JSJ decompositions of toroidal 3-manifolds obtained by Dehn surgeries on pretzel knots

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joint with I D Jong

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Introduction

K : a knot in S^3

$E_K = S^3 - N^\circ(K)$: the exterior of K

A slope: an isotopy class of a simple closed curve on ∂E_K.

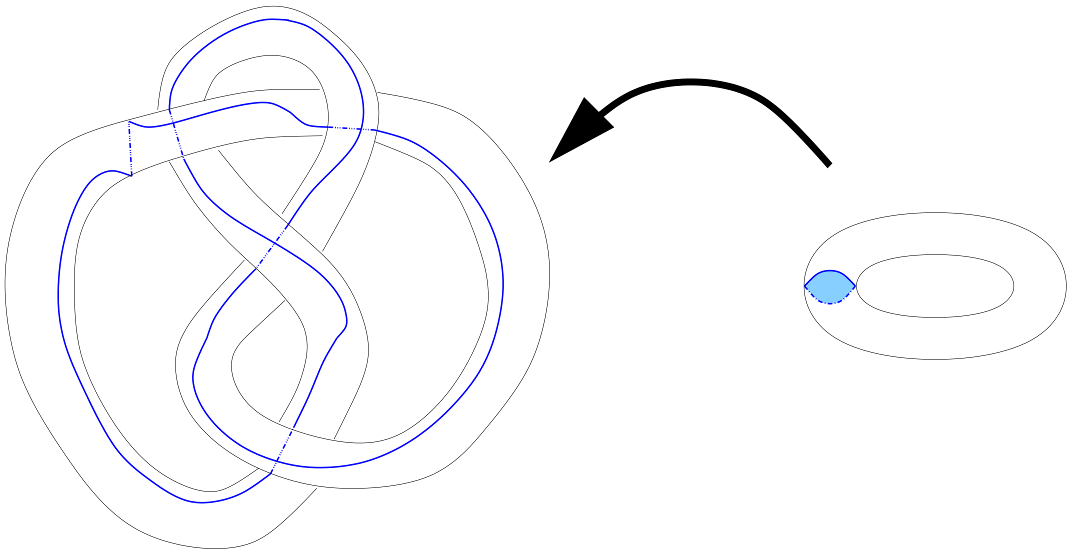
We identify slopes on ∂E_K with irreducible fractions by

$$H_1(\partial E_K, \mathbb{Z})/\pm \ni \pm(p[m] + q[l]) \longleftrightarrow p/q \in \mathbb{Q} \cup \{1/0\}$$

(m :meridian, l: longitude)

Dehn surgery

The *Dehn surgery* along p/q is an operation of attaching a solid torus to E_K so that the meridian of the solid torus is attached along the slope p/q. Denote the obtained mfd by $K(p/q)$



Example $K(1/0) = S^3$

Exceptional surgery

A knot K is called *hyperbolic* if E_K has a complete finite volume hyperbolic metric. If K is hyperbolic, it is known that $K(p/q)$ has hyperbolic metric except finite number of slopes (Thurston).

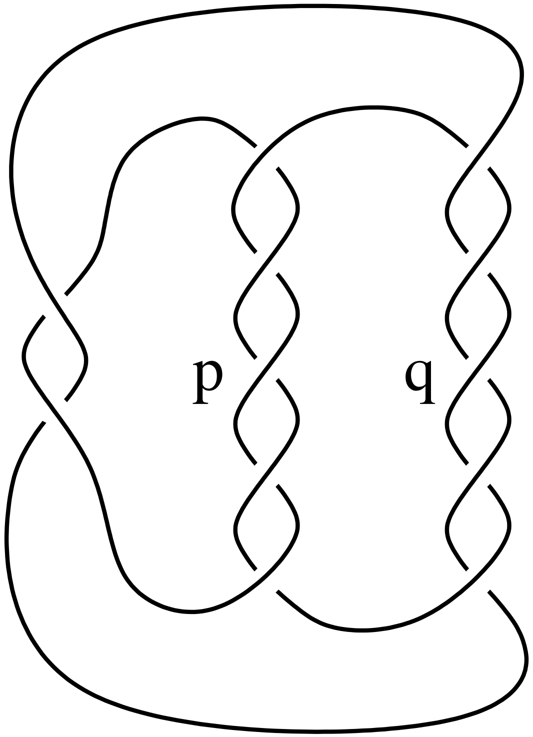
By the geometrization theorem, there are three types of exceptional surgeries:

- *toroidal* if $K(p/q)$ contains an incompressible torus,
- *Seifert* if $K(p/q)$ has a Seifert fibered structure,
- *reducible* if $K(p/q)$ contains a sphere which does not bound an embedded 3-ball.

Today we focus on toroidal surgeries.

Toroidal surgery of $(-2, p, q)$-pretzel knots

The $(-2, p, q)$-pretzel knot has a toroidal surgery.



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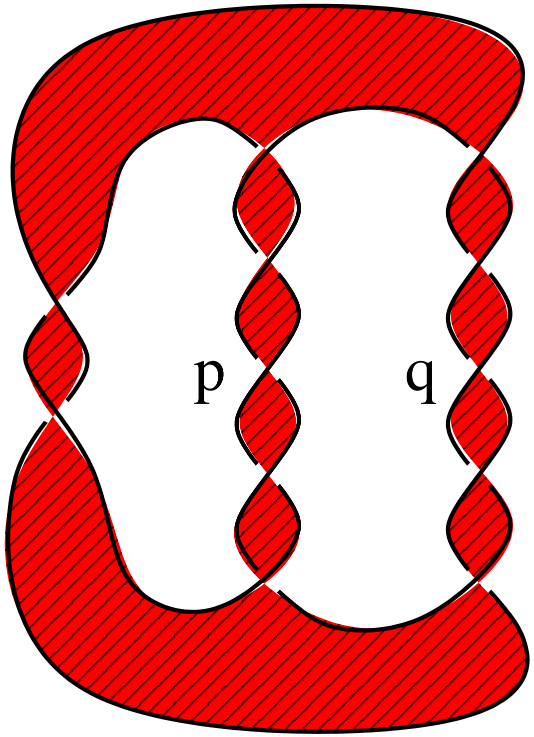


The red surface is diffeo to a once punctured Klein bottle. The boundary slope of this surface is $2(p+q)$.

Consider Dehn surgery along $2(p+q)$-slope.

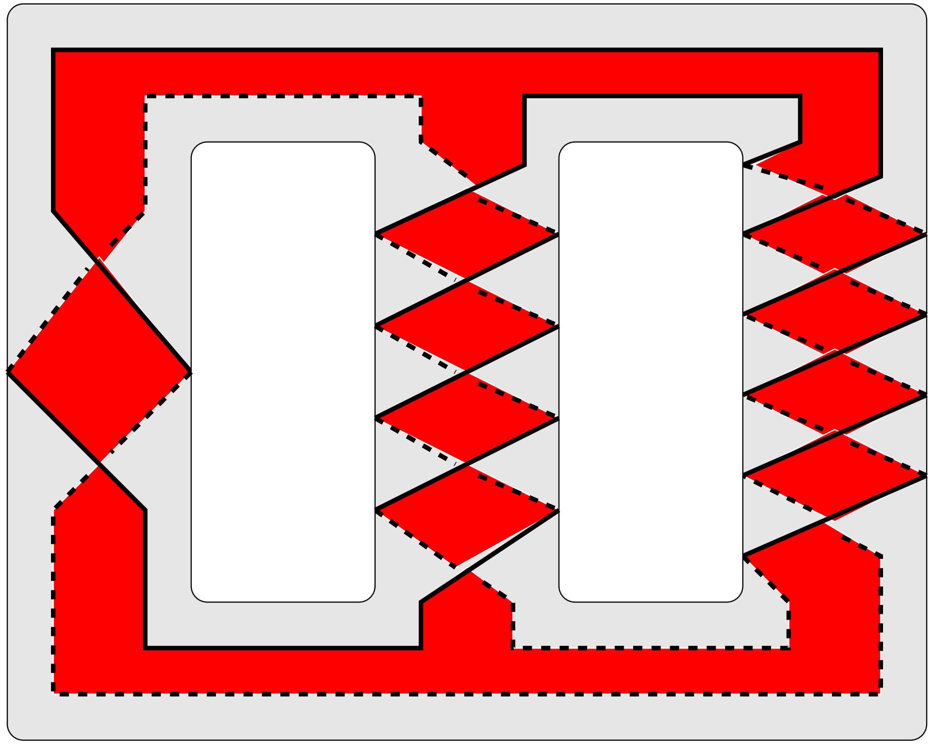
The nbd of the red surface is diffeo to

(I-bundle over a once-punctured Klein bottle)

