A method for finding ideal points from an ideal triangulation and its application

Yuichi Kabaya

(Tokyo Institute of Technology)

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0. Background

N : a compact orientable 3-manifold whose boundary is a torus
(e.g. knot complement)

$X(N) = \mathrm{Hom}(\pi_1(N), \mathrm{PSL}(2,\mathbb{C}))/\sim$: character variety

A point at infinity of $X(N)$ is called *ideal point*.

Why should we study ideal points?

Ideal points of the character variety of a 3-manifold

$\Longrightarrow$

- Incompressible surfaces, their boundary slopes
- The Culler-Shalen norm
  - Information about cyclic (finite) surgeries.

Ideal points of the character variety of a 3-manifold have much information about the manifold.

Using a presentation of the fundamental group, we can obtain defining equations of $X(N)$. But in general it is difficult to analyze $X(N)$ and find ideal points.

Motivation

We want to give a method for finding ideal points from combinatorial data of the manifold.

1. The character variety of a 3-manifold

N : a compact 3-manifold

$R(N) = \mathrm{Hom}(\pi_1(N), \mathrm{PSL}(2, \mathbb{C}))$

$\mathrm{PSL}(2, \mathbb{C})$ acts on $R(N)$ by conjugation. Let $X(N)$ be the algebraic quotient: $X(N) = R(N)//\mathrm{PSL}(2, \mathbb{C})$. We call $X(N)$ the *character variety* of N.

Fact 1 $X(N)$ *is an affine algebraic set. The quotient map* $t : R(N) \to X(N)$ *is a regular map. For* $\gamma \in \pi_1(N)$, $\mathrm{tr}(\rho(\gamma))$ $([\rho] \in X(N))$ *is a regular function on* $X(M)$.

2. Ideal points and valuations

C : an affine algebraic curve.

$\widetilde{C}$: the projective smooth curve which is birational equivalent to C

The points $\widetilde{C}-C$ are called *ideal points*. Ideal points represent 'points at infinity' of C.

Valuations

We denote the function field of C by $\mathbb{C}(C)$. A (discrete) *valuation* of $\mathbb{C}(C)$ is a map $v : \mathbb{C}(C) - \{0\} \to \mathbb{Z}$ satisfying

1. $v(xy) = v(x) + v(y)$,
2. $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in \mathbb{C}(C)$,

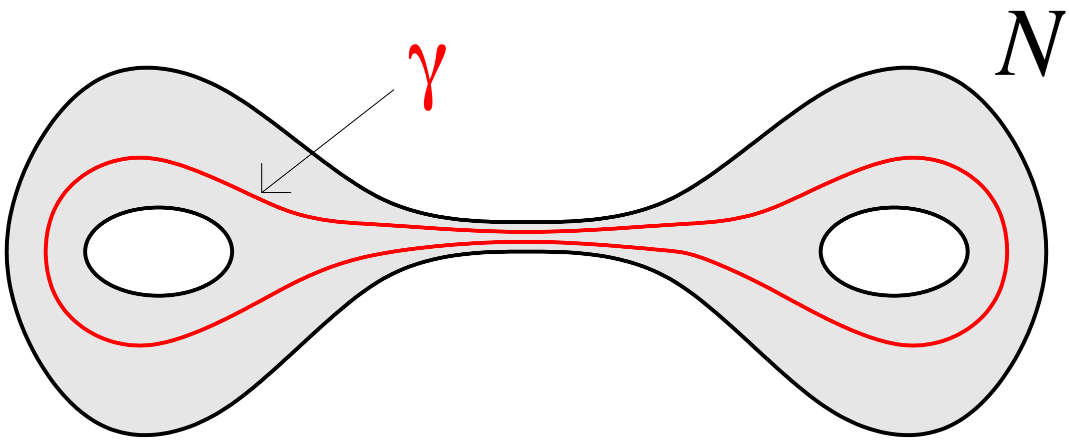
Let p be a point on $\widetilde{C}$. Take a local coordinate t at p. $f \in \mathbb{C}(C)$ can be represented by $f = at^n$ $(a \in \mathbb{C}(C), a(p) \neq 0)$. We can define the valuation associated to p by $v(f) = n$. This gives a correspondence between the valuations of $\mathbb{C}(\widetilde{C})$ and the points of $\widetilde{C}$.

Geometric meaning of ideal point

$x \in \widetilde{C}$: an ideal point of $C \subset X(N)$

$[\rho_i] \in C \subset X(N)$: a sequence of points s.t. $[\rho_i] \to x$

Then $\mathrm{tr}(\rho_i(\gamma))$ diverges for some loop $\gamma \in \pi_1(N)$.



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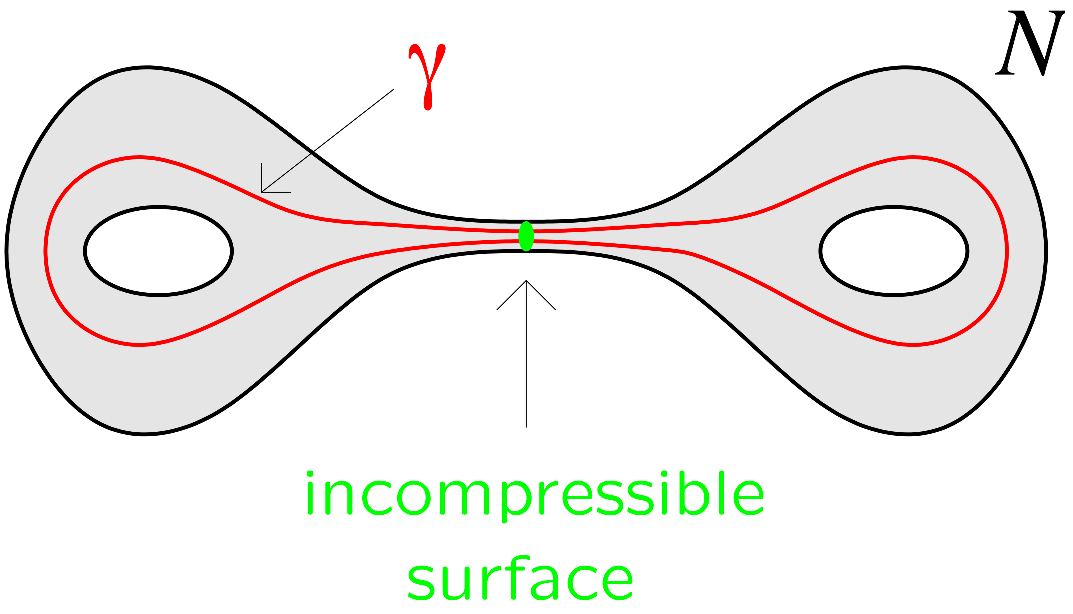


Definition

A properly embedded orientable surface $S \subset N$ is called *incompressible* if $\pi_1(S) \to \pi_1(N)$ is injective. $\partial S \subset \partial N$ is called *boundary slope*.

Theorem(Culler-Shalen)

For each ideal point of a curve of $X(N)$, we can construct an incompressible surface.

Known result

- If we know the A-polynomial of N, we can find ideal points easily. A side of the Newton polygon of A-polynomial corresponds to an ideal points of $X(N)$. But in general it is difficult to compute A-polynomial.

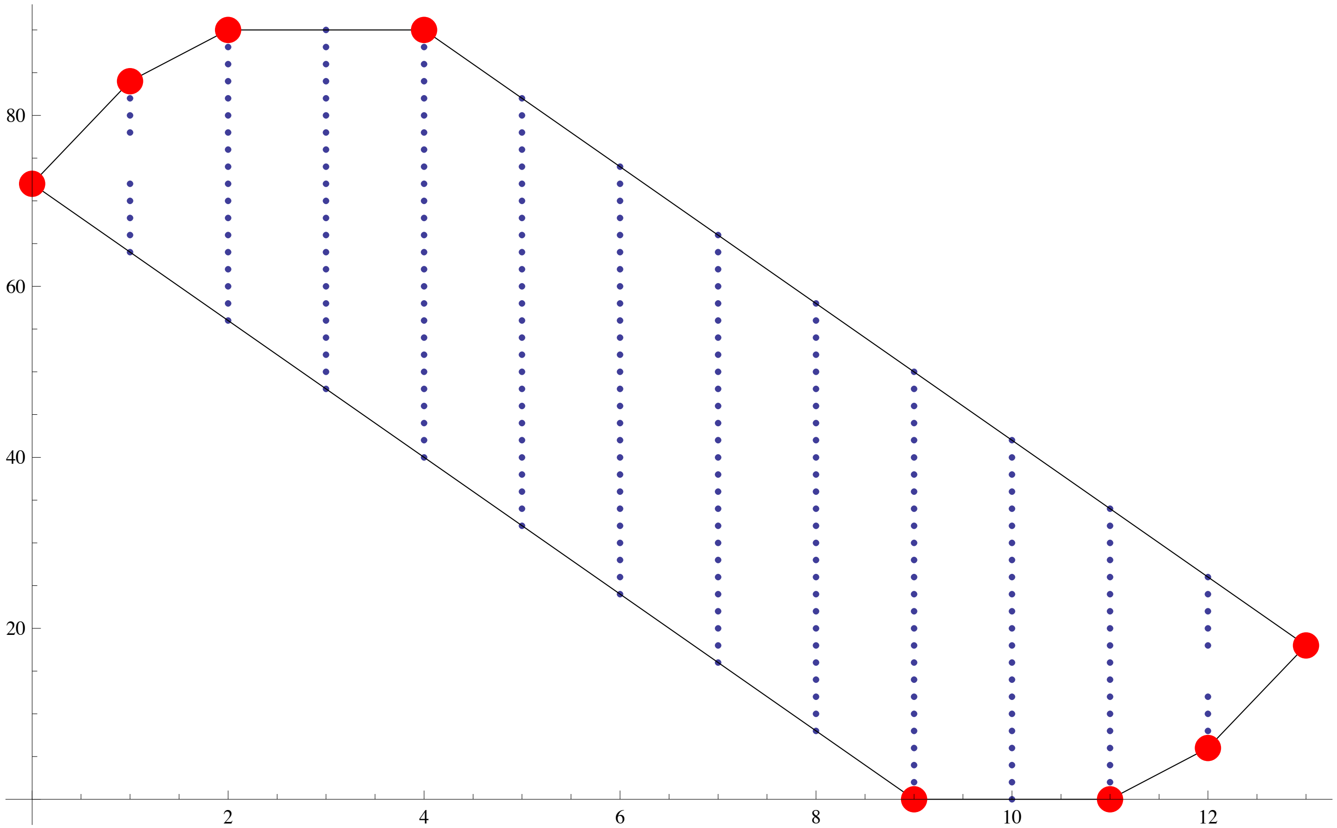
For example, the A-polynomial of 10_4 knot complement is

