have

$$(4.6) \quad \mathcal{F}'^\perp = \mathcal{B}_T.$$

Let  $X$  and  $X'$  be disjoint with  $\omega_2 = X \cup X'$ ,  $|X| = \mathfrak{c}$  and  $|X'| = \kappa$ . Let

$$\mathcal{F} = \mathcal{F}' \cup \{B(f) : f \in X\}.$$

Clearly  $\mathcal{F}$  is an ad family. By (4.6) we have  $\mathcal{F}^\perp \subseteq \mathcal{B}_T$ .

We claim  $\mathfrak{a}^+(\mathcal{F}) = \kappa$ : Since  $\mathcal{F} \cup \{B(f) : f \in X'\}$  is a mad family by Lemma 1, we have  $\mathfrak{a}^+(\mathcal{F}) \leq \kappa$ . Again by Lemma 1, if  $\mathcal{G} \subseteq \mathcal{F}^\perp$  is an ad family of cardinality  $< \kappa$ , then there is  $f \in X'$  such that  $B(f)$  is ad from every  $B \in \mathcal{G}$ . Thus  $\mathfrak{a}^+(\mathcal{F}) \geq \kappa$ .  $\square$

## 5. Destructibility of mad families

For a poset  $\mathbb{P}$ , a mad family  $\mathcal{F}$  in  $[T]^{\aleph_0}$  is said to be  $\mathbb{P}$ -*destructible* if

$$V^{\mathbb{P}} \models \mathcal{F} \text{ is not mad in } [T]^{\aleph_0}.$$

Otherwise it is  $\mathbb{P}$ -*indestructible*.

The results in Section 3 can be also formulated in terms of destructibility of mad families.

**Theorem 11.** (1) (CH) *There is an ad family  $\mathcal{F} \subseteq \mathcal{A}_T$  which cannot be extended to a  $\mathcal{C}_\omega$ -indestructible mad family in any generic extension of the ground model of the form  $V^{\mathcal{C}_\kappa}$ .*

(2) *Let  $W = V^{\mathcal{C}_{\omega_1}}$ . Then, in  $W$ , there is an ad family  $\mathcal{F} \subseteq \mathcal{N}\mathcal{D}_T$  of cardinality  $\aleph_1$  such that, in any generic extension of  $W$  by a c.c.c. poset  $\mathbb{P}$  with  $\mathbb{P} \in V$ ,  $\mathcal{F}$  cannot be extended to a  $\mathcal{C}_\omega$ -indestructible mad family.*

(3) *Let  $W = V^{\mathcal{C}_{\omega_1}}$ . Then, in  $W$ , there is an ad family  $\mathcal{F} \subseteq \mathcal{N}_T$  of cardinality  $\aleph_1$  such that, in any generic extension of  $W$  by a c.c.c. poset  $\mathbb{P}$  with  $\mathbb{P} \in V$ ,  $\mathcal{F}$  cannot be extended to a  $\mathcal{R}_\omega$ -indestructible mad family.*

*Proof.* (1): The family  $\mathcal{F}$  as in Theorem 4 will do. Since we have  $\mathcal{F}' \subseteq \mathcal{N}\mathcal{D}_T$  for any mad  $\mathcal{F}'$  extending  $\mathcal{F}$  in  $V^{\mathcal{C}_\kappa}$ , a further Cohen real over  $V^{\mathcal{C}_\kappa}$  introduces a branch almost avoiding all elements of  $\mathcal{F}'$ . Thus  $\mathcal{F}'$  is no longer mad in  $V^{\mathcal{C}_\kappa * \mathcal{C}_\omega}$ .

(2): By Theorem 5 and by an argument similar to the proof of (1).

(3): In  $W$ , let  $\mathcal{F}$  be as in the proof of Theorem 6. Then any mad  $\mathcal{F}' \supseteq \mathcal{F}$  on  $T$  in any  $W^\mathbb{P}$  for  $\mathbb{P}$  as above is included in  $\mathcal{N}_T$  by  $\mathcal{O}_T \subseteq \mathcal{N}_T$ . Hence, in  $W^{\mathbb{P} * \mathcal{R}_\omega}$ , the random real  $f$  over  $W^\mathbb{P}$  introduces the branch  $B(f)$  almost avoiding all elements of  $\mathcal{F}'$ . Thus  $\mathcal{F}'$  is no longer mad in  $W^{\mathbb{P} * \mathcal{R}_\omega}$ .  $\square$

## 6. $\kappa$ -almost decided and $\lambda$ -minimal mad families

In this final section we collect several other constructions of mad families with some additional properties.