After Dehn surgery, it becomes

(I-bundle over a Klein bottle)

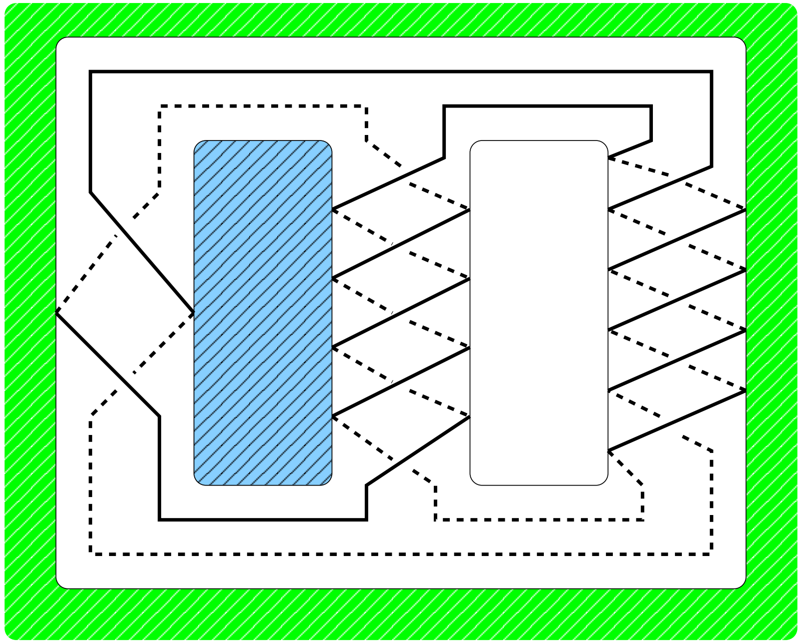
The boundary of this manifold is a torus. That is the incompressible torus.



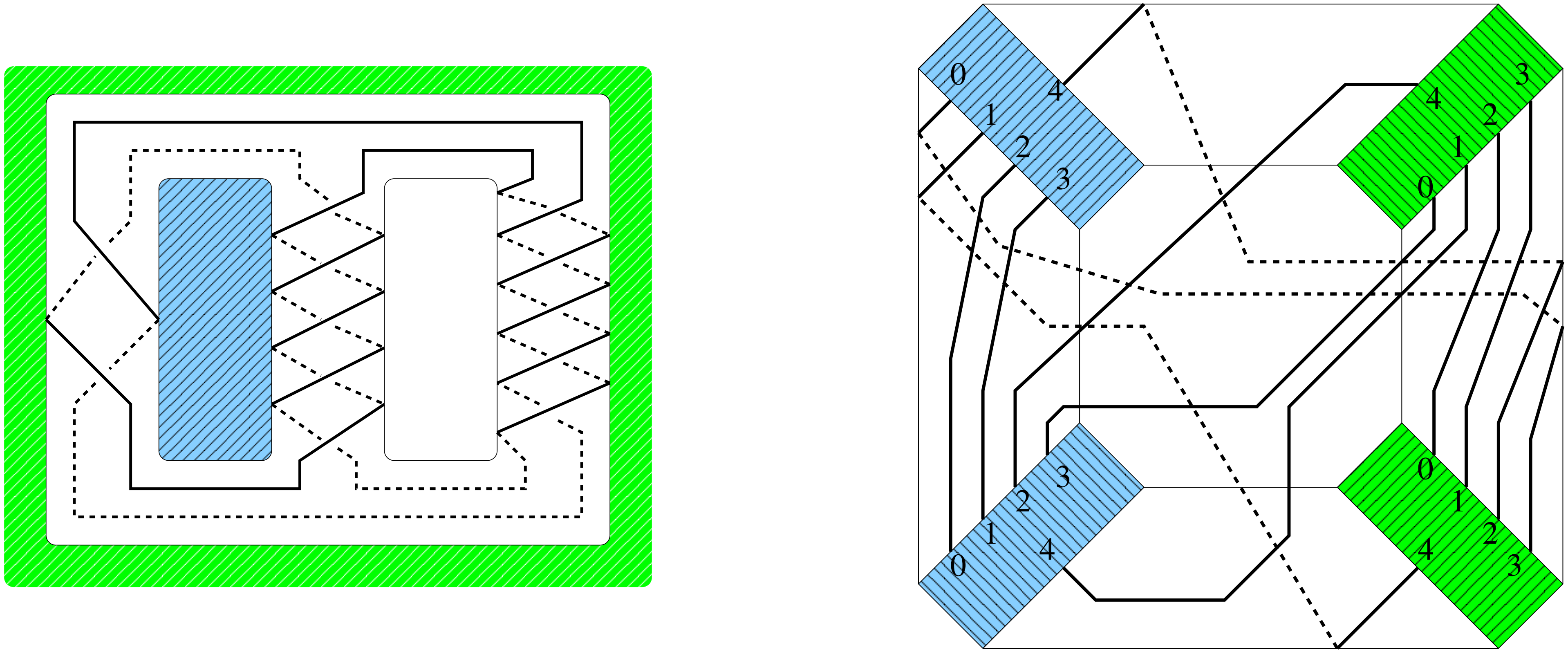
How is the counterpart?

The counterpart is obtained by 2-handle addition along the knot to the "outside".