$$A_{10_4}(L, M) =$$

For example, the A-polynomial of 10_4 knot complement is

$$A_{10_4}(L,M) = M^{72} + L(-9M^{64} + 17M^{66} + 9M^{68} - 8M^{70} + 4M^{72} + 2M^{78} - 3M^{80} + 2M^{82} - M^{84}) + L^2(36M^{56} - 126M^{58} + 33M^{60} + 202M^{62} - 75M^{64} - 52M^{66} + 90M^{68} - 37M^{70} + 3M^{72} + 35M^{74} - 30M^{76} - 10M^{78} + 15M^{80} - 8M^{82} + 3M^{84} - 3M^{86} + 3M^{88} - M^{90}) + L^3(-84M^{48} + 406M^{50} - 382M^{52} - 762M^{54} + 1043M^{56} + 601M^{58} - 941M^{60} + 154M^{62} + 466M^{64} - 215M^{66} + 74M^{68} + 16M^{70} - 153M^{72} + 81M^{74} - 23M^{76} - 11M^{78} + 26M^{80} - 5M^{82} - 16M^{84} + 19M^{86} - 10M^{88} + 2M^{90}) + L^4(126M^{40} - 742M^{42} + 1119M^{44} + 1145M^{46} - 3579M^{48} - 579M^{50} + 4865M^{52} - 455M^{54} - 3019M^{56} + 1125M^{58} + 1368M^{60} + 83M^{62} - 414M^{64} - 801M^{66} + 338M^{68} + 390M^{70} - 299M^{72} - 70M^{74} + 195M^{76} - 80M^{78} - 13M^{80} + 4M^{82} + 21M^{84} - 19M^{86} + 7M^{88} - M^{90}) + L^5(-126M^{32} + 840M^{34} - 1650M^{36} - 691M^{38} + 5456M^{40} - 1609M^{42} - 9172M^{44} + 4253M^{46} + 8723M^{48} - 3703M^{50} - 4963M^{52} + 2030M^{54} + 4235M^{56} + 938M^{58} - 4105M^{60} - 1534M^{62} + 3017M^{64} + 491M^{66} - 1783M^{68} + 377M^{70} + 569M^{72} - 291M^{74} - 135M^{76} + 177M^{78} - 66M^{80} + 9M^{82}) + L^6(84M^{24} - 602M^{26} + 1391M^{28} - 60M^{30} - 4273M^{32} + 3336M^{34} + 7458M^{36} - 8485M^{38} - 8520M^{40} + 9533M^{42} + 7490M^{44} - 5697M^{46} - 5824M^{48} + 3647M^{50} + 8652M^{52} - 2065M^{54} - 9238M^{56} + 2013M^{58} + 6185M^{60} - 2615M^{62} - 2243M^{64} + 1709M^{66} + 288M^{68} - 678M^{70} + 266M^{72} - 36M^{74}) + L^7(-36M^{16} + 266M^{18} - 678M^{20} + 288M^{22} + 1709M^{24} - 2243M^{26} - 2615M^{28} + 6185M^{30} + 2013M^{32} - 9238M^{34} - 2065M^{36} + 8652M^{38} + 3647M^{40} - 5824M^{42} - 5697M^{44} + 7490M^{46} + 9533M^{48} - 8520M^{50} - 8485M^{52} + 7458M^{54} + 3336M^{56} - 4273M^{58} - 60M^{60} + 1391M^{62} - 602M^{64} + 84M^{66}) + L^8(9M^8 - 66M^{10} + 177M^{12} - 135M^{14} - 291M^{16} + 569M^{18} + 377M^{20} - 1783M^{22} + 491M^{24} + 3017M^{26} - 1534M^{28} - 4105M^{30} + 938M^{32} + 4235M^{34} + 2030M^{36} - 4963M^{38} - 3703M^{40} + 8723M^{42} + 4253M^{44} - 9172M^{46} - 1609M^{48} + 5456M^{50} - 691M^{52} - 1650M^{54} + 840M^{56} - 126M^{58}) + L^9(-1 + 7M^2 - 19M^4 + 21M^6 + 4M^8 - 13M^{10} - 80M^{12} + 195M^{14} - 70M^{16} - 299M^{18} + 390M^{20} + 338M^{22} - 801M^{24} - 414M^{26} + 83M^{28} + 1368M^{30} + 1125M^{32} - 3019M^{34} - 455M^{36} + 4865M^{38} - 579M^{40} - 3579M^{42} + 1145M^{44} + 1119M^{46} - 742M^{48} + 126M^{50}) + L^{10}(2 - 10M^2 + 19M^4 - 16M^6 - 5M^8 + 26M^{10} - 11M^{12} - 23M^{14} + 81M^{16} - 153M^{18} + 16M^{20} + 74M^{22} - 215M^{24} + 466M^{26} + 154M^{28} - 941M^{30} + 601M^{32} + 1043M^{34} - 762M^{36} - 382M^{38} + 406M^{40} - 84M^{42}) + L^{11}(-1 + 3M^2 - 3M^4 + 3M^6 - 8M^8 + 15M^{10} - 10M^{12} - 30M^{14} + 35M^{16} + 3M^{18} - 37M^{20} + 90M^{22} - 52M^{24} - 75M^{26} + 202M^{28} + 33M^{30} - 126M^{32} + 36M^{34}) + L^{12}(-M^6 + 2M^8 - 3M^{10} + 2M^{12} + 4M^{18} - 8M^{20} + 9M^{22} + 17M^{24} - 9M^{26}) + L^{13}(M^{18}).$$

The Newton polygon of A_{10_4} is



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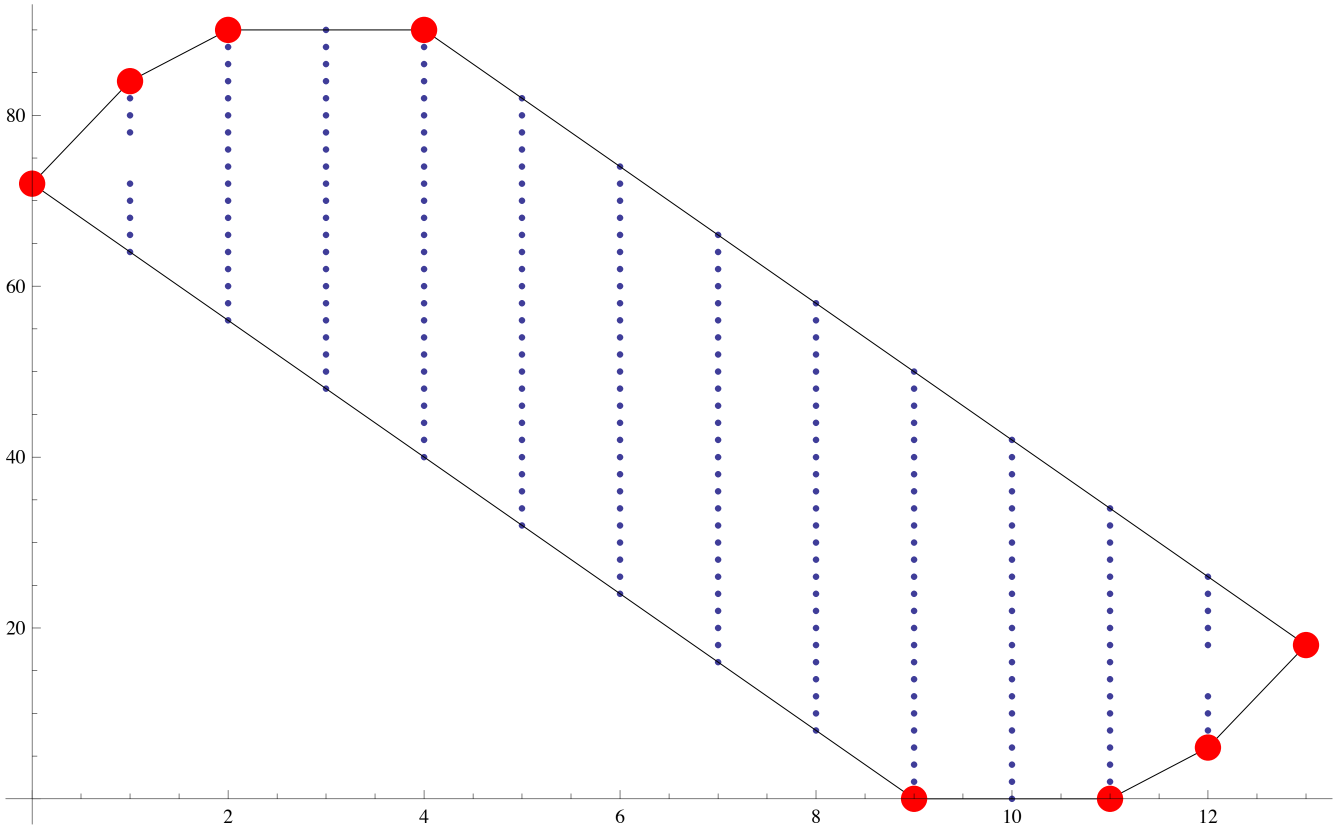

There are ideal points corresponding to -12, -6, 0 and 8 slopes.

