Exotic components in linear slices of quasi-Fuchsian punctured torus groups

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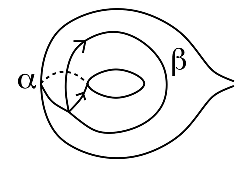
Basic definitions

We assume that S is a once-punctured torus.(We will generalize results in the last.)

- $\alpha, \beta \in \pi_1(S)$: generators s.t. the commutator $[\alpha, \beta]$ to be peripheral.



We often denote the simple closed curve freely homotopic to α (resp. β) by the same symbol.

- $X(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \mathrm{tr}(\rho([\alpha, \beta])) = -2\}/\sim$ where $\rho \sim \rho'$ if they are conjugate. ($\mathrm{PSL}_2\mathbb{C}$-character variety)
- $AH(S) = \{[\rho] \in X(S) \mid \text{discrete, faithful}\}$
- $QF(S) = \{[\rho] \in AH(S) \mid \rho(\pi_1(S)) \text{ is quasi-Fuchsian}\}$
- The *complex length* $\lambda_\gamma(\rho)$, for $\gamma \in \pi_1(S)$ and $[\rho] \in X(S)$, is a complex number characterized by $\mathrm{tr}(\rho(\gamma)) = 2\cosh(\lambda_\gamma(\rho)/2)$, $\mathrm{Re}(\lambda_\gamma(\rho)) > 0$ and $-\pi < \mathrm{Im}(\lambda_\gamma(\rho)) \le \pi$. The real (resp. imaginary) part of $\lambda_\gamma(\rho)$ is the translation length (resp. rotation angle) of $\rho(\gamma)$.
- For a real number $l > 0$, we define the *linear slice* by
$$X(l) = \{[\rho] \mid \lambda_\alpha(\rho) \equiv l\}.$$
- $QF(l) = X(l) \cap QF(S)$.

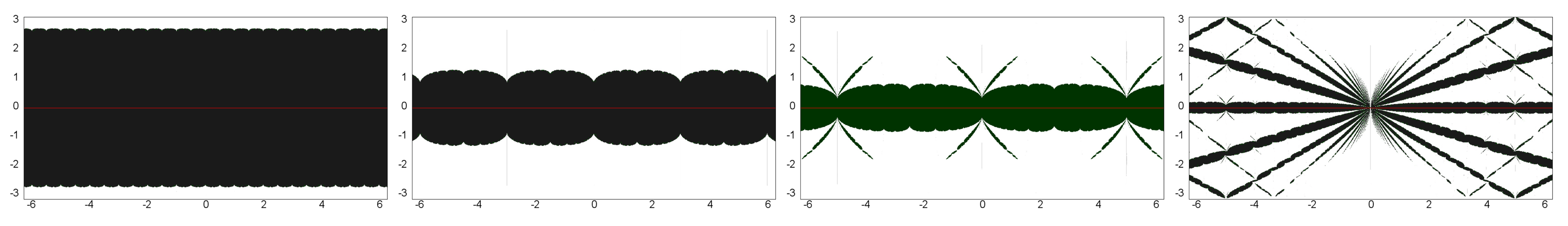
For a summary, see **Diagram 1**.

Known facts and experimental pictures

- $X(S)$ is complex 2-dimensional, $X(l)$ is complex 1-dimensional.
- $AH(S)$ is closed, and $QF(S)$ is open in $X(S)$.
- $\overline{QF(S)} = AH(S)$ (Minsky [Mi], density theorem for once punctured torus.)
- $QF(l)$ is a union of (open) disks (McMullen [Mc], disk convexity of QF)
- $QF(l)$ contains at least one component containing all Fuchsian representation ρ satisfying $\lambda_\alpha(\rho) = l$. We call this component the *standard component*. (This component is called BM-slice and extensively studied in [KS].)
- If $l > 0$ is sufficiently small, $QF(l)$ has only one component (Otal).

We will see that $X(l)$ can be identified with $\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \le \pi\}$. In the following pictures, $QF(l)$ is indicated by shaded regions, and the red lines correspond to Fuchsian representations.

Fig. 1. Pictures of $QF(l)$



$l = 0.5$ $l = 3.0$ $l = 5.0$ $l = 20.0$

These pictures suggest that $QF(l)$ has more components as l getting longer. In fact:

Theorem A (Komori-Yamashita)

If l is sufficiently large, $QF(l)$ has more than one component.

We give another proof of this theorem by using *complex projective structures*, with a few new observations and a possible generalization.