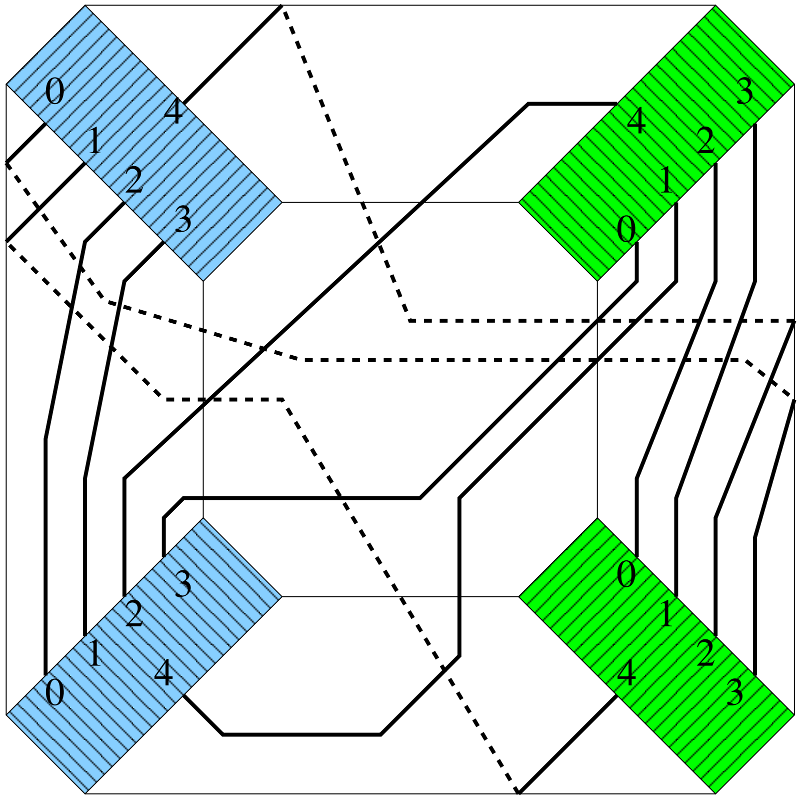
Heegaard diagram of the outside

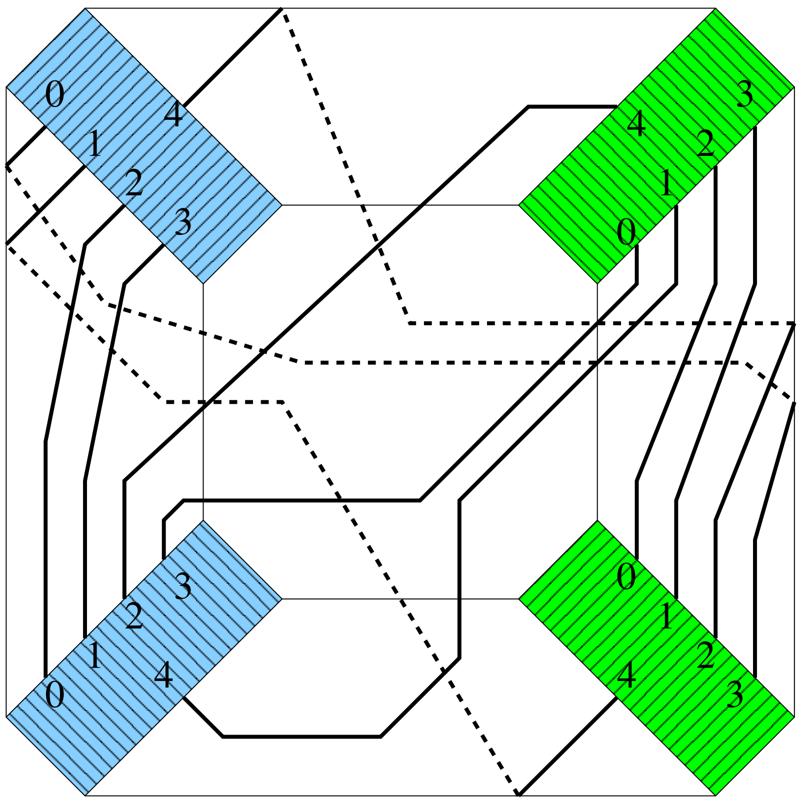


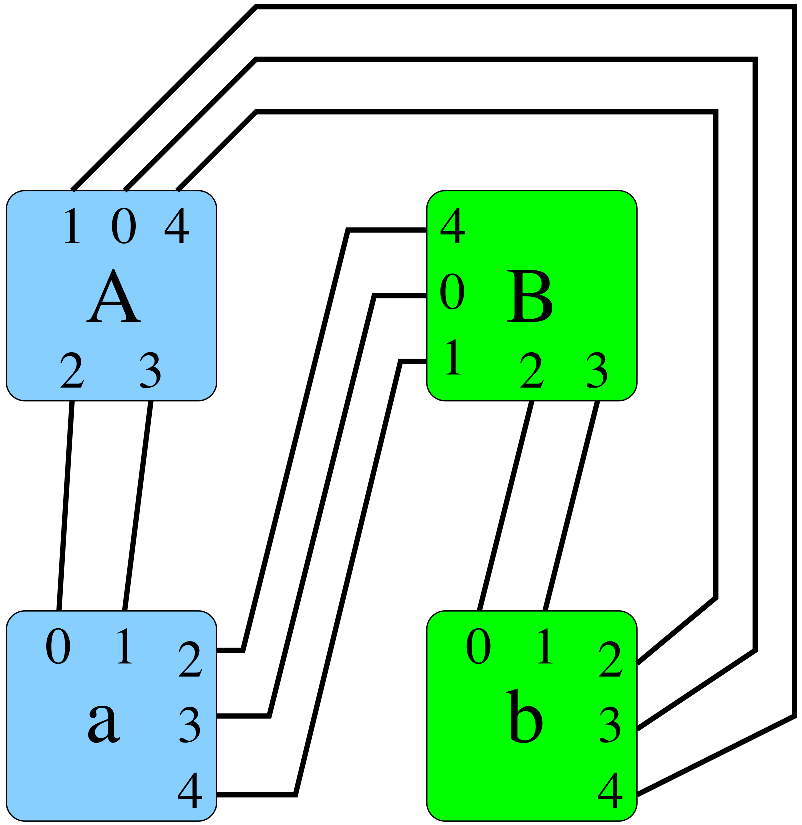
Heegaard diagram of the outside



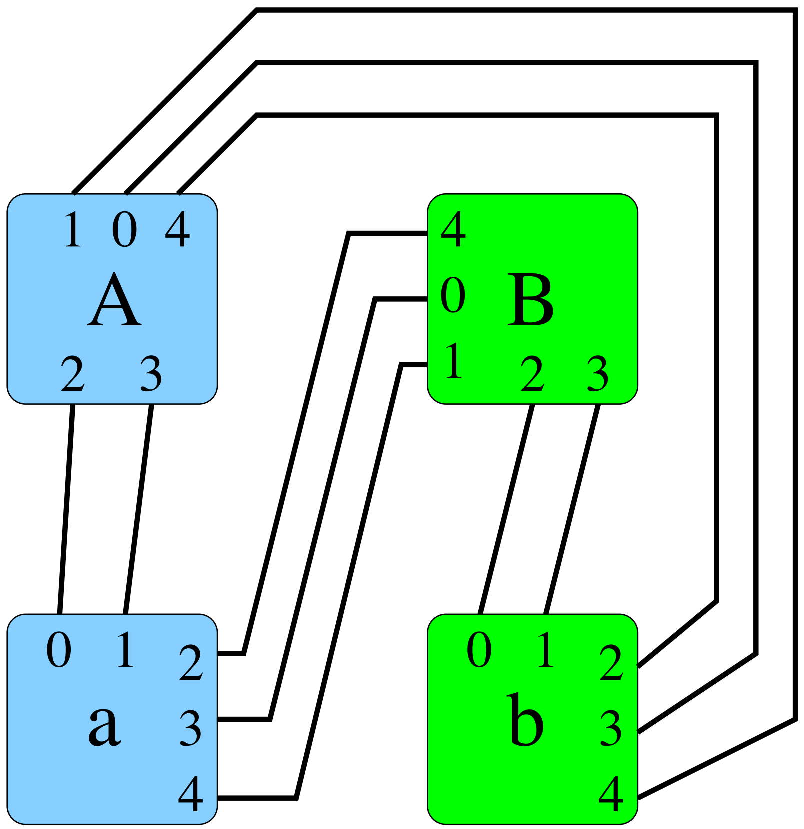
Cut along the blue disk and the green disk. (These are meridian disks of the Heegaard diagram.)