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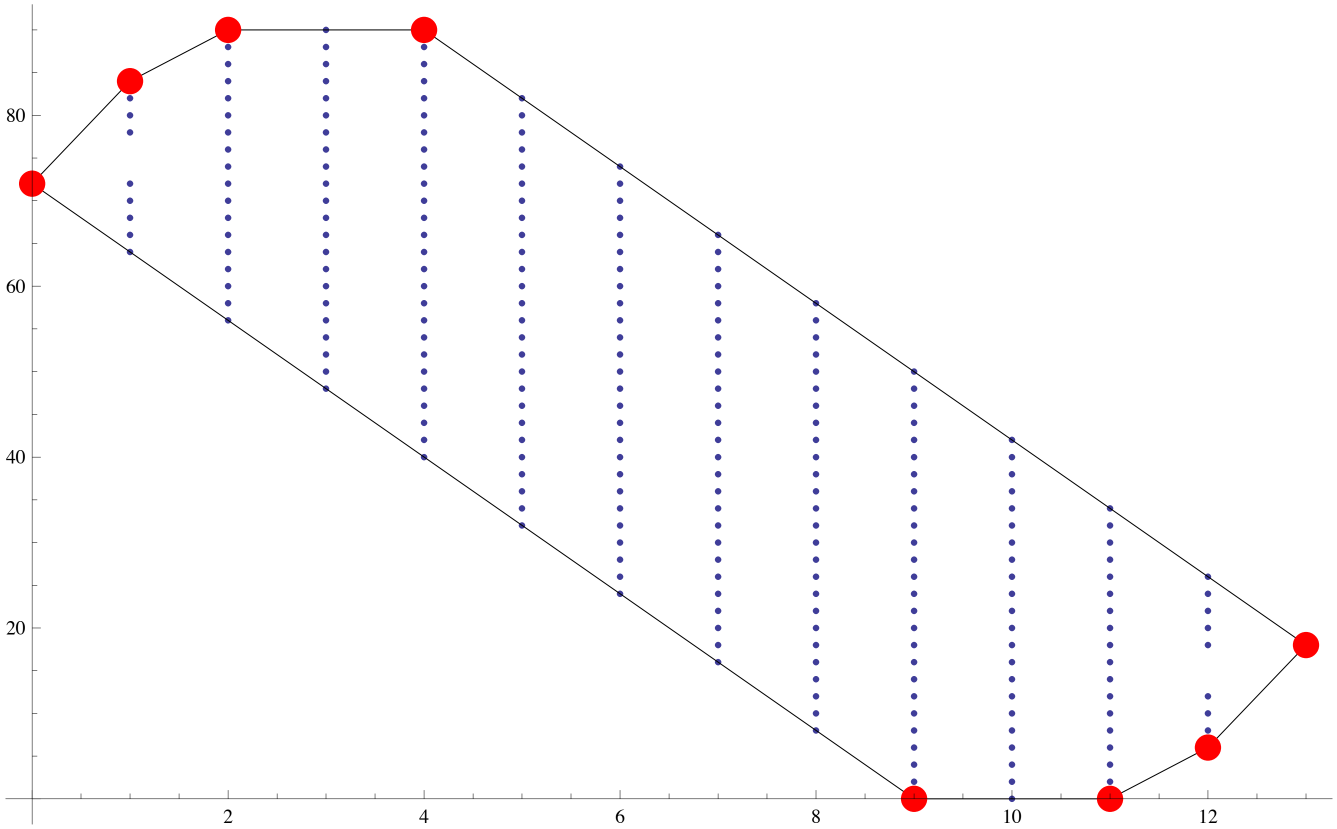


There are ideal points corresponding to -12, -6, 0 and 8 slopes.

We want to obtain this polygon directly from the combinatorial data of N.

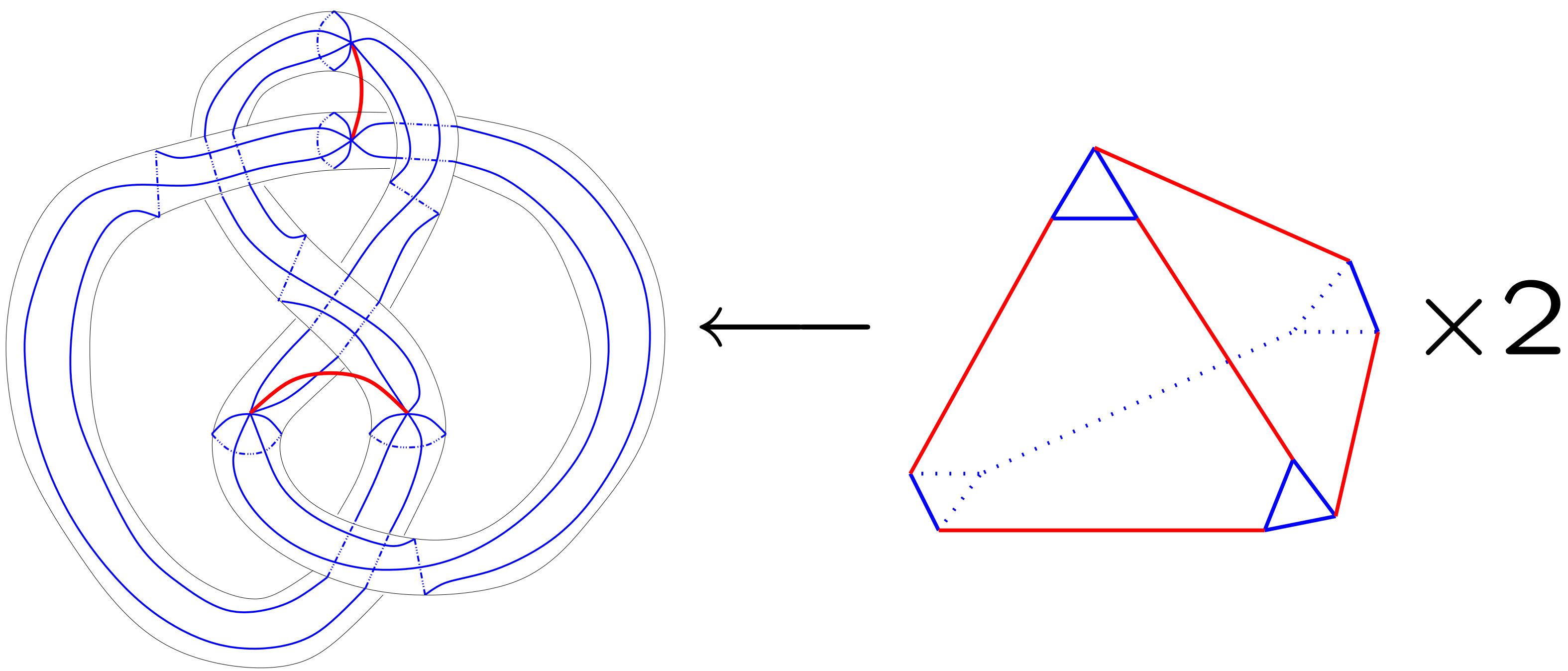
3. Ideal triangulation

N : a compact 3-manifold whose boundary is a torus.

Definition 2 *A (topological) ideal triangulation of* N *is a cell complex* K *formed by gluing 3-simplices along their faces in pair so that* $K - Nbd(K^{(0)})$ *is homeomorphic to* N*.*

In other words, ideal triangulation is a decomposition of N into 3-simplices with trncated vertices so that each truncated vertices are attached to ∂N. We call a 3-simplex with 4 vertices deleted *ideal tetrahedron*.

Example: Complement of the figure eight knot

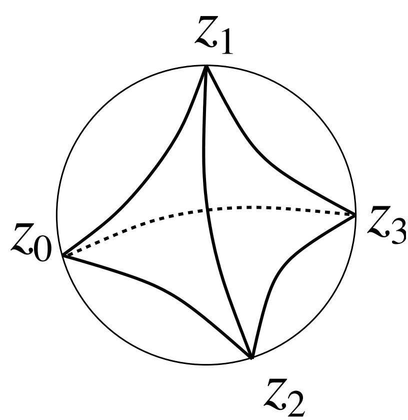


4. Complex parameter of ideal tetrahedron

$\mathbb{H}^3 = \{(x, y, t) | t > 0\}$: the upper-half space model of 3-dimensional hyperbolic space.

The ideal boundary of $\mathbb{H}^3$ is represented by $\mathbb{C}P^1$.