Given an ad family  $\mathcal{F}$  on  $T$  let  $\mathcal{I}(\mathcal{F})$  be the ideal on  $T$  generated by  $\mathcal{F} \cup [T]^{<\omega}$ , i.e. for  $S \subseteq T$  we have  $S \in \mathcal{I}(\mathcal{F})$  if  $S \subseteq^* \cup \mathcal{F}'$  for some finite subfamily  $\mathcal{F}'$  of  $\mathcal{F}$ .

Let  $\mathcal{F}$  be mad family on  $T$  and  $\mathcal{B} \subseteq \mathcal{F}$ . Clearly  $\mathcal{B}^\perp \supseteq \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$ . We say that  $\mathcal{B}$  *almost decides*  $\mathcal{F}$  if  $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$ . A mad family  $\mathcal{F}$  is said to be  $\kappa$ -almost decided if every  $\mathcal{B} \in [\mathcal{F}]^\kappa$  almost decides  $\mathcal{F}$ .

**Theorem 12.** *Assume that MA( $\sigma$ -centered) holds. Then there is a  $\mathfrak{c}$ -almost decided mad family  $\mathcal{F}$  on  $T$ .*

*Proof.* Let  $\langle B_\beta : \beta < \mathfrak{c} \rangle$  be an enumeration of  $[T]^{\aleph_0}$ . We define  $A_\alpha$ ,  $\alpha < \mathfrak{c}$  inductively such that

$$(6.1) \quad \{A_n : n \in \omega\} \text{ is a partition of } T \text{ into infinite subsets;}$$

For all  $\alpha \in \mathfrak{c} \setminus \omega$

$$(6.2) \quad A_\alpha \text{ is ad from } A_\beta \text{ for all } \beta < \alpha;$$

$$(6.3) \quad \text{For } \beta < \alpha, \text{ if } B_\beta \notin \mathcal{I}(\{A_\delta : \delta < \alpha\}) \text{ then } |A_\alpha \cap B_\beta| = \aleph_0;$$

*Claim.* The construction of  $A_\alpha$ ,  $\alpha < \mathfrak{c}$  as above is possible.

$\vdash$  Suppose that  $\alpha \in \mathfrak{c} \setminus \omega$  and  $A_\beta$ ,  $\beta < \alpha$  have been constructed according to (6.1), (6.2) and (6.3). Let

$$S_\alpha = \{\beta < \alpha : B_\beta \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})\}.$$

Let  $\mathbb{P}_\alpha = \{\langle \varphi, s \rangle : \varphi \in \text{Fn}(T, 2), s \in [\alpha]^{<\aleph_0}\}$  be the poset with the ordering defined by

$$\begin{aligned} \langle \varphi', s' \rangle \leq_{\mathbb{P}_\alpha} \langle \varphi, s \rangle &\Leftrightarrow \\ \varphi &\subseteq \varphi', s \subseteq s' \text{ and} \\ \forall t \in \text{dom}(\varphi') \setminus \text{dom}(\varphi) &(\varphi'(t) = 1 \rightarrow t \notin A_\delta \text{ for all } \delta \in s) \end{aligned}$$

for  $\langle \varphi, s \rangle, \langle \varphi', s' \rangle \in \mathbb{P}_\alpha$ .

$\mathbb{P}_\alpha$  is  $\sigma$ -centered since  $\langle \varphi, s \rangle, \langle \varphi', s' \rangle \in \mathbb{P}_\alpha$  are compatible if  $\varphi = \varphi'$ .

For  $\beta < \alpha$ , let

$$C_\beta = \{\langle \varphi, s \rangle \in \mathbb{P}_\alpha : \beta \in s\}$$

and, for  $\beta \in S_\alpha$  and  $n \in \omega$ , let

$$D_{\beta,n} = \{\langle \varphi, s \rangle \in \mathbb{P}_\alpha : \exists t \in \text{dom}(\varphi) (\ell(t) \geq n \wedge \varphi(t) = 1 \wedge t \in B_\beta)\}.$$

It is easy to see that  $C_\beta$ ,  $\beta < \alpha$  and  $D_{\beta,n}$ ,  $\beta \in S_\alpha$ ,  $n \in \omega$  are dense in  $\mathbb{P}_\alpha$ . Let

$$\mathcal{D} = \{C_\beta : \beta < \alpha\} \cup \{D_{\beta,n} : \beta \in S_\alpha, n \in \omega\}.$$

Since  $|\mathcal{D}| < \mathfrak{c}$ , we can apply MA( $\sigma$ -centered) to obtain a  $(\mathcal{D}, \mathbb{P}_\alpha)$ -generic filter  $G$ . Let

$$A_\alpha = \{t \in T : \varphi(t) = 1 \text{ for some } \langle \varphi, s \rangle \in G\}.$$

Then this  $A_\alpha$  is as desired.  $\dashv$

Let  $\mathcal{F} = \{A_\alpha : \alpha < \mathfrak{c}\}$ .  $\mathcal{F}$  is infinite by (6.2) and mad by (6.3).