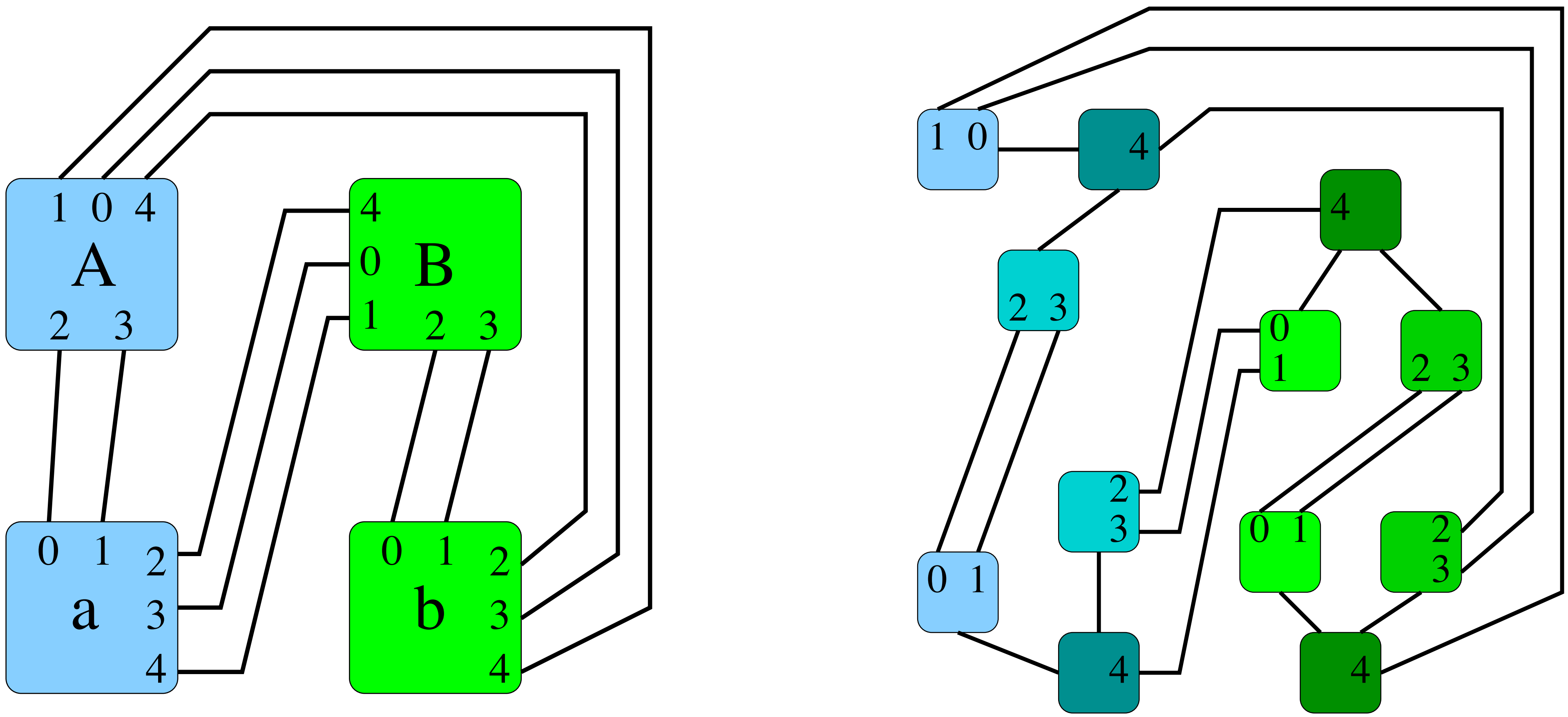

0
4
1
2
3
3
4
2
1
0
3
2
1
4
0
0
1
4
2
3



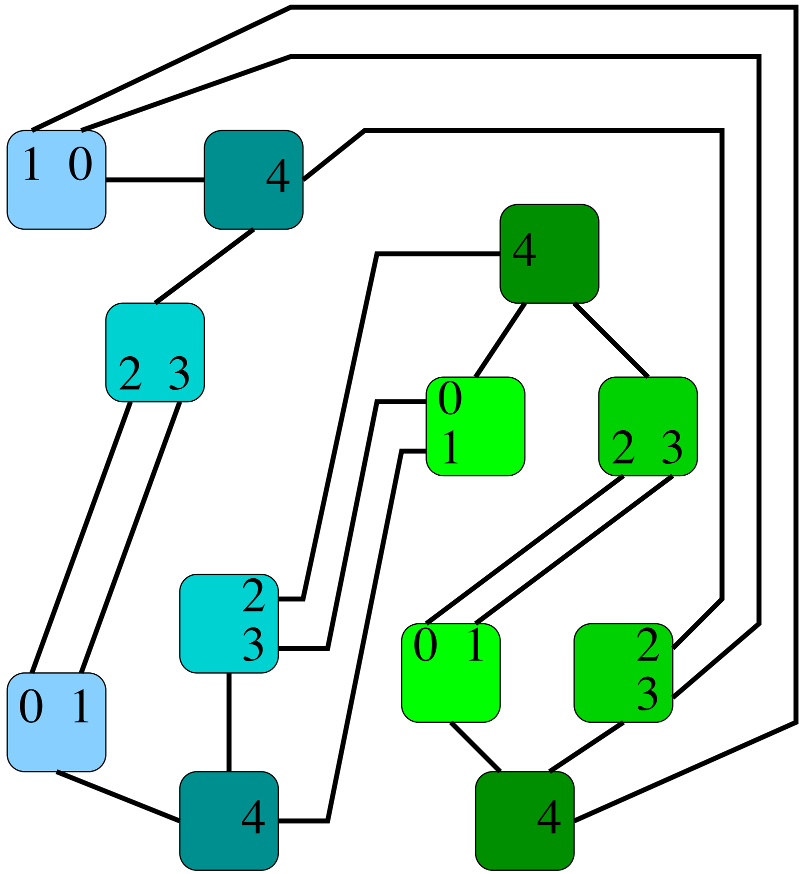


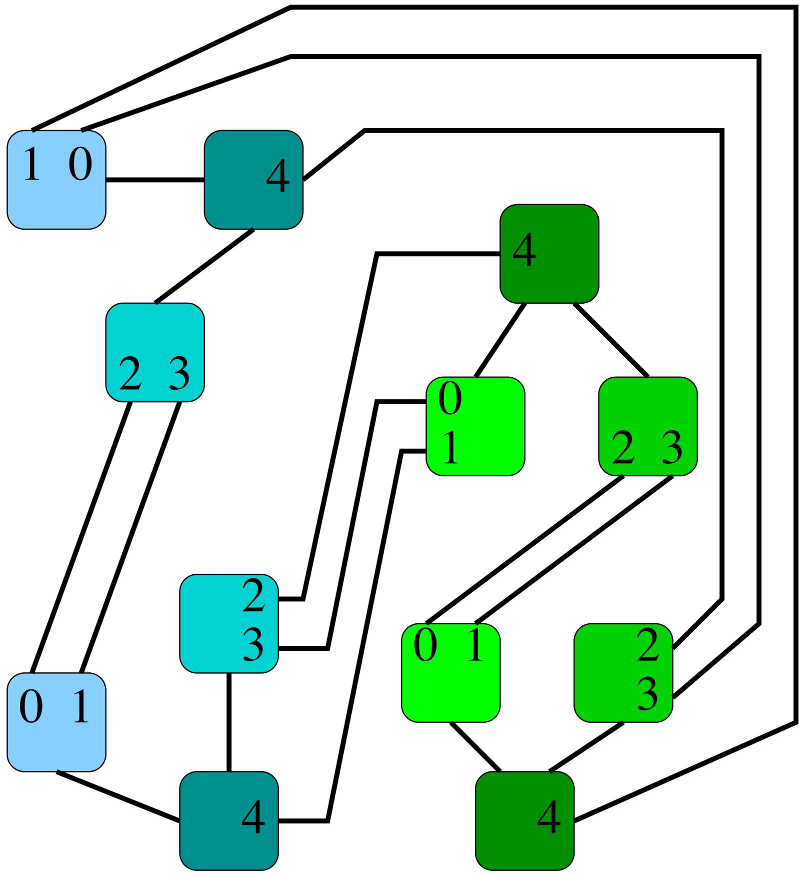
Deform the diagram.