Complex Fenchel-Nielsen coordinates

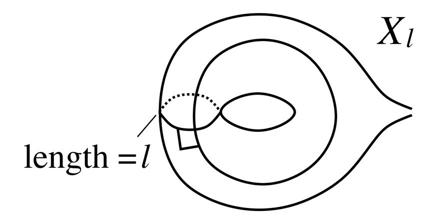
We give an explicit description of $X(l)$ as a subset of $\mathbb{C}$.

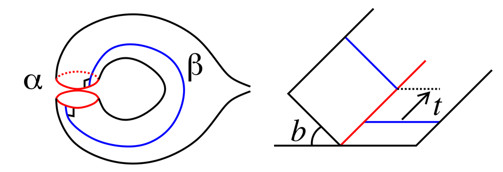
- $X_{SL}(S)$: the $\mathrm{SL}_2\mathbb{C}$-character variety is defined similarly as $\mathrm{PSL}_2\mathbb{C}$ case.
- We have the following isomorphism:
$$\begin{array}{ccc} X_{SL}(S) & \xrightarrow{\cong} & \{(x, y, z) \mid x^2 + y^2 + z^2 - xyz = 0\} \\ \cup\!\!\!\!\!\! & & \cup\!\!\!\!\!\! \\ [\rho] & \longmapsto & (\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta)), \mathrm{tr}(\rho(\alpha\beta))) \end{array}$$

$X(S)$ is the quotient of $X_{SL}(S)$ by the action of $(\mathbb{Z}/2\mathbb{Z})^2$. (Explicitly, it is generated by $(x, y, z) \mapsto (-x, y, -z)$ and $(x, y, z) \mapsto (x, -y, -z)$.)

- Fix $l > 0$. There exists a bijection $\{\tau \mid -\pi < \mathrm{Im}(\tau) \le \pi\} \to X(l)$ defined by
$$\tau \mapsto \left(2\cosh\left(\frac{l}{2}\right), \frac{2\cosh(\tau/2)}{\tanh(l/2)}, \frac{2\cosh((\tau + l)/2)}{\tanh(l/2)}\right).$$
- The Dehn twist along α acts on $X(l)$ as $(l, \tau) \mapsto (l, \tau + l)$.

Let $\tau = t + b\sqrt{-1}$. Let X_l be the marked hyperbolic surface such that $l_\alpha(X_l) = l$ and the geodesics homotopic to α and β are orthogonal. Let $\mathrm{tw}_{t\cdot\alpha}(X_l)$ be the hyperbolic surface obtained from X_l by twisting distance t along α. The representation corresponding to (l, τ) coincides with the holonomy of the *pleated surface* obtained from $\mathrm{tw}_{t\cdot\alpha}(X_l)$ by bending along α with angle b.

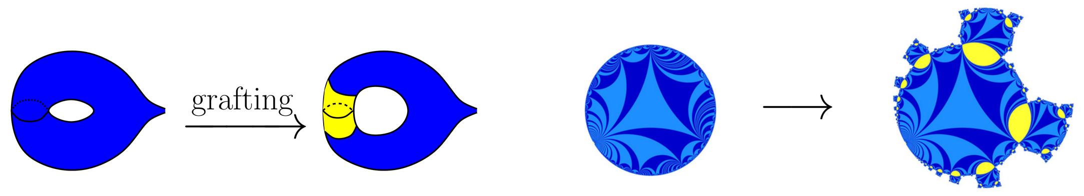




$\mathbb{C}P^1$-structures and grafting

A *complex projective structure* (or *$\mathbb{C}P^1$-structure*) is a geometric structure locally modelled on $\mathbb{C}P^1$ whose transition functions are in $\mathrm{PSL}_2\mathbb{C}$. By analytic continuation of the local structure, we obtain a *developing map* $\mathrm{dev} : \widetilde{S} \to \mathbb{C}P^1$ and a *holonomy representation* $\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ so that dev is ρ-equivariant. Conversely such pair (dev, ρ) determines a $\mathbb{C}P^1$-structure.

A hyperbolic structure on S gives a $\mathbb{C}P^1$-structure given by the uniformization $\widetilde{S} \cong \mathbb{H}^2 \subset \mathbb{C}P^1$. We can construct a $\mathbb{C}P^1$-structure from a (marked) hyperbolic structure X by inserting annulus of height b along a simple closed geodesic γ, denote it by $\mathrm{Gr}_{b\cdot\gamma}(X)$. This construction is called a *grafting*.