We show that  $\mathcal{F}$  is  $\mathfrak{c}$ -almost decided. First, note that we have  $\mathfrak{a} = \mathfrak{c}$  by the assumptions of the theorem. By (6.3), we have:

$$(6.4) \quad \text{For any } B \in [T]^{\aleph_0}, \text{ if } B \notin \mathcal{I}(\{A_\alpha : \alpha < \mathfrak{c}\}) \text{ then} \\ |\{\alpha < \mathfrak{c} : |A_\alpha \cap B| < \aleph_0\}| < \mathfrak{c}.$$

Suppose that  $\mathcal{H} \in [\mathcal{F}]^{\mathfrak{c}}$  and  $B \in \mathcal{H}^\perp$ . Then  $|\{\alpha < \mathfrak{c} : |A_\alpha \cap B| < \aleph_0\}| = \mathfrak{c}$  and so  $B \in \mathcal{I}(\mathcal{F})$  by (6.4). Thus there is a finite  $\mathcal{F}' \subset \mathcal{F}$  such that  $B \subset^* \cup \mathcal{F}'$  and  $F \cap B$  is infinite for each  $F \in \mathcal{F}'$ . But  $B \in \mathcal{H}^\perp$  so  $\mathcal{F}' \cap \mathcal{H} = \emptyset$ . Thus  $\mathcal{F}'$  witnesses that  $B \in \mathcal{I}(\mathcal{F} \setminus \mathcal{H})$  which was to be proved.  $\square$

For a mad family  $\mathcal{F}$  on  $T$ ,  $\mathcal{C} \subseteq \mathcal{F}$  is said to be *minimal in  $\mathcal{F}$*  if  $\mathfrak{a}^+(\mathcal{F} \setminus \mathcal{C}) = |\mathcal{C}|$ . A mad family  $\mathcal{F}$  is said to be  $\lambda$ -*minimal* if every  $\mathcal{C} \in [\mathcal{F}]^\lambda$  is minimal in  $\mathcal{F}$ .

**Lemma 5.** *Suppose that  $\mathcal{F}$  is a mad family on  $T$ .*

- (1) *If  $\mathcal{F}$  is  $|\mathcal{F}|$ -minimal then  $|\mathcal{F}| = \mathfrak{a}$ .*
- (2) *If  $\mathcal{B} \subseteq \mathcal{F}$  almost decides  $\mathcal{F}$  and  $\mathcal{F} \setminus \mathcal{B}$  is infinite then  $\mathcal{F} \setminus \mathcal{B}$  is minimal in  $\mathcal{F}$ .*
- (3) *If  $\mathcal{F}$  is  $\kappa$ -almost decided for  $\kappa = |\mathcal{F}|$  then  $\mathcal{F}$  is  $\lambda$ -minimal for all  $\omega \leq \lambda < \kappa$ .*
- (4) *If  $|\mathcal{F}| = \mathfrak{a}$  and  $\mathcal{F}$  is  $\mathfrak{a}$ -almost decided then  $\mathcal{F}$  is  $\mathfrak{a}$ -minimal.*

*Proof.* (1): If  $\mathcal{F}$  is  $|\mathcal{F}|$ -minimal then  $\mathcal{F}$  itself is minimal in  $\mathcal{F}$ . Thus  $\mathfrak{a} = \mathfrak{a}^+(\emptyset) = \mathfrak{a}^+(\mathcal{F} \setminus \mathcal{F}) = |\mathcal{F}|$ .

(2): First, note that, for any infinite ad  $\mathcal{F}$ , we have  $\mathfrak{a}(\mathcal{I}(\mathcal{F})) = |\mathcal{F}|$ .

Suppose that  $\mathcal{F}$  is a mad family on  $T$  and  $\mathcal{B} \subseteq \mathcal{F}$  almost decides  $\mathcal{F}$ , i.e.  $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$ . Hence

$$\mathfrak{a}^+(\mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{B})) = \mathfrak{a}^+(\mathcal{B}) = \mathfrak{a}(\mathcal{B}^\perp) = \mathfrak{a}(\mathcal{I}(\mathcal{F} \setminus \mathcal{B})) = |\mathcal{F} \setminus \mathcal{B}|.$$

(3): Suppose that  $\kappa = |\mathcal{F}|$  and  $\mathcal{F}$  is  $\kappa$ -almost decided. If  $\mathcal{C} \in [\mathcal{F}]^\lambda$  for some  $\omega \leq \lambda < \kappa$  then  $|\mathcal{F} \setminus \mathcal{C}| = \kappa$  and hence  $\mathcal{F} \setminus \mathcal{C}$  almost decides  $\mathcal{F}$ . By (2) it follows that  $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$  is minimal in  $\mathcal{F}$ .