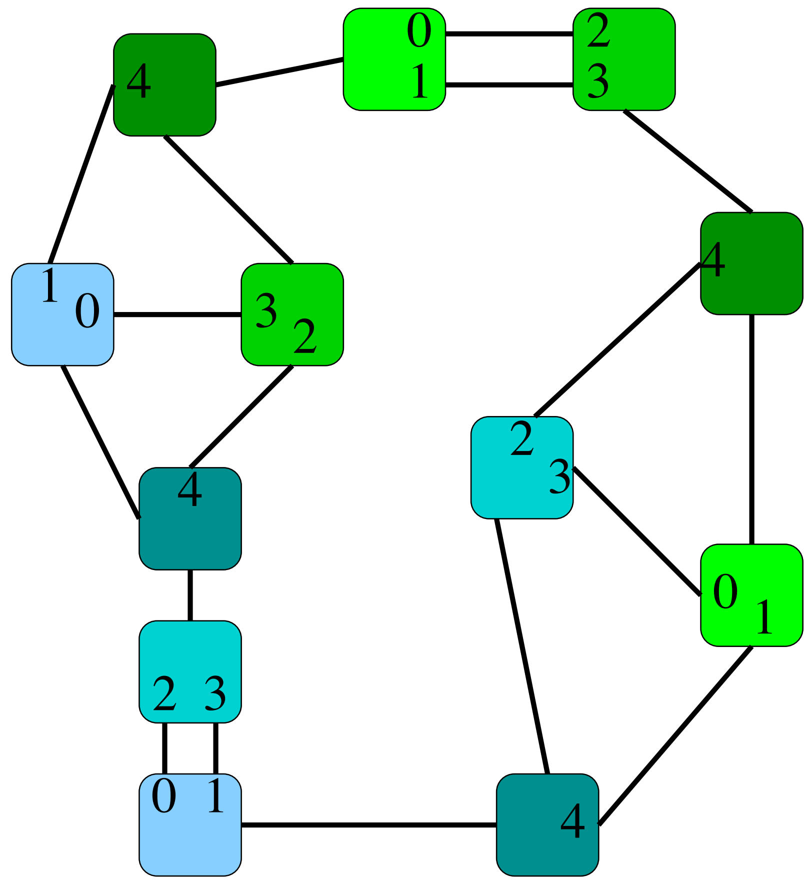

1 0 4
A
2 3
4
0
B
1 2 3
0 1 2
a
3
4
0 1 2
b
3
4



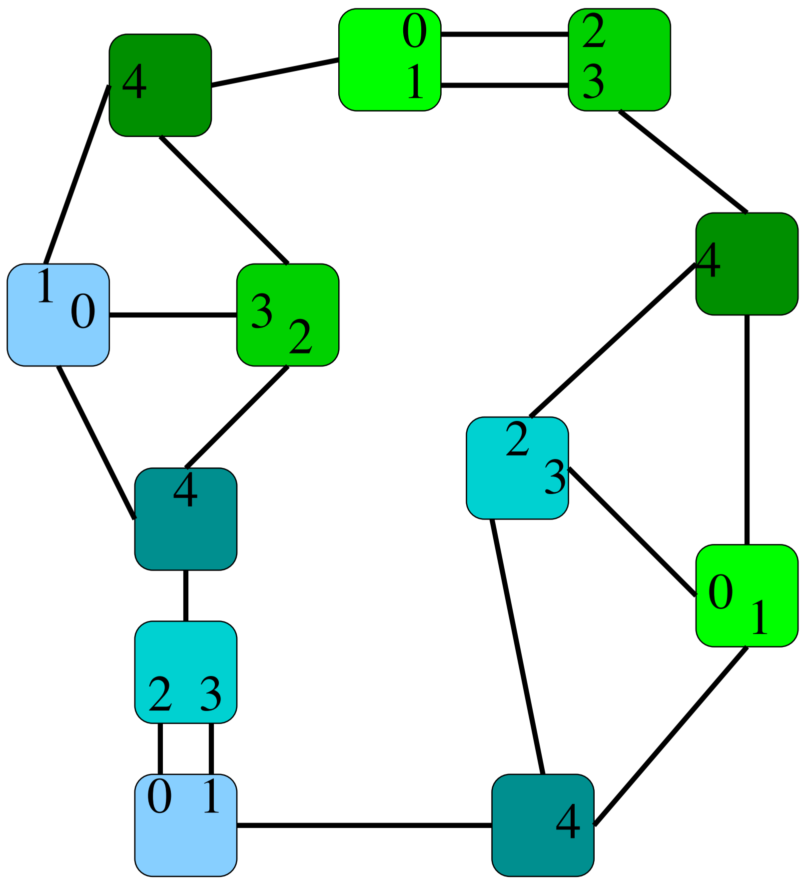
"Triangulate" the diagram. (Make the diagram to be a trivalent graph keeping the gluing pattern compatible.)