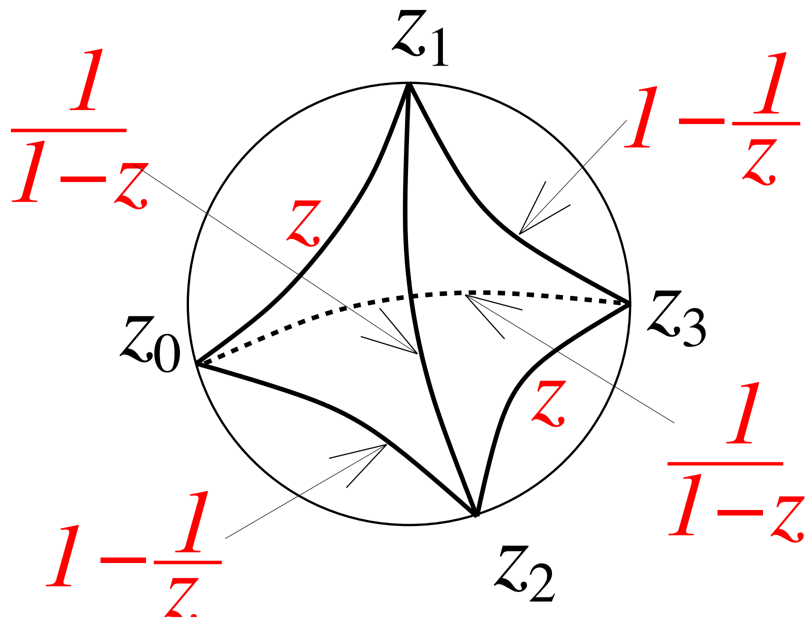
A (geometric) *ideal tetrahedron* is the convex hull of 4 distinct points (z_0, z_1, z_2, z_3) in $\mathbb{H}^3$. $(z_i \in \mathbb{C}P^1)$



The shape of a geometric ideal tetrahedron is parametrized by the cross ratio. For the edges (z_0, z_1) and (z_2, z_3), we associate a complex parameter given by cross ratio:

$$z = [z_0 : z_1 : z_2 : z_3] = \frac{(z_2 - z_1)(z_3 - z_0)}{(z_2 - z_0)(z_3 - z_1)}.$$

For the edges (z_1, z_2) and (z_0, z_3), the complex parameters are equal to $\frac{1}{1-z}$. For the edges (z_1, z_3) and (z_0, z_2), the complex parameters are equal to $1 - 1/z$.



5. Deformation variety $\mathcal{D}(M)$

K : an ideal triangulation of N

We give a complex parameter for each ideal tetrahedron.

$K = \Delta(z_1) \cup \cdots \cup \Delta(z_n)$

Put one ideal tetrahedron $\Delta(z_1)$ in $\mathbb{H}^3$. Then put an adjacent ideal tetrahedron in $\mathbb{H}^3$. By continuing this process, we obtain a map $D : \widetilde{K - K^{(1)}} \to \mathbb{H}^3$.

We can extend D to $D : \widetilde{K - K^{(0)}} \to \mathbb{H}^3$ by considering *gluing equations*.

At each 1-simplex e_k of K, there are adjacent edges of ideal tetrahedra. We denote the product of the complex parameters associated to these edges by R_k:

$$R_k = \prod_{j=1}^{n} (z_j)^{p_{k,j}} \left(\frac{1}{1-z_j} \right)^{p'_{k,j}} \left(1 - \frac{1}{z_j} \right)^{p''_{k,j}}$$
$$= \prod_{j=1}^{n} (-1)^{p''_{k,j}} (z_j)^{r'_{k,j}} (1-z_j)^{r''_{k,j}}$$
$$(r'_{k,j} = p_{k,j} - p''_{k,j}, \quad r''_{k,j} = p''_{k,j} - p'_{k,j}).$$

We call the equations $R_k = 1 (k = 1, \ldots, n)$ *gluing equations*. If $(z_1, \ldots, z_n)$ satisfies the gluing equations, the above map D extends to $D : \widetilde{K - K^{(0)}} \cong \widetilde{N} \to \mathbb{H}^3$. This map is called the *developing map*.

Let $\Delta(z_1)$ be one of the ideal tetrahedron in K. For $\gamma \in \pi_1(N) \cong \pi_1(K - K^{(0)})$, $D(\Delta(z_1))$ and $D(\gamma\Delta(z_1))$ are isometric ideal tetrahedron in $\mathbb{H}^3$. Because there is a unique element of $\mathrm{PSL}(2, \mathbb{C})$ which takes given 3 distinct points to other given 3 distinct points, we can associate a unique element of $\mathrm{PSL}(2, \mathbb{C})$ for γ. This gives the *holonomy representation* $\pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$.

Let

$$\mathcal{D}(N,K)(=\mathcal{D}(N))$$
$$=\{(z_1,\ldots,z_n)\in(\mathbb{C}-\{0,1\})^n | R_1(z)=1,\ldots,R_{n-1}(z)=1\} \tag{1}$$

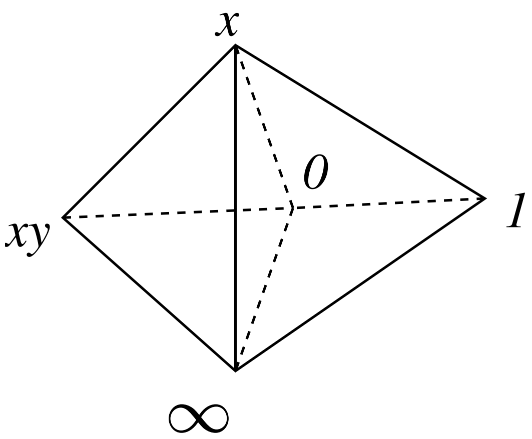
We call $\mathcal{D}(N)$ the *deformation variety*. By using developing maps and their holonomy representations, we can construct a map $\mathcal{D}(N)\to X(N)$.

Fact The map $\mathcal{D}(N)\to X(N)$ is algebraic.

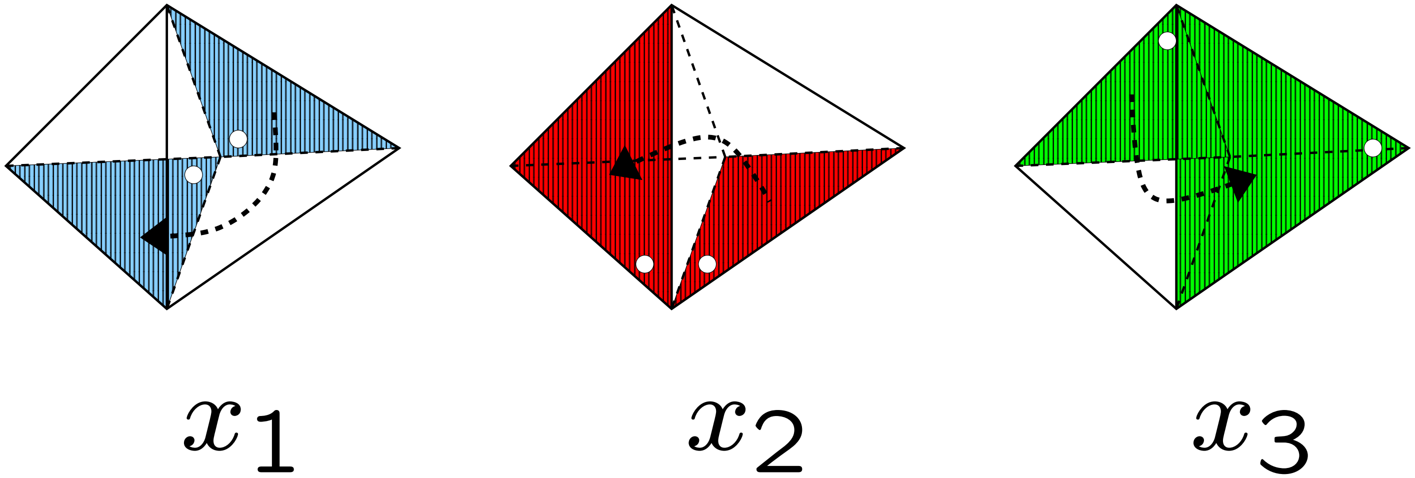
So we can analyze ideal points of $X(M)$ via ideal points of $\mathcal{D}(M)$.

Example (figure eight knot complement)

Put 2 ideal tetrahedra of the 4_1 knot complement as:



The generators of fundamental group is given by



These generators satisfies

$$\pi_1(S^3 - 4_1) \cong \langle x_1, x_2, x_3 | x_3 x_2 x_3^{-1} x_1^{-1} = 1, x_2^{-1} x_1 x_2 x_1^{-1} x_3^{-1} = 1 \rangle.$$

The representation is given by

$$\rho(x_1) = \frac{\pm 1}{\sqrt{y(1-x)}} \begin{pmatrix} -y(1-x) & 0 \\ 1 & -1 \end{pmatrix}$$
$$\rho(x_2) = \frac{\pm 1}{\sqrt{x(1-y)}} \begin{pmatrix} x(1-y) & xy \\ 0 & 1 \end{pmatrix}.$$

The gluing equation is:

$$xy(1-x)(1-y) = 1.$$

Remark

We choose a set of generators $\mathcal{M}, \mathcal{L}$ of $H_1(\partial N; \mathbb{Z})$. We can choose a pair of integers (m'_j, m''_j) and (l'_j, l''_j) so that

$$M = \pm \prod_{j=1}^{n} z_j^{m'_j} (1 - z_j)^{m''_j}, \quad L = \pm \prod_{j=1}^{n} z_j^{l'_j} (1 - z_j)^{l''_j}.$$

represent the squares of eigenvalues of $\rho(\mathcal{M})$ and $\rho(\mathcal{L})$, where ρ is a holonomy representation associated to $(z_1, \ldots, z_n) \in \mathcal{D}(N)$. By taking some conjugation, we have

$$\rho(\mathcal{M}) = \begin{pmatrix} \sqrt{M} & * \\ 0 & \sqrt{M}^{-1} \end{pmatrix}, \quad \rho(\mathcal{L}) = \begin{pmatrix} \sqrt{L} & * \\ 0 & \sqrt{L}^{-1} \end{pmatrix}.$$

Define $m = (m'_1, m''_1 \ldots, m_n,' m''_n)$ and $l = (l'_1, l''_1, \ldots, l'_n, l''_n)$.

Notation

Let $x = (x'_1, x''_1, \ldots, x'_n, x''_n)$ and $y = (y'_1, y''_1, \ldots, y'_n, y''_n)$. We define the symplectic form of $\mathbb{R}^{2n}$ by

$$x \wedge y = \sum_{j=1}^{n} (x'_j y''_j - x''_j y'_j).$$

Let $r_k = (r'_{k,1}, r''_{k,1}, \ldots, r'_{k,n}, r''_{k,n})$. Let $[R] = \text{span}_{\mathbb{R}}\langle r_1, \ldots, r_{n-1}\rangle$. We denote the orthogonal complement of $[R]$ with respect to $\wedge$ by $[R]^{\perp}$.