(4): Suppose that  $|\mathcal{F}| = \mathfrak{a}$  and  $\mathcal{F}$  is  $\mathfrak{a}$ -almost decided. Suppose that  $\mathcal{C} \in [\mathcal{F}]^{\mathfrak{a}}$ . If  $|\mathcal{F} \setminus \mathcal{C}| < \mathfrak{a}$ , then clearly  $\mathfrak{a}^+(\mathcal{F} \setminus \mathcal{C}) = \mathfrak{a} = |\mathcal{C}|$ . Hence  $\mathcal{C}$  is minimal in  $\mathcal{F}$ . If  $|\mathcal{F} \setminus \mathcal{C}| = \mathfrak{a}$  then  $\mathcal{F} \setminus \mathcal{C}$  almost decides  $\mathcal{F}$ . Thus, by (2),  $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$  is again minimal in  $\mathcal{F}$ .  $\square$

**Corollary 2.** *Assume that MA( $\sigma$ -centered) holds. Then there is a mad family  $\mathcal{F}$  on  $T$  which is  $\lambda$ -minimal for all  $\omega \leq \lambda \leq \mathfrak{c}$ .*

*Proof.* By Theorem 12 and Lemma 5, (3), (4).  $\square$

Theorem 12 can be further improved to the following theorem:

**Theorem 13.** *Assume that MA( $\sigma$ -centered) holds. Let  $\kappa = \mathfrak{c}$ . Then there is a  $\mathcal{C}_\omega$ -indestructible mad family  $\mathcal{F}$  (of size  $\kappa$ ) such that*

$$(6.5) \quad V^{\mathcal{C}_\omega} \models \mathcal{F} \text{ is } \kappa\text{-almost decided on } T.$$

*Proof.* Let  $\langle (t_\beta, B_\beta) : \beta < \kappa \rangle$  be an enumeration of

$$T \times \{B : B \text{ is a nice } \mathcal{C}_\omega\text{-name of an element of } [T]^{\aleph_0} \text{ in } V^{\mathcal{C}_\omega}\}.$$

Let  $A_\alpha$ ,  $\alpha < \kappa$  be then defined inductively just as in the proof of Theorem 12 with

$$(6.3)' \quad \text{For } \beta < \alpha, \text{ if } t \Vdash_{\mathcal{C}_\omega} \text{“} B_\alpha \notin \mathcal{I}(\{A_\delta : \delta < \alpha\}) \text{” then } t \Vdash_{\mathcal{C}_\omega} \text{“} |A_\alpha \cap B_\beta| = \aleph_0 \text{”}$$

in place of (6.3).  $\square$

**Corollary 3.** *For any cardinal  $\kappa \geq \mathfrak{c}$  in the ground model  $V$  there is a cardinal preserving generic extension  $W$  of  $V$  such that, in  $W$ ,  $\kappa < \mathfrak{c}$  and there is a  $\kappa$ -almost decided mad family  $\mathcal{F}$  of size  $\kappa$  (furthermore  $\mathcal{F}$  is  $\lambda$ -minimal for all  $\omega \leq \lambda \leq \kappa$ ).*

*Proof.* First extend  $V$  to a model  $V'$  of  $\kappa = \mathfrak{c}$  and MA( $\sigma$ -centered). In  $V'$ , let  $\mathcal{F}$  be as in Theorem 13. Then  $\mathcal{F}$  is as desired in  $V^{\mathcal{C}_\mu}$  for any  $\mu > \kappa$ . The claim in the parentheses follows from Lemma 5, (3) and (6.3)'.  $\square$

## References

1. T. Bartoszyński and H. Judah, *Set Theory: on the structure of the real line*, A K Peters, (1995).
2. A. Blass, Combinatorial cardinal characteristics of the continuum, in: M. Foreman and A. Kanamori (eds.), *Handbook of Set Theory*, Springer London, (2010), Vol. 1, 395–490.
3. J. Brendle, Around splitting and reaping, *Commentat. Math. Univ. Carol.* **39** (1998), 269–279.

4. J. Brendle, Mob families and mad families, *Arch. Math. Logic* **37** (1998), 183–197.
5. S. Fuchino, S. Geschke and L. Soukup, Almost disjoint families on large underlying sets, *RIMS Kokyuroku*, No.1530 (2007), 5-16.
6. M. Hrušák, MAD families and the rationals, *Commentat. Math. Univ. Carol.* **42,2** (2001), 343–350.
7. T. Jech, *Set theory, The Third Millennium Edition*, Springer-Verlag (2002).
8. K. Kunen, *Set Theory*, North-Holland (1980).
9. T. Leathrum, A special class of almost disjoint families, *J. Symb. Log.* **60** (1995), 879–891.
10. J. Steprāns, Combinatorial consequences of adding Cohen reals, in: H. Judah, ed., *Set theory of the reals*, *Isr. Math. Conf. Proc.* **6**, (1993), 583–617.