As in the definition of Teichmüller space, we let $P(S)$ be the set of marked $\mathbb{C}P^1$-structures. The grafting operation is generalized for measured laminations. We denote the Teichmüller space of S by $\mathcal{T}(S)$.

Theorem (Thurston, Kamishima-Tan)

The grafting map
$$\begin{array}{rcl} \mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) & \to & P(S) \\ \cup\!\!\!\!\!\! & & \cup\!\!\!\!\!\! \\ (\mu, X) & \mapsto & \mathrm{Gr}_\mu(X) \end{array}$$
is a homeomorphism. We call this *Thurston coordinates*.

Let $\overline{\mathbb{H}} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) \ge 0\}$. The *complex earthquake* $\overline{\mathbb{H}} \to P(S)$ for the simple closed curve α is defined by
$$\mathrm{Eq}_{(t+b\sqrt{-1})\cdot\alpha}(X_l) = \mathrm{Gr}_{b\cdot\alpha}(\mathrm{tw}_{t\cdot\alpha}(X_l)).$$
(Instead of α, complex earthquake is defined for any non-zero measured lamination. The domain $\overline{\mathbb{H}}$ can be extended to include some domain in the lower half plane [Mc].) The holonomy gives a map $\mathrm{hol} : P(S) \to X(S)$, which is known to be a local homeomorphism. By the definition of the grafting, we have
$$\mathrm{hol}(\mathrm{Eq}_{(t+b\sqrt{-1})\cdot\alpha}(X_l)) = t + b\sqrt{-1} \in X(l) \mod 2\pi\sqrt{-1}\mathbb{Z}.$$
In summary:

Diagram 1

$$\begin{array}{ccccccl} QF(S) \subset AH(S) & \subset & X(S) & \xleftarrow{\mathrm{hol}} & P(S) & & : \text{complex 2-dim} \\ \cup & & \cup & & \cup & & \\ QF(l) & \subset & X(l) & \leftarrow & \mathrm{Eq}_{\overline{\mathbb{H}}\cdot\alpha}(X_l) & \leftarrow \overline{\mathbb{H}} & : \text{complex 1-dim} \end{array}$$

Consider the set of $\mathbb{C}P^1$-structures with quasi-Fuchsian holonomy, i.e. $\mathrm{hol}^{-1}(QF(S))$. There exists a *standard component* Q_0 consisting of $\mathbb{C}P^1$-structures with injective developing maps. By Goldman's results, any other component is obtained from Q_0 by *2π-grafting* along a multicurve μ on S. We denote the corresponding component by Q_μ. Let $\mathcal{ML}_{\mathbb{Z}}(S) (\subset \mathcal{ML}(S))$ be the set of multicurves on S.

Theorem (Goldman)
$$\mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} Q_\mu$$