6. Ideal points of $\mathcal{D}(M)$

Let p be an ideal point of $\mathcal{D}(N)$ and v be the associated valuation. Then v satisfies

$$\begin{aligned}
0 = v(1) = v(R_k) &= v(\pm \prod_{j=1}^{n} (z_j)^{r'_{k,j}} (1 - z_j)^{r''_{k,j}}) \\
&= \sum_{j=1}^{n} \Big(r'_{k,j} v(z_j) + r''_{k,j} v(1 - z_j) \Big) \qquad (2) \\
&= (r'_{k,1}, r''_{k,1}, \ldots, r'_{k,n}, r''_{k,n}) \\
&\qquad \wedge (-v(1 - z_1), v(z_1), \ldots, -v(1 - z_n), v(z_n)).
\end{aligned}$$

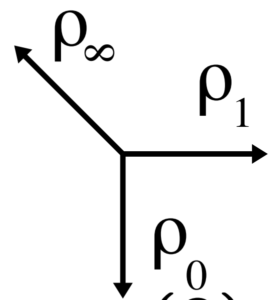
The equation (2) means

$$(-v(1 - z_1), v(z_1), \ldots, -v(1 - z_n), v(z_n)) \in [R]^{\perp}$$

Here $(v(z_j), v(1-z_j))$ behave

$$(v(z_j), v(1-z_j)) = \begin{cases} (0, c) & \text{if } z_j \to 1 \\ (c, 0) & \text{if } z_j \to 0 \\ (-c, -c) & \text{if } z_j \to \infty. \end{cases}$$

for some positive integer c. So $(v(1-z_1), -v(z_1), \dots, v(1-z_n), -v(z_n))$ is in



$$[R]^{\perp} \cap \underbrace{\left\{\mathbb{R}_{\geq 0}(1,0) \cup \mathbb{R}_{\geq 0}(0,-1) \cup \mathbb{R}_{\geq 0}(-1,1)\right\}^n}_{(*)} \tag{3}$$

Let $I = (i_1, \ldots, i_n) \in \{1, 0, \infty\}^n$ and call it a *degeneration index*. A degeneration index I describes how each ideal tetrahedron degenerates ($z_j \to 1, 0, \infty$). For a degeneration index I, we define

$$r(I)_{k,j} = \begin{cases} r''_{k,j} & \text{if } i_j = 1 \\ r'_{k,j} & \text{if } i_j = 0 \\ -r'_{k,j} - r''_{k,j} & \text{if } i_j = \infty \end{cases}$$

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$r(I)_{k,j}$ represents the main contribution from k-th simplex on gluing equation at 1-simplex e_k.

$$(z_j)^{r'_{k,j}} (1 - z_j)^{r''_{k,j}} \qquad z_j \to 1$$

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$$(z_j)^{r'_{k,j}}(1 - z_j)^{r''_{k,j}} \quad z_j \to \infty$$

We define

$$R(I) = \begin{pmatrix} r(I)_{1,1} & \dots & r(I)_{1,n} \\ \vdots & & \vdots \\ r(I)_{n-1,1} & \dots & r(I)_{n-1,n} \end{pmatrix},$$

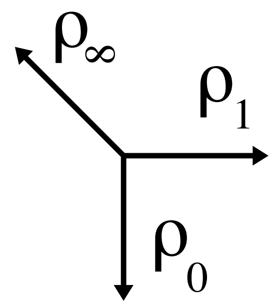
and

$$d(I)_j = (-1)^{j+1} \text{det} \begin{pmatrix} r(I)_{1,1} & \dots & \widehat{r(I)_{1,j}} & \dots & r(I)_{1,n} \\ \vdots & & \vdots & & \vdots \\ r(I)_{n-1,1} & \dots & \widehat{r(I)_{n-1,j}} & \dots & r(I)_{n-1,n} \end{pmatrix}$$

where the hat means removing the column. Then we define a *degeneration vector* by

$$d(I) = (d(I)_1, d(I)_2, \dots, d(I)_n) \in \mathbb{Z}^n \subset \mathbb{R}^n.$$

We put $\rho_1 = (1, 0)$, $\rho_0 = (0, -1)$ and $\rho_\infty = (-1, 1)$. Let



$$S(I) = \{(t_1\rho_{i_1}, \dots, t_n\rho_{i_n}) | t_j \in \mathbb{R}\} \subset \mathbb{R}^{2n},$$
$$H(I) = \{(t_1\rho_{i_1}, \dots, t_n\rho_{i_n}) | t_j \in \mathbb{R}, t_j \geq 0\} \subset \mathbb{R}^{2n}.$$

By using these notation, the necessary condition which the valuation satisfies is

$$(v(1 - z_1), -v(z_1), \dots, v(1 - z_n), -v(z_n)) \in \bigcup_I H(I) \bigcap [R]^\perp.$$

We can easily show that

$$(d(I)_1\rho_{i_1}, d(I)_2\rho_{i_2}, \dots, d(I)_n\rho_{i_n}) \in S(I) \cap [R]^\perp$$

If every coefficient of $d(I)$ is positive or 0, this satisfies the necessary condition.

This necessary condition was given in

T. Yoshida, *On ideal points of deformation curves of hyperbolic* 3*-manifolds with one cusp*, Topology 30 (1991), no. 2, 155–170.

Our main theorem gives a sufficient condition that $d(I)$ actually corresponds to ideal point:

Theorem 3 (K.) *Let* $I = (i_1, \ldots, i_n)$ *be an element of* $\{1, 0, \infty\}^n$*. If* $d(I)_j > 0$ *for all* j*, then there are ideal points of* $\mathcal{D}(N)$ *corresponding to* I*. The number of ideal points corresponding* I *is* $\gcd(d(I)_1, \ldots, d(I)_n)$*. (Similar satatement is valid in the case of* $d(I)_j < 0$ *for all* j*.)*

Remark

$v(M)$ and $v(L)$ can be easily calculated by

$$|v(M)| = |m \wedge x|, \quad |v(L)| = |l \wedge x|$$

where we denote $x = (d'_1(I)\rho_{i_1}, \ldots, d'_n(I)\rho_{i_n}) \in \mathbb{Z}^{2n}$. In fact, we have

$$v(M) = v(\prod_{j=1}^{n} z_j^{m'_j}(1 - z_j)^{m''_j}) = m \wedge x.$$

If $m \wedge x$ or $l \wedge x$ is nonzero, v corresponds to an ideal point of $X(N)$ because the character of $\mathcal{M}$ or $\mathcal{L}$ diverges. If $v(M^pL^q) = 0$, $\mathcal{M}^p\mathcal{L}^q$ is the boundary slope of the ideal point by Culler-Shalen theory.

Idea of the proof

We assume $d(I) > 0$. Let $c = \text{gcd}(d(I)_1, \ldots, d(I)_n)$ and $d'(I)_j = d(I)_j/c$. We can embedded $\mathcal{D}(N)$ in the weighted projective space $\mathbb{C}P(d'_1, \ldots, d'_n, -1)$ by.

$$\begin{aligned} 1 - z_j = a_k t^{d'_j} \quad & \text{if} \quad i_j = 1, \\ z_j = a_j t^{d'_j} \quad & \text{if} \quad i_j = 0, \\ 1/z_j = a_j t^{d'_j} \quad & \text{if} \quad i_j = \infty \end{aligned}$$

where $[a_1, \ldots, a_n, t] \in \mathbb{C}P(d'_1, \ldots, d'_n, -1)$.

Then the gluing equations $R_k = 1$ are replaced to

$$R_k(a_1, \ldots, a_n, t) = \prod_{j=1}^{n} a_j^{r(I)_{k,j}} (1 - a_j t^{d_j})^{\overline{r(I)}_{k,j}} = \pm 1 \quad (k = 1, \ldots n-1) \tag{4}$$

At infinity ($t = 0$), these equations are

$$\prod_{j=1}^{n} a_j^{r(I)_{k,j}} = \pm 1 \quad (k = 1, \ldots, n-1).$$

By using elementary transformations, $R(I)$ reduces to upper triangular matrix. So the above equations reduce to the form $(a_j)^m = 1$. These solution gives ideal points of $\mathcal{D}(N)$.

(For details, see arXiv:GT0706.0971)

Remark

If we calculate degeneration vectors for all $I = (i_1, \ldots, i_n) \in \{1, 0, \infty\}^n$, we can find ideal points satisfying our theorem.

We need 3^n times calculations. But if we find once, it is easy to verify the calculation.

If the number of tetrahedra is less than 15 or 16, we can calculate $d(I)$ for all I by using computer.

Unfortunately, there are many ideal points which can not be detected by our method. It is difficult to analyze an ideal point at which some ideal tetrahedra do not degenerate.

Remark

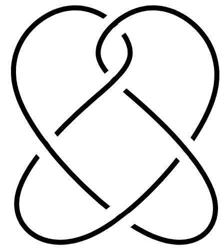
If we calculate degeneration vectors for all $I = (i_1, \ldots, i_n) \in \{1, 0, \infty\}^n$, we can find ideal points satisfying our theorem.

We need 3^n times calculations. But if we find once, it is easy to verify the calculation.

If the number of tetrahedra is less than 15 or 16, we can calculate $d(I)$ for all I by using computer. ($3^{15} = 14,348,907$)

Unfortunately, there are many ideal points which can not be detected by our method. It is difficult to analyze an ideal point at which some ideal tetrahedra do not degenerate.

