## 6 Erdős-角谷 and Erdős-Komjáth Theorems

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**Proposition 29** (somewhere in [Erdős-Hajnal-Máté-Rado book]) Let $\kappa$ be an infinite cardinal, $\lambda \geq \kappa^{++}$ a regular cardinal and $\mu \geq \kappa^{+}$ a cardinal. For any $f : \lambda \times \mu \to \kappa$, there are $\eta < \xi < \mu$ and a stationary $K \subseteq \lambda$ such that $f(\alpha,\eta) = f(\beta,\xi)$ for all $\alpha$, $\beta \in K$. partition-th

**Proof** Suppose that $f : \lambda \times \mu \to \kappa$. By replacing $f$ with $f\restriction \lambda \times \kappa^{+}$, we may assume that $\mu = \kappa^{+}$. For each $\alpha \in \lambda$, let $n_\alpha \in \kappa$ be such that

(∗) $\{\eta < \kappa^{+} : f(\alpha,\eta) = n_\alpha\}$ is unbounded in $\mu = \kappa^{+}$.

By Fodor's lemma, there is a stationary $K_1 \subseteq \lambda$ and $n^* < \kappa$ such that $n_\alpha = n^*$ for all $\alpha \in K_1$.

For each $\alpha \in K_1$, let $\eta_\alpha < \kappa^{+}$ be such that $f(\alpha,\eta_\alpha) = n^*$. By Fodor's lemma, there is a stationary $K_2 \subseteq K_1$ and $\eta < \kappa^{+}$ such that $\eta_\alpha = \eta$ for all $\alpha \in K_2$.

Now by (∗), we can find $\xi_\alpha \in \kappa^{+} \setminus \eta{+}1$ for each $\alpha \in K_2$ such that $f(\alpha,\xi_\alpha) = n^*$. Again by Fodor's lemma, there is a stationary $K \subseteq K_2$ and $\xi < \kappa^{+}$ such that $\xi_\alpha = \xi$ for all $\alpha \in K$. By the definition of $\xi_\alpha$'s and $K$, we have $\eta < \xi < \kappa^{+}$ and $f(\alpha,\eta) = f(\beta,\xi) = n^*$ for all $\alpha$, $\beta \in K$. Hence these $\eta$, $\xi$, $K$ are as desired. ❑

**Theorem 30** (Erdős and 角谷) For a field $F$ of cardinality $\leq \kappa$ and a vector space $V$ over $F$ of cardinality $\geq \kappa^{++}$, there is no partition of $V$ into $\kappa$ independent subsets. Erdos-Kakutani

**Proof** Otherwise, there would be a function $h : V \to \kappa$ such that $h^{-1}{}''\{n\}$ is an independent subset of $V$ for all $n < \kappa$. Let $V = V' \oplus V''$ where $|\,V'\,| = \kappa^{++}$ and $|\,V''\,| = \kappa^{+}$. Let $f : V' \times V'' \to \kappa$ be defined by $f(a,b) = h(a+b)$.

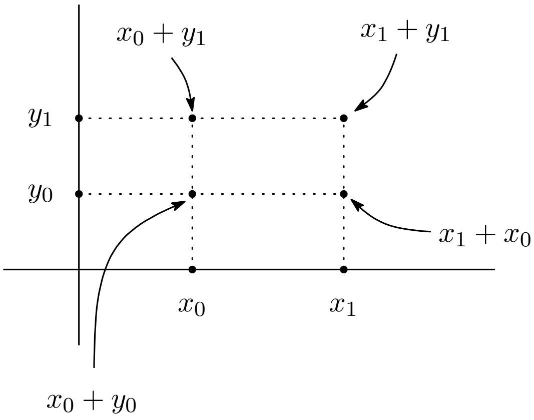
By Proposition 29, there are $a$, $b \in V'$, $a \neq b$ and $c$, $d \in V''$, $c \neq d$ such that $f(a,c) = f(b,d) = f(a,d) = f(b,c) = n$ for some $n < \kappa$. Then $a+c$, $b+d$, $a+d$ and $b+c$ are all in $h^{-1}{}''\{n\}$. But

$$(a+c)+(b+d)-(a+d)-(b+c) = 0.$$

This is a contradiction to the assumption that $h^{-1}{}''\{n\}$ is independent. ❑

**Theorem 31** (Erdős and Komjáth) Suppose that $|\,\mathbb{R}\,| \geq \kappa^{++}$. Then, for any partition $X_\alpha$, $\alpha < \kappa$ of $\mathbb{R}^2$, there is $\alpha < \kappa$ such that $X_\alpha$ contains four verteces of a rectangle. Erdos-Komjath

**Proof** Let $X_\alpha$, $\alpha < \kappa$ be an arbitrary partition of $\mathbb{R}^2$. Let $X = \{(x,0) : x \in \mathbb{R}\}$ and $Y = \{(0,y) : y \in \mathbb{R}\}$. By assumption, $|\,X\,| = |\,Y\,| \geq \kappa^{++}$. Let $f : X \times Y \to \kappa$ be such that $\vec{x}+\vec{y} \in X_\alpha$ for $\alpha = f(\vec{x},\vec{y})$, for all $\vec{x} \in X$ and $\vec{y} \in Y$. Then by Proposition 29, there are $\vec{x}_0, \vec{x}_1 \in X \setminus Y$, $\vec{x}_0 \neq \vec{x}_1$, $\vec{y}_0, \vec{y}_1 \in Y$, $\vec{y}_0 \neq \vec{y}_1$ and $\alpha \in \omega$ such that $f(\vec{x}_i,\vec{y}_j) = \alpha$ for all $i$, $j \in 2$. This means $\{\vec{x}_i + \vec{y}_j : i, j \in 2\} \subseteq X_\alpha$. But by the choice of $X$ and $Y$, $\vec{x}_i + \vec{y}_j$, $i$, $j \in 2$ are vertices of a rectangle.