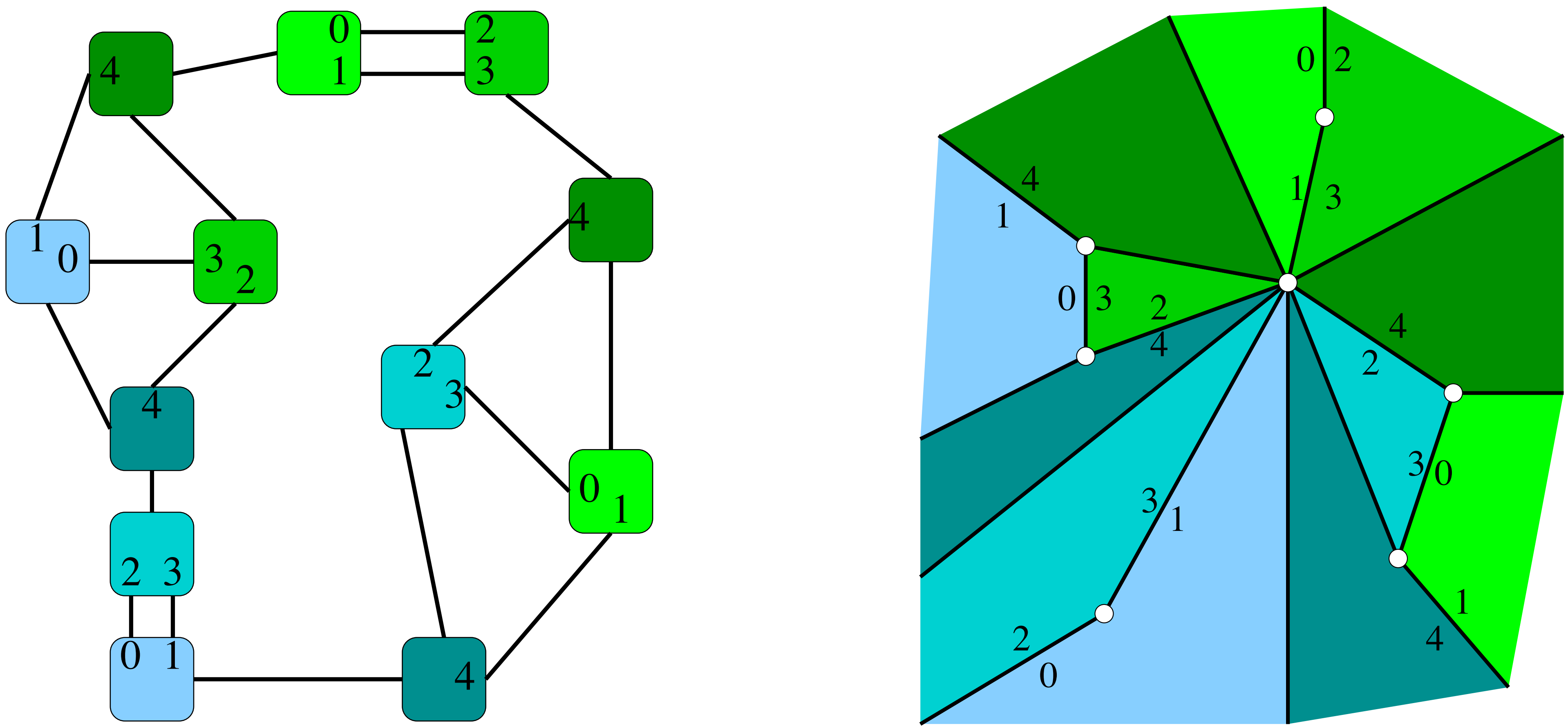

1 0
4
4
2 3
0 1
2 3
2 3
0 1
2 3
0 1
4
4





Deform the “triangulated” diagram.


4
0
1
2
3
1
0
3
2
4
4
2
3
0
1
2
3
0
1
4



Taking the dual of the graph, we obtain a polyhedral decomposition with all faces are triangles. Then subdivide the polyhedron into tetrahedra. This data can be used in SnapPea.

We study the manifold by using SnapPea

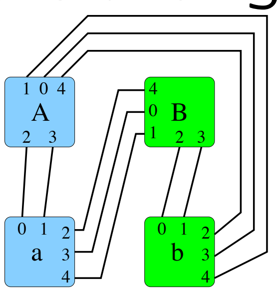
1. Simplify triangulation,

2. Study whether it has a hyperbolic structure,

If the manifold has a hyperbolic structure, we further study

3. Compute the canonical triangulation

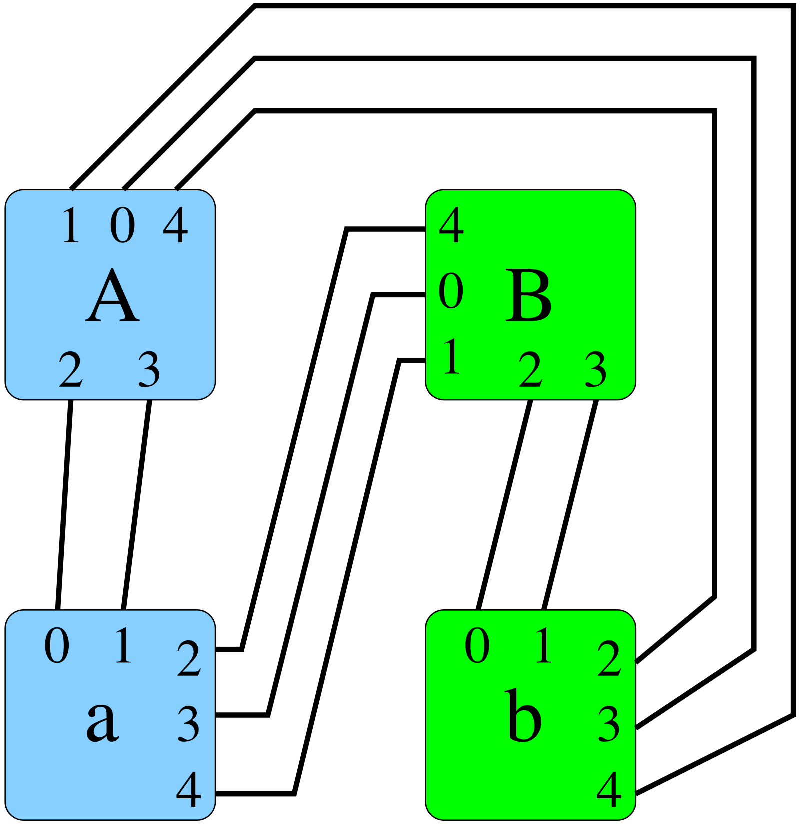
4. Compare with known manifolds (eg. census manifolds).

I wrote a program which produce SnapPea's triangulation data from a Heegaard diagram in the form: 

. By using this program, we study the "outside" of the toroidal surgery.

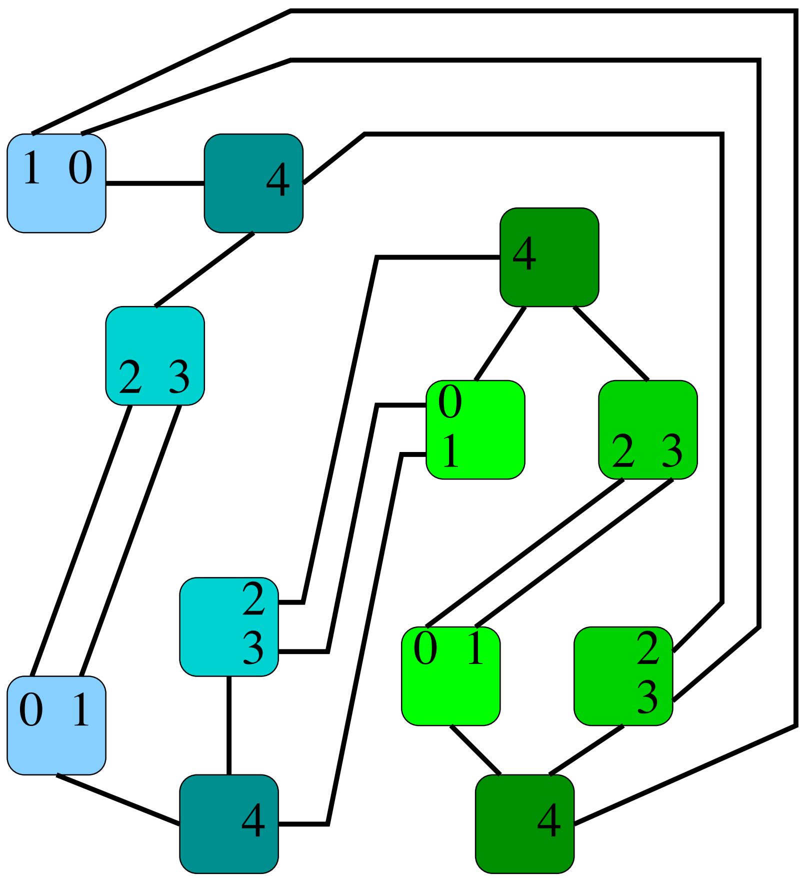
What the program carry out

0. Input data




```
a: AABBB
   23401
A: BBAAB
   34012
b: BBAAA
   23401
B: aaBBa
   34012
```

1. Construct a triangulation



SnapPea's triangulation data

6	8	7	1	0	5	7	5
0132	0132	0213	0132	0132	0321	0132	0321
7	11	0	2	1	0	8	6
0132	0213	0132	0132	0132	0213	0132	0132
8	9	1	9	2	0	4	7
0132	0321	0132	0321	0132	0132	0213	0132
9	11	10	4	3	2	10	2
0132	0132	0213	0132	0132	0321	0132	0321
10	8	3	5	4	3	11	9
0132	0213	0132	0132	0132	0213	0132	0132
11	6	4	6	5	3	1	10
0132	0321	0132	0321	0132	0132	0213	0132

(In the case of above figure, there are 12 tetrahedra.)