7. Examples The complement of the 5_2 knot.



The degeneration indexes satisfying the condition of the Theorem are

$$(I =) \quad (\infty, \infty, 0), (1, 0, 0), (\infty, 1, \infty), (0, 0, \infty), (1, 1, 1), (0, \infty, 1)$$

and the corresponding degeneration vectors are

$$(d(I) =) \quad (2, 1, 1), (1, 2, 1), (1, 1, 1), (1, 1, 1), (2, 2, 1),$$

respectively. $(v(M), v(L))$ are

$$(0, 1), (0, -1), (4, 1), (-4, -1), (10, 1), (-10, -1).$$

and the corresponding boundary slopes are

$$0, 0, -4, -4, -10, -10$$

Some $(-2, p, q)$-pretzel knots

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
(-2,3,7)-	$(\infty, 1, 0)$	$(1, 1, 2)$	$(-1, 16)$	16
pretzel	$(\infty, 0, 1)$	$(1, 2, 1)$	$(1, -16)$	16
(m016)	$(1, 1, \infty)$	$(1, 2, 2)$	$(-4, 74)$	$37/2$
	$(1, \infty, 1)$	$(1, 2, 2)$	$(4, -74)$	$37/2$
	$(0, \infty, 0)$	$(1, 1, 1)$	$(1, -20)$	20
	$(0, 0, \infty)$	$(1, 1, 1)$	$(-1, 20)$	20
(-2,3,13)-	$(1, \infty, \dots)$	$(1, 5, 1, 1, 3, 7, 7)$	$(-1, 16)$	16
pretzel	$(\infty, 0, \dots)$	$(8, 7, 4, 6, 2, 8, 1)$	$(1, -16)$	16
(v0959)	$(1, \infty, \dots)$	$(1, 6, 2, 8, 4, 8, 9)$	$(-10, 302)$	$151/5$
	$(\infty, \infty, \dots)$	$(1, 1, 5, 7, 3, 8, 1)$	$(10, -302)$	$151/5$
(-2,5,5)-	$(0, 1, \dots)$	$(1, 2, 1, 1, 1, 1, 2)$	$(-1, 14)$	14
pretzel	$(\infty, \infty, \dots)$	$(2, 1, 1, 1, 1, 2, 1)$	$(1, -14)$	14
(v2642)	$(0, 1, \dots)$	$(4, 3, 1, 1, 1, 2, 1)$	$(-2, 30)$	15
	$(\infty, 1, \dots)$	$(4, 1, 3, 4, 2, 6, 1)$	$(2, -30)$	15
	$(\infty, 1, \dots)$	$(1, 4, 2, 4, 3, 1, 6)$	$(-2, 30)$	15
	$(\infty, \infty, \dots)$	$(3, 4, 1, 1, 1, 1, 2)$	$(2, -30)$	15

Other knots (alternating non-Montesinos knots)

- 8_{17}

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
8_{17}	$(0, 1, \infty, \dots)$	$-(4, 4, 1, 3, 1, \dots)$	$(-1, -14)$	-14
(12	$(\infty, \infty, 1, \dots)$	$(3, 1, 3, 4, 2, \dots)$	$(1, 14)$	-14
ideal	$(\infty, 0, 0, \dots)$	$(1, 1, 1, 1, 1, \dots)$	$(1, -2)$	2
tet.)	$(0, 0, \infty, \dots)$	$(5, 2, 3, 2, 1, \dots)$	$(-1, 14)$	14
	$(\infty, \infty, 0, \dots)$	$-(3, 3, 4, 3, 1, \dots)$	$(1, -14)$	14

Other knots (alternating non-Montesinos knots)

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knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
8_{17}	$(0, 1, \infty, \dots)$	$-(4, 4, 1, 3, 1, \dots)$	$(-1, -14)$	-14
(12	$(\infty, \infty, 1, \dots)$	$(3, 1, 3, 4, 2, \dots)$	$(1, 14)$	-14
ideal	$(\infty, 0, 0, \dots)$	$(1, 1, 1, 1, 1, \dots)$	$(1, -2)$	2
tet.)	$(0, 0, \infty, \dots)$	$(5, 2, 3, 2, 1, \dots)$	$(-1, 14)$	14
	$(\infty, \infty, 0, \dots)$	$-(3, 3, 4, 3, 1, \dots)$	$(1, -14)$	14

Fact

- The *diameter* of the boundary slope set $diam(K)$ is defined by the maximum distance between boundary slopes. Let $cr(K)$ be the crossing number of K. For alternating Montesinos knot, we have

$$diam(K) = 2cr(K). \quad \text{(Ichihara-Mizushima)}$$

- $10_{79}, 10_{80}$ (alternating non-Montesinos knots)

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
10_{79}	$(0, 1, 1, \dots)$	$(2, 3, 3, \dots)$	$(-3, -10)$	$-10/3$
(14	$(\infty, 0, \infty, \dots)$	$-(2, 1, 1, \dots)$	$(1, 0)$	0
ideal	$(0, \infty, 1, 1, \dots)$	$(1, 1, 1, \dots)$	$(1, 0)$	0
tet.)	$(0, 1, \infty, \dots)$	$-(1, 1, 2, \dots)$	$(-3, 10)$	$10/3$
	$(\infty, \infty, \infty, \dots)$	$(2, 1, 2, \dots)$	$(1, -6)$	6
10_{80}	$(0, 0, 0, \dots)$	$-(1, 3, 1, \dots)$	$(-2, -26)$	-13
(14	$(0, \infty, 0, \dots)$	$(1, 1, 1, \dots)$	$(1, 8)$	-8
ideal	$(1, 1, 0, \dots)$	$(1, 4, 2, \dots)$	$(3, 20)$	$-20/3$
tet.)	$(\infty, 0, \infty, \dots)$	$-(2, 3, 2, \dots)$	$(-3, -20)$	$-20/3$
	$(1, \infty, 0, \dots)$	$(1, 2, 1, \dots)$	$(-1, -2)$	-2
	$(0, \infty, \infty, \dots)$	$-(1, 1, 2, \dots)$	$(1, 2)$	-2

• $10_{79}, 10_{80}$ (alternating non-Montesinos knots)

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
10_{79}	$(0, 1, 1, \ldots)$	$(2, 3, 3, \ldots)$	$(-3, -10)$	$-10/3$
(14	$(\infty, 0, \infty, \ldots)$	$-(2, 1, 1, \ldots)$	$(1, 0)$	0
ideal	$(0, \infty, 1, 1, \ldots)$	$(1, 1, 1, \ldots)$	$(1, 0)$	0
tet.)	$(0, 1, \infty, \ldots)$	$-(1, 1, 2, \ldots)$	$(-3, 10)$	$10/3$
	$(\infty, \infty, \infty, \ldots)$	$(2, 1, 2, \ldots)$	$(1, -6)$	6
10_{80}	$(0, 0, 0, \ldots)$	$-(1, 3, 1, \ldots)$	$(-2, -26)$	-13
(14	$(0, \infty, 0, \ldots)$	$(1, 1, 1, \ldots)$	$(1, 8)$	-8
ideal	$(1, 1, 0, \ldots)$	$(1, 4, 2, \ldots)$	$(3, 20)$	$-20/3$
tet.)	$(\infty, 0, \infty, \ldots)$	$-(2, 3, 2, \ldots)$	$(-3, -20)$	$-20/3$
	$(1, \infty, 0, \ldots)$	$(1, 2, 1, \ldots)$	$(-1, -2)$	-2
	$(0, \infty, \infty, \ldots)$	$-(1, 1, 2, \ldots)$	$(1, 2)$	-2

Fact

• It is known that all alternating Montesinos knots have only even boundary slopes.

More examples...