❑

**Proposition 32** For any field $F$ of cardinality $\leq \kappa$ and a vector space $V$ over $F$ of cardinality $\leq \kappa^+$, there is a partition of $V \setminus \{0_V\}$ into $\kappa$ independent sets. indep-part

**Proof** Suppose that $F$ is a field of cardinality $\leq \kappa$ and $V$ a vector space over $F$ of cardinality $\leq \kappa^+$ The assertion of the proposition clearly holds if $|V| \leq \kappa$ since then we can take each $X_\xi$ to be a singleton. Thus we may assume that $\dim V = \kappa^+$. Fix a basis $\{x_\alpha : \alpha < \kappa^+\}$ of the linear space $V$. Let

$$V_\beta = [\{x_\alpha : \alpha < \beta\}]_V$$

for $\beta < \kappa^+$ where $[X]_V$ for $X \subseteq V$ denotes the subspace of $V$ spanned by $X$. Let $\{x_{\beta,\xi} : \xi < \kappa\}$ be an enumeration of $S_\beta = V_{\beta+1} \setminus V_\beta$.

For $\xi < \kappa$, let

$$X_\xi = \{x_{\beta,\xi} : \beta < \kappa^+\}$$

Then clearly we have $V \setminus \{0_V\} = \bigcup_{\xi<\kappa} X_\xi$. We show that $X_\xi$ is independent for all $\xi < \kappa$. For $a \in V \setminus \{0_V\}$, let $o(a)$ be the ordinal $\beta < \kappa^+$ such that $a \in S_\beta$. Suppose that $\xi < \kappa$ and $\sum_{k<n} q_k z_k = 0$ for some $n \in \omega \setminus 1$, $q_k \in K \setminus \{0\}$ and $z_k \in X_\xi$ for $k < n$. By definition of $X_\xi$ we may assume that $o(z_0) > o(z_1), \ldots, o(z_{n-1})$. But then

$$z_0 = -\frac{1}{q_0}\left(\sum_{k\in n\setminus\{0\}} q_k z_k\right)$$

while

$$o(z_0) > \max\{o(z_1), \ldots, o(z_{n-1})\} \geq o\left(-\frac{1}{q_0}\left(\sum_{k\in n\setminus\{0\}} q_k z_k\right)\right).$$

This is a contradiction. ❑ (Proposition 32)

**Theorem 33** The following are equivalent: char-HC

(0) CH;

(1) $\mathbb{R} \setminus \{0\}$ can be partitioned into countably may subsets $X_k$, $k \in \omega$ such that each $X_k$ is independent over $\mathbb{Q}$;

(2) (Erdős and Kakutani) For any countable subfield $Q$ of $\mathbb{R}$, $\mathbb{R}\setminus\{0\}$ can be partitioned into countably may subsets $X_k$, $k \in \omega$ such that each $X_k$ is independent over $Q$;

(3) (Erdős and Komjáth) $\mathbb{R}^2$ can be partitioned into countably many subsets $X_k$, $k \in \omega$ such that none of $X_k$'s contains all vertices of a rectangle.

**Proof** Note that $\mathbb{R}$ and $\mathbb{R}^2$ can be seen as vector spaces over a subfield $Q \leq \mathbb{R}$. Hence "(0) $\Rightarrow$ (2)" and "(0) $\Rightarrow$ (3)" follow form Proposition 32. "(2) $\Rightarrow$ (1)" is clear. "(1) $\Rightarrow$ (0)" follows from Theorem 30 and "(3) $\Rightarrow$ (0)" from Theorem 31. ❑ (Theorem 33)

$2^{\aleph_0} = \aleph_2$ can be also characterized similarly.

**Theorem 34** The following are equivalent: char-continuum=aleph-2

(0) $2^{\aleph_0} = \aleph_2$

(1) There is no partition $X_k$, $k \in \omega$ of $\mathbb{R} \setminus \{0\}$, such that every $X_k$, $k \in \omega$ is linearly independent over $\mathbb{Q}$ but there is a partition $X_\alpha$, $\alpha < \omega_1$ of $\mathbb{R} \setminus \{0\}$ such that every $X_\alpha$, $\alpha < \omega_1$ is linearly independent over $\mathbb{Q}$;

(2) For any countable subfield $Q$ of $\mathbb{R}$, there is no partition $X_k$, $k \in \omega$ of $\mathbb{R} \setminus \{0\}$, such that every $X_k$, $k \in \omega$ is linearly independent over $Q$ but there is a partition $X_\alpha$, $\alpha < \omega_1$ of $\mathbb{R} \setminus \{0\}$ such that every $X_\alpha$, $\alpha < \omega_1$ is linearly independent over $Q$;

(3) $\mathbb{R}^2$ can not be partitioned into countably many subsets $X_k$, $k \in \omega$ such that none of $X_k$'s contains all vertices of a rectangle but ther is a partition of $\mathbb{R}^2$ int $\omega_1$ sets such that none of these sets contains all vertices of any recutangle.