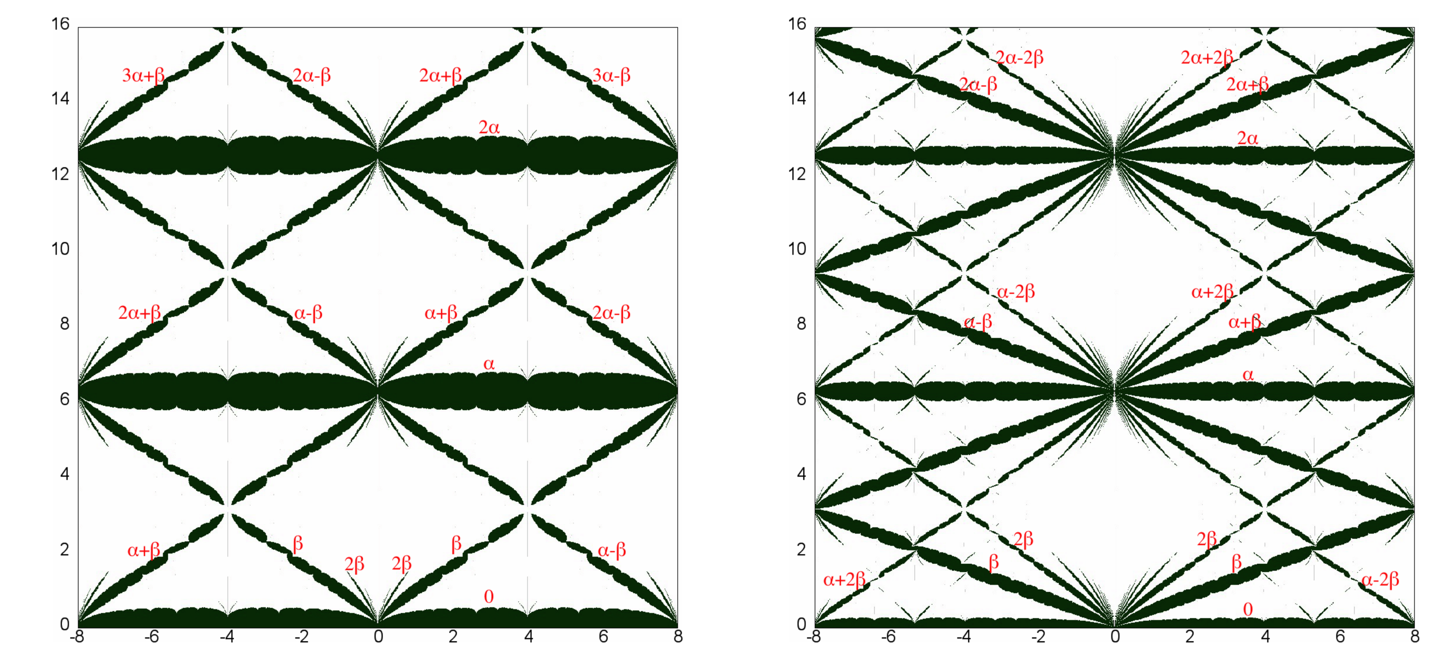
Our proof of Theorem A

Let D_γ be the Dehn twist along a simple closed geodesic γ. Fix $X \in \mathcal{T}(S)$. Consider the sequence
$$\left\{\left(\frac{2\pi}{n} D_\beta^n(\alpha), X\right)\right\}_{n=1,2,\cdots} \subset \mathcal{ML}(S) \times \mathcal{T}(S)$$
in the Thurston coordinates. This converges to $(2\pi \cdot \beta, X)$, which is in Q_β. Thus there exists N such that $(\frac{2\pi}{n} D_\beta^n(\alpha), X) \in Q_\beta$ for any $n \ge N$. Applying D_β^{-n}, we conclude that
$$\left(\frac{2\pi}{n}\alpha, D_\beta^{-n}(X)\right) \in Q_{D_\beta^n(\beta)} = Q_\beta \quad (\forall n \ge N).$$
Now we have $\mathrm{hol}(\frac{2\pi}{n}\alpha, D_\beta^{-n}(X)) \in X(l_\alpha(D_\beta^{-n}(X)))$ for any n. If $\mathrm{hol}(\frac{2\pi}{n}\alpha, D_\beta^{-n}(X))$ is in the standard component (i.e. the component containing Fuchsian representations), it must be in $Q_{m\cdot\alpha}$ for some $m \in \mathbb{Z}_{\ge 0}$. Since $\beta \ne m\alpha$, $\mathrm{hol}(\frac{2\pi}{n}\alpha, D_\beta^{-n}(X))$ is not in the standard component for $n \ge N$.

Further observations

- $l_\alpha(D_\beta^{-n}(X))$ is getting longer as $n \to \infty$, but $l_\beta(D_\beta^{-n}(X))$ is constant. As a consequence, the Fenchel-Nielsen twist of $D_\beta^{-n}(X)$ with respect to α is relatively small.
- The argument above does work if we replace 2π with $2k\pi$ ($k \in \mathbb{N}$), of course we need larger l. The wrapping of α of the representation is k. This should be compared with the result of Evans-Holt [EH]
- Since $QF(l)$ is invariant under translation $\tau \mapsto \tau + l$, if there is a non-standard component, there are infinitely many. Even after taking the quotient by this translation, $QF(l)$ may have arbitrary many components from the above observation. Moreover, by using Bromberg's model near the Maskit slice [B], there might be infinitely many components even after taking the quotient.

From the above observations with Ito's results [I], the Goldman's classification of $\mathrm{hol}^{-1}(QF(S))$ looks like this: (we abbreviate $Q_{\alpha+\beta}$ as $\alpha + \beta$.)



$l = 8.0$ $l = 16.0$

Theorem A can be generalized as follows.

Generalization

Let X be a hyperbolic surface and α a simple closed geodesic on X. If $l_\alpha(X)$ is sufficiently large, the complex earthquake $\mathrm{Eq}_{\overline{\mathbb{H}}\cdot\alpha}(X)$ has a non-empty intersection with Q_μ for some $\mu \notin 2\pi\mathbb{Z}_{\ge 0} \cdot \alpha$.

Actually, we can take a simple closed geodesic β intersecting α, and a similar argument does work.

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