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
8_{16}	$(\infty,1,1,1,\infty,\infty,0,0,1,\infty,0)$	$-(1,2,1,1,1,1,1,1,1,1,1)$	$(-1,2)$	2
	$(1,\infty,\infty,0,\infty,\infty,1,0,0,1,1)$	$(1,1,1,2,1,1,1,1,1,1,1)$	$(1,-2)$	2
	$(1,\infty,1,\infty,\infty,0,\infty,0,0,0,\infty)$	$(2,2,1,2,1,1,2,1,1,1,1)$	$(1,-6)$	6
	$(1,0,1,1,0,\infty,0,0,\infty,0,\infty)$	$-(1,2,2,2,1,1,1,1,2,1,1)$	$(-1,6)$	6
	$(1,\infty,1,\infty,0,\infty,1,0,0,0,\infty)$	$(3,4,2,4,1,4,1,1,2,1,3)$	$(1,-16)$	16
	$(1,0,1,1,\infty,0,0,0,1,0,\infty)$	$-(2,4,3,4,4,1,2,1,1,3,1)$	$(-1,16)$	16
8_{17}	$(0,1,\infty,\infty,\infty,0,0,\infty,1,\infty,\infty,1)$	$-(4,4,1,3,1,2,2,3,1,3,1,2)$	$(-1,-14)$	−14
	$(\infty,\infty,1,0,0,\infty,\infty,1,\infty,0,1,\infty)$	$(3,1,3,4,2,1,2,1,2,1,1,1)$	$(1,14)$	−14
	$(\infty,0,0,0,\infty,1,0,0,0,\infty,0,1)$	$(1,1,1,1,1,1,1,3,1,1,2,1)$	$(1,-2)$	2
	$(0,0,\infty,0,0,1,\infty,1,1,\infty,0,0)$	$(5,2,3,2,1,3,4,2,4,3,1,3)$	$(-1,14)$	14
	$(\infty,\infty,0,\infty,1,\infty,0,0,0,0,\infty,1)$	$-(3,3,4,3,1,1,2,1,2,4,2,1)$	$(1,-14)$	14
8_{18}	$(\infty,\infty,\infty,1,\infty,1,\infty,\infty,0,\infty,0,0,0)$	$(3,2,3,1,4,3,1,1,1,3,2,1,1)$	$(-1,-14)$	−14
	$(0,0,1,1,1,\infty,\infty,0,0,1,\infty,0,0)$	$-(2,3,2,1,1,3,1,4,1,3,3,1,1)$	$(1,14)$	−14
	$(0,1,0,\infty,1,\infty,1,0,\infty,\infty,0,\infty,0)$	$-(1,1,4,2,3,1,1,3,2,2,2,6,3)$	$(1,14)$	−14
	$(0,1,1,1,1,\infty,0,1,\infty,0,\infty,\infty,0)$	$-(1,1,2,4,1,3,3,1,1,2,2,3,1)$	$(1,14)$	−14
	$(0,0,\infty,1,1,0,1,0,\infty,\infty,1,\infty,1)$	$-(1,1,2,2,2,1,2,4,3,2,3,1,2)$	$(1,14)$	−14
	$(\infty,\infty,0,0,0,0,0,\infty,0,\infty,0,0,\infty)$	$(2,2,1,1,1,2,3,1,4,3,1,1,3)$	$(-1,-14)$	−14
	$(0,1,0,0,\infty,\infty,1,\infty,1,\infty,0,0,\infty)$	$(2,4,1,2,3,2,1,3,2,1,1,3,6)$	$(-1,-14)$	−14
	$(1,0,\infty,0,\infty,\infty,1,\infty,0,0,0,1,\infty)$	$(3,2,1,3,4,2,2,2,2,1,1,2,1)$	$(-1,-14)$	−14
	$(0,0,\infty,\infty,1,0,\infty,0,\infty,1,\infty,0,1)$	$-(2,2,2,1,3,3,4,1,1,1,1,3,1)$	$(-1,14)$	14
	$(\infty,0,\infty,0,\infty,1,\infty,\infty,1,0,0,1,0)$	$(1,2,2,1,1,1,4,3,1,3,2,1,3)$	$(1,-14)$	14
	$(\infty,\infty,0,\infty,0,1,0,0,0,\infty,0,1,0)$	$(6,2,2,1,1,2,1,1,1,2,2,1,3)$	$(1,-14)$	14
	$(1,1,0,0,\infty,\infty,1,\infty,1,0,\infty,1,0)$	$(1,2,4,3,2,3,3,3,3,1,1,2,1)$	$(1,-14)$	14
	$(1,1,0,\infty,1,0,0,1,\infty,0,0,\infty,0)$	$-(1,1,1,2,2,1,1,3,3,2,2,4,1)$	$(-1,14)$	14
	$(\infty,1,0,\infty,1,0,1,0,\infty,\infty,1,0,1)$	$-(1,4,2,3,3,1,3,2,3,3,1,1,2)$	$(-1,14)$	14
	$(0,1,1,1,\infty,\infty,0,1,1,1,\infty,0,1)$	$-(2,2,2,1,1,2,1,1,1,2,6,3,1)$	$(-1,14)$	14
	$(0,1,0,0,0,0,0,\infty,1,0,1,0,\infty)$	$(2,1,1,3,3,2,1,2,2,1,1,1,4)$	$(1,-14)$	14

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
10_{79}	$(0,1,1,1,0,\infty,0,1,\infty,\dots)$	$(2,3,3,3,2,2,1,2,1,4,1,4,3,5)$	$(-3,-10)$	$-10/3$
	$(\infty,0,\infty,1,\infty,\infty,\infty,\dots)$	$-(2,1,1,2,2,1,1,2,1,2,2,1,1,1)$	$(1,0)$	0
	$(0,\infty,1,1,\infty,0,1,\infty,\dots)$	$(1,1,1,1,2,2,1,1,2,1,1,1,1,2)$	$(1,0)$	0
	$(0,1,\infty,\infty,\infty,1,0,\dots)$	$-(1,1,2,1,2,5,3,5,2,2,2,1,1,2)$	$(-3,10)$	$10/3$
	$(\infty,\infty,\infty,1,\infty,\infty,\infty,0,\dots)$	$(2,1,2,1,2,1,1,2,1,3,3,1,1,2)$	$(1,-6)$	6
10_{80}	$(0,0,0,1,1,0,1,\dots)$	$-(1,3,1,3,5,1,3,1,2,1,2,3,2,1)$	$(-2,-26)$	-13
	$(0,\infty,0,\infty,\infty,\infty,0,1,\dots)$	$(1,1,1,1,1,2,1,1,1,1,1,2,1,1)$	$(1,8)$	-8
	$(1,1,0,\infty,\infty,\infty,0,\dots)$	$(1,4,2,3,5,3,8,4,2,3,1,2,3,1)$	$(3,20)$	$-20/3$
	$(\infty,0,\infty,1,1,\infty,1,1,\dots)$	$-(2,3,2,3,6,1,9,1,3,4,1,5,5,2)$	$(-3,-20)$	$-20/3$
	$(1,\infty,0,0,1,1,1,\dots)$	$(1,2,1,2,1,1,2,3,1,3,1,1,1,1)$	$(-1,-2)$	-2
	$(0,\infty,\infty,\infty,\infty,\infty,0,\dots)$	$-(1,1,2,1,1,2,1,2,1,4,1,2,1,1)$	$(1,2)$	-2
10_{82}	$(0,\infty,\infty,0,1,0,0,\dots)$	$-(7,3,5,1,1,5,2,1,1,7,3,3,2)$	$(-2,-26)$	-13
	$(1,0,0,\infty,\infty,\infty,0,\dots)$	$(1,5,1,2,2,3,2,1,1,3,7,2,3)$	$(2,26)$	-13
	$(0,0,0,\infty,\infty,\infty,0,\dots)$	$(1,3,1,1,1,2,2,1,1,2,6,1,2)$	$(1,12)$	-12
	$(0,\infty,\infty,\infty,1,0,0,\dots)$	$-(4,2,3,1,1,3,2,1,1,6,2,2,1)$	$(-1,-12)$	-12
	$(0,0,\infty,1,1,\infty,0,\dots)$	$-(4,2,2,1,1,1,2,1,1,1,5,1,1)$	$(1,-2)$	2
	$(\infty,1,1,\infty,0,0,0,\dots)$	$(1,1,1,4,1,1,2,1,1,5,1,1,1)$	$(-1,2)$	2
	$(1,1,\infty,\infty,1,\infty,1,\dots)$	$-(1,4,1,3,2,3,2,1,1,3,3,3,2)$	$(-1,6)$	6
	$(0,\infty,0,1,1,1,0,\dots)$	$-(3,2,1,2,2,1,2,1,1,1,3,1,1)$	$(1,-6)$	6
	$(0,\infty,1,1,0,0,0,\dots)$	$(4,1,2,5,1,2,2,1,1,3,4,2,3)$	$(1,-14)$	14
	$(1,1,\infty,\infty,1,\infty,0,\dots)$	$-(1,4,1,7,2,3,2,1,1,7,3,3,2)$	$(-1,14)$	14
10_{85}	$(0,\infty,0,\infty,\infty,0,0,\dots)$	$-(6,2,2,2,8,4,6,10,2,6,6,2,4)$	$(-2,-40)$	-20
	$(1,0,1,0,\infty,\infty,1,\dots)$	$(3,2,1,4,2,2,1,1,1,3,1,5,1)$	$(2,30)$	-15
	$(1,0,1,0,\infty,\infty,0,\dots)$	$(2,2,1,3,2,1,1,2,1,2,2,3,1)$	$(1,14)$	-14
	$(1,0,1,\infty,0,\infty,0,\dots)$	$-(1,2,1,1,1,1,4,3,1,1,3,1,2)$	$(1,2)$	-2
10_{90}	$(0,1,1,0,0,\infty,1,\dots)$	$-(2,1,1,5,2,3,2,4,2,1,1,1,2,1,3)$	$(-2,-14)$	-7
	$(0,\infty,\infty,\infty,\infty,0,1,\dots)$	$(1,4,1,3,6,2,1,3,1,1,1,2,1,1,7)$	$(1,-12)$	12
	$(0,1,1,\infty,0,0,1,\dots)$	$-(3,2,1,3,1,2,1,1,1,1,1,6,6,6,1)$	$(-1,12)$	12
	$(0,1,1,\infty,0,0,1,\dots)$	$(3,2,1,3,1,2,1,1,1,1,2,6,7,5,1)$	$(-1,18)$	18
	$(0,\infty,\infty,\infty,\infty,0,1,\dots)$	$-(1,4,1,3,6,2,1,3,1,2,2,1,1,1,10)$	$(1,-18)$	18