2. Simplify triangulation and compute hyp str

Using `remove_finite_vertices()` and `basic_simplification()`, we reduce the number of tetrahedra.

```
   1    1    1    1
 1023 2031 0132 1302
   0    0    0    0
 1302 1023 2031 0132
   0    0    0    0
```

2. Simplify triangulation and compute hyp str

Using `remove_finite_vertices()` and `basic_simplification()`, we reduce the number of tetrahedra.

```
   1    1    1    1
1023 2031 0132 1302      0.500000000000 + 0.866025403784 I
   0    0    0    0
1302 1023 2031 0132      0.500000000000 + 0.866025403784 I
   0    0    0    0
```

By using `find_complete_hyperbolic_structure()`, we can compute the shape of tetrahedra.

Results of calculation

pretzel	census name	hyp. volume of the mfd
$(-2, 5, 5)$	$m003$	2.0298..
$(-2, 5, 7)$	$m019$	2.9441..
$(-2, 5, 9)$	$m044$	3.2756..
$(-2, 5, 11)$	$m072$	3.4245..
$(-2, 5, 13)$	$s011$	3.5023..
$(-2, 5, 15)$	$v0011$	3.5477..
$(-2, 7, 7)$	$m159$	3.8216..
$(-2, 7, 9)$	$m230$	4.1487..
$(-2, 7, 11)$	$s190$	4.3000..
$(-2, 7, 13)$	$v0354$	4.3810..
$(-2, 9, 9)$	$s309$	4.4769..
$(-2, 9, 11)$	$v0642$	4.6301..

Note $m003$ is known as figure-eight sister mfd.

Digression: volume of $\mathrm{PSL}(2,\mathbb{C})$-representation at infinity

Let $\mathbb{H}^3$ be the hyperbolic 3-space and ω be the volume form. Let M be a closed 3-manifold and $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. Take an equivariant map $D : \widetilde{M} \to \mathbb{H}^3$. The pull-back of the volume form $D^*\omega$ reduces to a 3-form on M. We define

$$Vol(\rho) = \int_M D^*\omega \in \mathbb{R}$$

This dose not depend on the choice of D. For manifold with torus boundary, we can also define volume of a $\mathrm{PSL}(2,\mathbb{C})$-representation by giving a boundary condition.

The volume of a representation can be easily calculated by using SnapPea.

By Culler-Shalen theory, it is known that there is an incompressible surface corresponding to an ideal point of $\mathrm{PSL}(2,\mathbb{C})$-representation space. (But it is not known that there is an ideal point corresponding to a given incompressible surface.) In our case, the incompressible torus actually corresponds to an ideal point. We study the limit of volume as representations approaching to the ideal point.

pretzel	census name	volume of the mfd	vol at ideal pt
$(-2,5,5)$	$m003$	2.0298..	2.0298..
$(-2,5,7)$	$m019$	2.9441..	2.9441..
$(-2,5,9)$	$m044$	3.2756..	3.2756..
$(-2,5,11)$	$m072$	3.4245..	3.4245..
$(-2,5,13)$	$s011$	3.5023..	3.5023..
$(-2,5,15)$	$v0011$	3.5477..	3.5477..
$(-2,7,7)$	$m159$	3.8216..	3.8216..
$(-2,7,9)$	$m230$	4.1487..	4.1487..
$(-2,7,11)$	$s190$	4.3000..	4.3000..
$(-2,7,13)$	$v0354$	4.3810..	4.3810..
$(-2,9,9)$	$s309$	4.4769..	4.4769..
$(-2,9,11)$	$v0642$	4.6301..	4.6301..

This calculation shows that the volume at an ideal point seems to be equal to the Gromov norm of the surgered manifold. This is a motivation of this research.

Problem

Does the toroidal surgery of the $(-2, p, q)$-pretzel knot produce hyperbolic manifold for any $p \geq q \geq 5$?

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Theorem

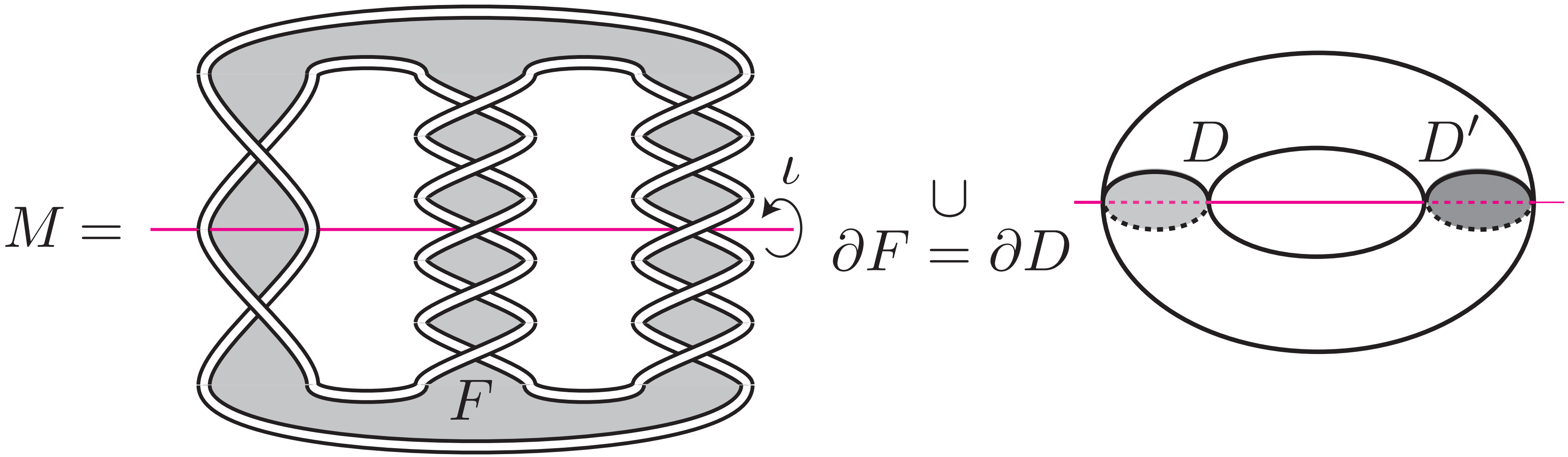
The toroidal surgery of the $(-2, 5, 2n + 3)$-pretzel knot produces hyperbolic manifold for any $n \geq 1$.

$K := P(-2, 5, 5)$, $M := K(20)$.

F: once punctured Klein bottle.