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
10_{91}	$(0, \infty, \infty, 1, 0, 0, \infty, \dots)$	$(1, 5, 3, 1, 1, 4, 11, 4, 7, 1, 1, 3, 6, 2, 5)$	$(-1, -26)$	-26
	$(1, 1, 1, 0, \infty, 0, 0, \dots)$	$-(4, 8, 5, 3, 2, 5, 1, 3, 6, 14, 1, 3, 8, 4, 1)$	$(1, 26)$	-26
	$(\infty, \infty, \infty, 0, \infty, 0, 1, \dots)$	$(1, 2, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 1)$	$(-1, 2)$	2
	$(1, 1, 1, 0, \infty, 0, \infty, \dots)$	$(1, 3, 2, 2, 2, 1, 3, 1, 1, 1, 1, 1, 1, 1, 1)$	$(-1, 6)$	6
	$(1, 0, 1, 0, 0, 0, 0, \dots)$	$-(2, 2, 3, 1, 4, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1)$	$(1, -6)$	6
	$(1, 0, 1, 0, 0, 0, 0, \dots)$	$(3, 4, 2, 2, 3, 1, 2, 1, 1, 1, 1, 6, 4, 1, 2)$	$(1, -10)$	10
	$(1, 0, 1, 0, 0, 0, 0, \dots)$	$-(3, 4, 5, 2, 6, 1, 2, 1, 2, 1, 1, 5, 4, 1, 3)$	$(1, -16)$	16
10_{93}	$(1, 0, \infty, \infty, \infty, 1, 1, \dots)$	$(2, 1, 3, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1)$	$(-1, 2)$	2
10_{94}	$(\infty, 1, 0, \infty, 1, 0, 0, \dots)$	$-(2, 1, 2, 4, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1)$	$(1, -6)$	6
	$(\infty, 1, 0, 1, \infty, 0, 0, \dots)$	$(3, 9, 2, 1, 10, 5, 3, 4, 1, 8, 1, 3, 5, 4)$	$(-1, 28)$	28
10_{98}	$(0, \infty, 1, 0, \infty, 1, 0, 1, \dots)$	$(2, 1, 1, 1, 2, 1, 1, 1, 2, 2, 1, 1, 2, 1, 1)$	$(1, 4)$	-4
10_{100}	$(\infty, 0, 0, 1, 1, \infty, 0, \dots)$	$(2, 1, 1, 1, 1, 2, 4, 1, 1, 2, 1, 1, 1, 1)$	$(-1, -12)$	-12
	$(\infty, 0, 0, \infty, 1, 0, 1, \dots)$	$-(2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1, 4, 1)$	$(1, 12)$	-12
	$(\infty, 0, 0, 0, 1, \infty, 0, \dots)$	$-(2, 1, 1, 1, 2, 2, 4, 1, 1, 1, 2, 2, 1, 4)$	$(-1, -6)$	-6
	$(\infty, 0, 0, \infty, 0, 0, 1, \dots)$	$(2, 1, 1, 4, 2, 1, 1, 2, 1, 2, 1, 2, 4, 1)$	$(1, 6)$	-6
10_{102}	$(\infty, 1, \infty, 1, 0, \infty, \infty, \dots)$	$(1, 2, 2, 2, 1, 1, 3, 1, 1, 2, 4, 3, 5, 1, 1)$	$(-2, -2)$	-1
	$(0, \infty, \infty, 1, 1, 0, 0, \dots)$	$-(4, 1, 3, 1, 3, 1, 6, 1, 1, 2, 1, 3, 6, 1, 1)$	$(1, -12)$	12
	$(\infty, 0, 1, 1, 1, 0, \dots)$	$(1, 4, 1, 1, 3, 3, 1, 1, 6, 2, 1, 1, 4, 1, 3)$	$(-1, 12)$	12
	$(0, \infty, \infty, 1, 1, 0, \dots)$	$(3, 1, 3, 1, 3, 1, 9, 2, 1, 2, 1, 3, 6, 1, 1)$	$(1, -18)$	18
	$(\infty, 0, 1, 1, 1, 0, \infty, \dots)$	$-(1, 4, 1, 1, 3, 3, 1, 1, 5, 2, 1, 1, 4, 2, 6)$	$(-1, 18)$	18
10_{103}	$(\infty, 1, \infty, 1, 0, \infty, 0, \dots)$	$-(3, 1, 4, 4, 2, 1, 1, 4, 4, 3, 2, 1, 1, 3, 2)$	$(-1, -20)$	-20
	$(0, \infty, 0, 0, \infty, 1, 0, \dots)$	$(2, 2, 2, 3, 1, 1, 3, 4, 4, 2, 1, 2, 1, 1, 3)$	$(1, 20)$	-20
	$(\infty, 1, \infty, 1, \infty, 1, 0, \dots)$	$(2, 1, 2, 4, 1, 1, 1, 4, 4, 3, 2, 1, 1, 3, 2)$	$(-1, -14)$	-14
	$(0, \infty, \infty, 0, 1, 1, 0, \dots)$	$-(2, 2, 1, 1, 1, 2, 3, 4, 4, 2, 1, 2, 1, 1, 3)$	$(1, 14)$	-14
	$(\infty, 1, \infty, 1, 0, \infty, 0, \dots)$	$(1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1)$	$(-1, -12)$	-12
	$(\infty, 1, \infty, 1, 0, \infty, \infty \dots,$	$-(2, 1, 2, 2, 2, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1)$	$(-1, -6)$	-6
	$(0, 1, 0, \infty, \infty, 0, 0, \dots)$	$(1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 3, 1, 1, 1, 2)$	$(1, 6)$	-6
10_{104}	$(0, 1, 0, 0, 1, 1, 0, \dots)$	$-(1, 2, 1, 3, 1, 2, 1, 1, 1, 2, 3, 1, 1, 1, 1)$	$(-1, 6)$	6
	$(\infty, \infty, 1, \infty, 0, \infty, 0, \dots)$	$-(1, 1, 2, 1, 1, 3, 2, 1, 6, 1, 1, 1, 3, 5, 3)$	$(1, -10)$	10
	$(\infty, 0, 0, 0, \infty, \infty, 0, \dots)$	$(1, 2, 1, 3, 1, 1, 2, 1, 1, 1, 6, 3, 3, 4, 1)$	$(-1, 10)$	10
	$(\infty, 1, 0, 0, 1, \infty, 0, \dots)$	$(2, 3, 1, 5, 1, 1, 2, 3, 1, 2, 2, 1, 1, 1, 2)$	$(-1, 14)$	14
	$(\infty, 0, 0, 0, 1, \infty, 0, \dots)$	$-(1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 9, 3, 3, 5, 1)$	$(-1, 16)$	16
	$(\infty, \infty, 1, \infty, \infty, \infty, 0, \dots)$	$(1, 1, 5, 1, 2, 3, 2, 1, 5, 1, 1, 1, 3, 6, 3)$	$(1, -16)$	16

knot	I	$d(I)$	$(v(M), v(L))$	boundary slope
10_{106}	$(0, 1, 1, \infty, 0, \infty, 1, \dots)$	$-(1, 2, 1, 3, 2, 1, 2, 1, 4, 1, 3, 1, 3, 3, 1)$	$(1, 14)$	-14
	$(1, 1, 0, 0, 1, \infty, 0, \dots)$	$(8, 2, 2, 1, 6, 1, 7, 3, 1, 3, 1, 1, 2, 4, 2)$	$(-1, -14)$	-14
	$(1, 1, \infty, 0, 1, 1, 0, \dots)$	$-(5, 2, 1, 1, 5, 1, 4, 3, 1, 3, 1, 1, 2, 4, 2)$	$(-1, -8)$	-8
	$(1, \infty, 0, 0, 1, \infty, 0, \dots)$	$-(2, 1, 2, 1, 3, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1)$	$(-1, -6)$	-6
	$(1, 0, 0, 0, 1, \infty, \infty, \dots)$	$(3, 3, 2, 1, 3, 1, 2, 3, 3, 3, 2, 5, 5, 1, 3)$	$(-3, -4)$	$-4/3$
	$(\infty, 1, 0, 0, \infty, \infty, 0, \dots)$	$-(4, 3, 4, 1, 2, 1, 4, 12, 2, 1, 2, 4, 3, 3, 3)$	$(3, 4)$	$-4/3$
	$(1, \infty, 0, 0, 0, \infty, 0, \dots)$	$(1, 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 1, 1, 2, 1)$	$(-1, 6)$	6
	$(1, \infty, 0, 0, 1, \infty, 0, \dots)$	$(1, 1, 2, 1, 1, 1, 2, 2, 1, 1, 1, 2, 1, 1, 2)$	$(-1, 6)$	6
10_{108}	$(0, \infty, 0, \infty, 0, 0, \infty, \dots)$	$(3, 1, 1, 2, 1, 3, 1, 1, 1, 3, 2, 2, 1, 1)$	$(1, 2)$	-2
	$(0, 0, 1, \infty, 1, 1, \infty, \dots)$	$(1, 1, 1, 1, 2, 2, 1, 1, 1, 1, 3, 1, 1, 2)$	$(-1, 0)$	0
	$(\infty, \infty, \infty, \infty, \infty, 0, \dots)$	$(1, 1, 2, 1, 1, 3, 2, 1, 1, 1, 5, 1, 2, 2)$	$(2, -22)$	11
	$(1, \infty, 0, \infty, 1, 0, 1, \dots)$	$-(3, 1, 6, 2, 1, 1, 1, 2, 1, 2, 2, 2, 2, 1)$	$(-2, 22)$	11

8. Other application

Finite surgery

Let K be a knot. Let γ be a simple closed curve on $\partial Nbd(K)$. The manifold $K(\gamma)$ oabtained by Dehn surgery along γ is called *finite* if $\pi_1(K(\gamma))$ is finite and then γ is called a *finite slope*.

Culler-Shalen norm

Let $\gamma \in H_1(\partial N, \mathbb{Z})$. *Culler-Shalen* norm measures the degree of the map $\mathrm{tr}(\gamma) : \widetilde{X(M)} \to \mathbb{C}P^1$. We have the following formula:

$$||\gamma|| = \sum_{p:\text{ideal point}} v_p(\mathrm{tr}(\gamma)) \qquad \cdots\cdots(a)$$

It is known that the norms for finite slopes are small.

Theorem(Futer, Ishikawa, K., Mattman, Shimokawa)

Let K be a $(-2, p, q)$ pretzel knot with p, q odd and $5 \leq p \leq q$. Then K admits no non-trivial finite surgery.

Outline of the proof

Most cases of the proof are shown by using 6-theorem (or 2π-theorem) and group theoretic argument. But these methods are not sufficient for $(-2, 5, 5)$ and $(-2, 5, 7)$. We use Culler-Shalen norm for these cases.

Let $S = \min\{||\gamma|| \mid \gamma$: non-trivial slope$\}$. In our case, if γ is finite unless γ is even and $||\gamma|| \leq S + 8$, or $||\gamma|| \leq 2S$ (Boyer-Zhang). Find ideal points using an ideal triangulation. Then we can estimate the Culler-Shalen norm by using (a). Then we apply Boyer-Zhang theorem.

Thank you!