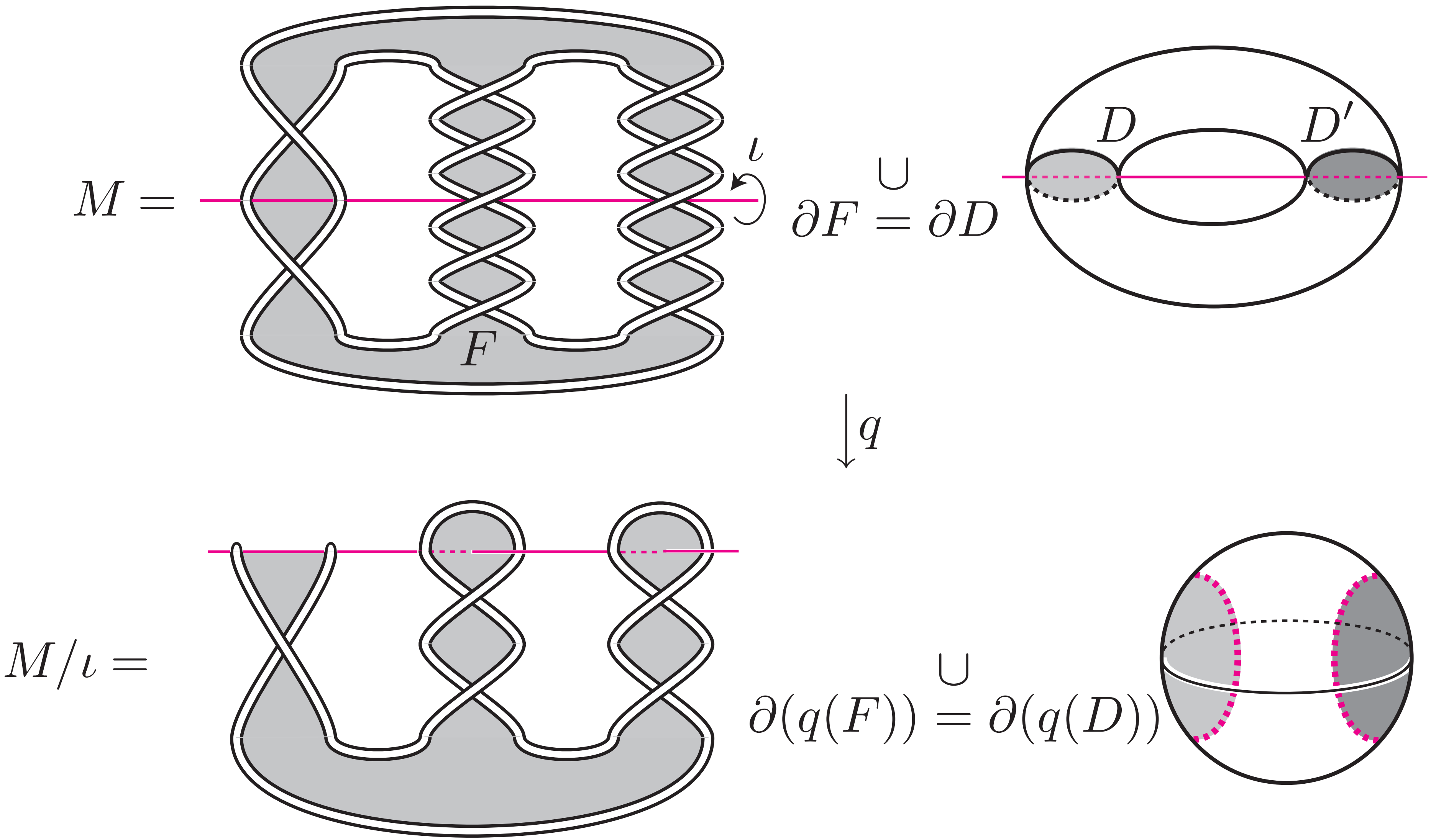
($\widehat{F} = F \cup D$: Klein bottle in $K(20)$)

We show that $M - N^\circ(\widehat{F})$ is the figure-eight sister.



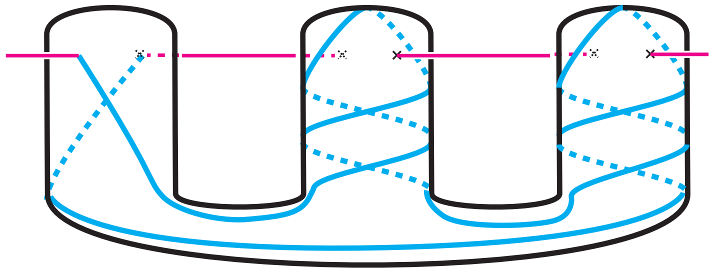
$\partial D'$ runs parallel with ∂D on the torus.

Take a quotient q w.r.t. the involution ι.

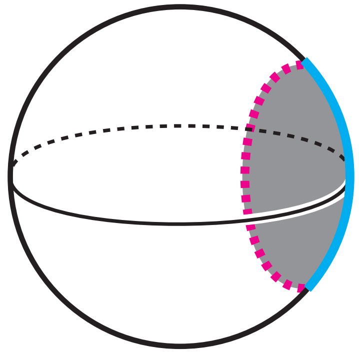


Remove an open nbd. of $q(\widehat{F})$ from M/ι.

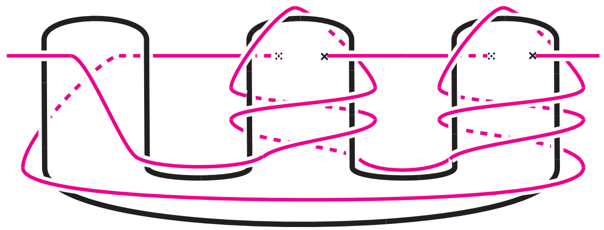
$(M/\iota) - N^\circ(q(\widehat{F})) =$



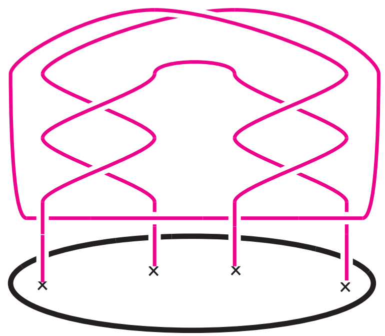
$\cup$



$=$

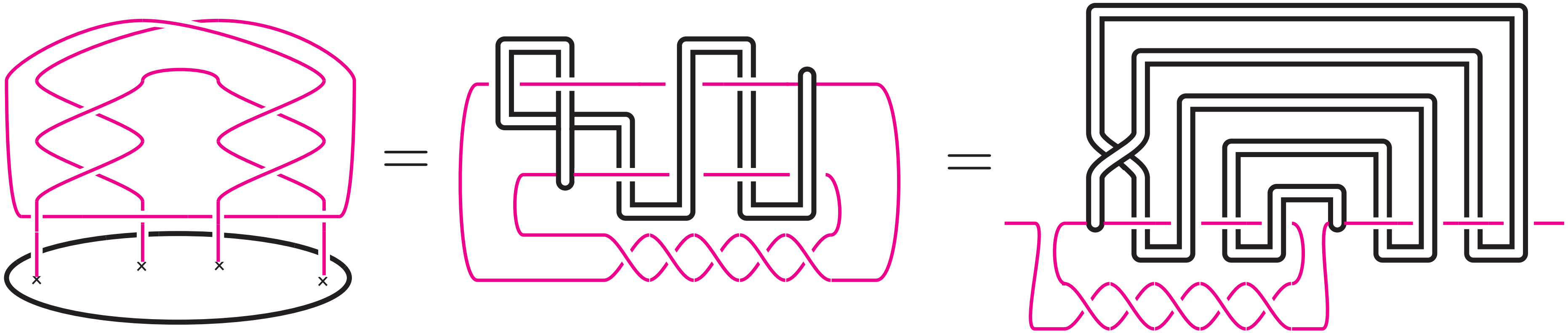


$=$



Deforming the tangle by isotopies as follows.

Note that the tangle represents $(M/\iota) - N^{\circ}(q(\widehat{F}))$.



Then $M - N^{\circ}(\widehat{F})$ is the exterior of K' contained in the lens space $L(5, 1)$.

Applying Kirby move, we see that $M - N^{\circ}(\widehat{F})$ is the figure-eight sister. $\square$