Quandleによるshadow coloringと PSL(2,C)表現の体積と Chern-Simons不変量

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Introduction

M : an oriented closed 3-manifold

$\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$: a rep. of the fund. group of M

$\mathrm{Vol}(M, \rho) \in \mathbb{R}$ and $\mathrm{CS}(M, \rho) \in \mathbb{R}/\pi^2\mathbb{Z}$ are invariants of the representation ρ.

When ρ is a discrete faithful rep. of a hyperbolic mfd M, then Vol and CS are the volume and the Chern-Simons invariant of the hyperbolic metric.

The definition of Vol and CS are generalized to the case of manifolds with torus boundary e.g. knot complements.

A formula of $i(\mathrm{Vol} + i\mathrm{CS}) \in \mathbb{C}/\pi^2\mathbb{Z}$ was given by Neumann in terms of triangulations of 3-manifolds.

We give a formula in terms of knot diagrams by using the quandle formed by parabolic elements of $\mathrm{PSL}(2,\mathbb{C})$.

Quandle str. on $\mathbb{C}^2 \setminus \{0\}$

Define a binary operation $*$ on $\mathbb{C}^2 \setminus \{0\}$ by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} * \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} 1 - x_2 y_2 & -x_2^2 \\ y_2^2 & 1 + x_2 y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This satisfies the quandle axioms:

1. $x * x = x$ for $x \in \mathbb{C}^2 \setminus \{0\}$
2. The inverse of $*y : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$ is given by

$$*^{-1}y : \begin{pmatrix} 1 + x_2 y_2 & x_2^2 \\ -y_2^2 & 1 - x_2 y_2 \end{pmatrix}$$

3. $(x * y) * z = (x * z) * (y * z)$

$\mathcal{P}$: the set of parabolic elements of $\mathrm{PSL}(2,\mathbb{C})$

($\cong$ the set of parabolic elements of $\mathrm{SL}(2,\mathbb{C})$ with trace 2)

$\mathcal{P}$ has a quandle str. by $x * y = y^{-1}xy$.

Define a map $\mathbb{C}^2 \setminus \{0\} \xrightarrow{2:1} \mathcal{P}$ by

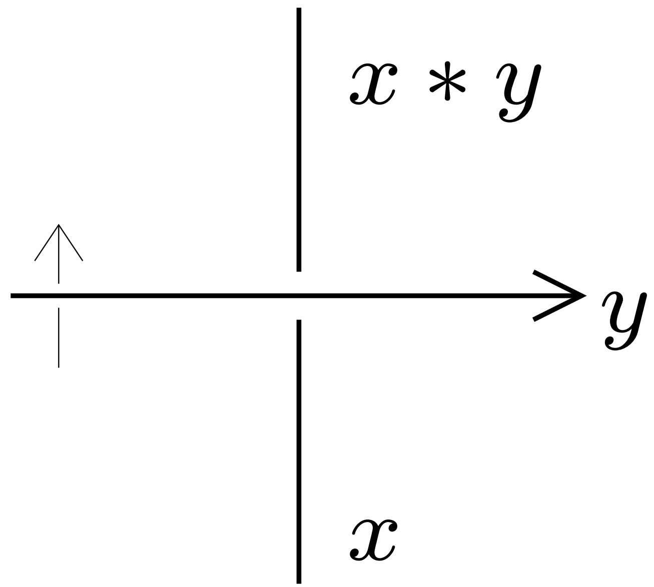
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - xy & -x^2 \\ y^2 & 1 + xy \end{pmatrix}$$

This map induces a quandle isomorphism $\mathcal{P} \cong (\mathbb{C}^2 \setminus \{0\})/\pm$

Arc coloring by $(\mathbb{C}^2 \setminus \{0\})/\pm$

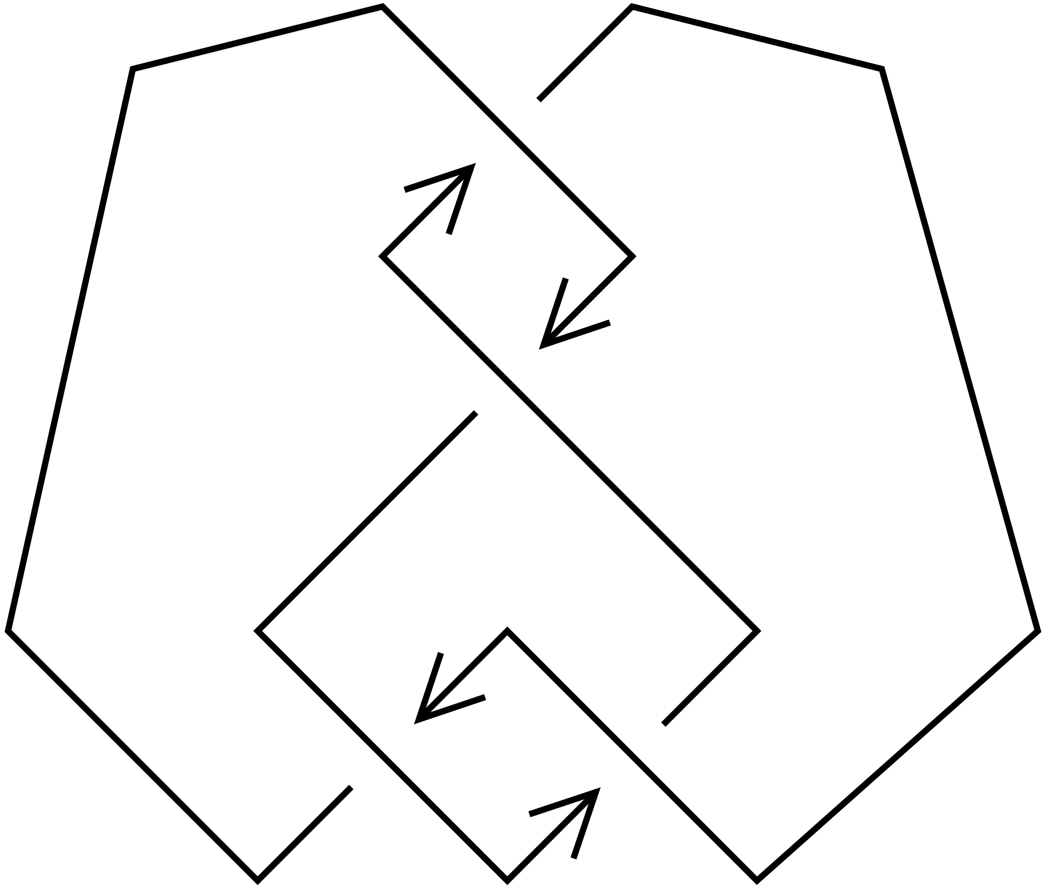
Let D be a diagram of a knot.

A map $\mathcal{A} : \{\text{arcs of } D\} \to (\mathbb{C}^2 \setminus \{0\})/\pm$ is called an *arc coloring* if it satisfies the following relation at each crossing.



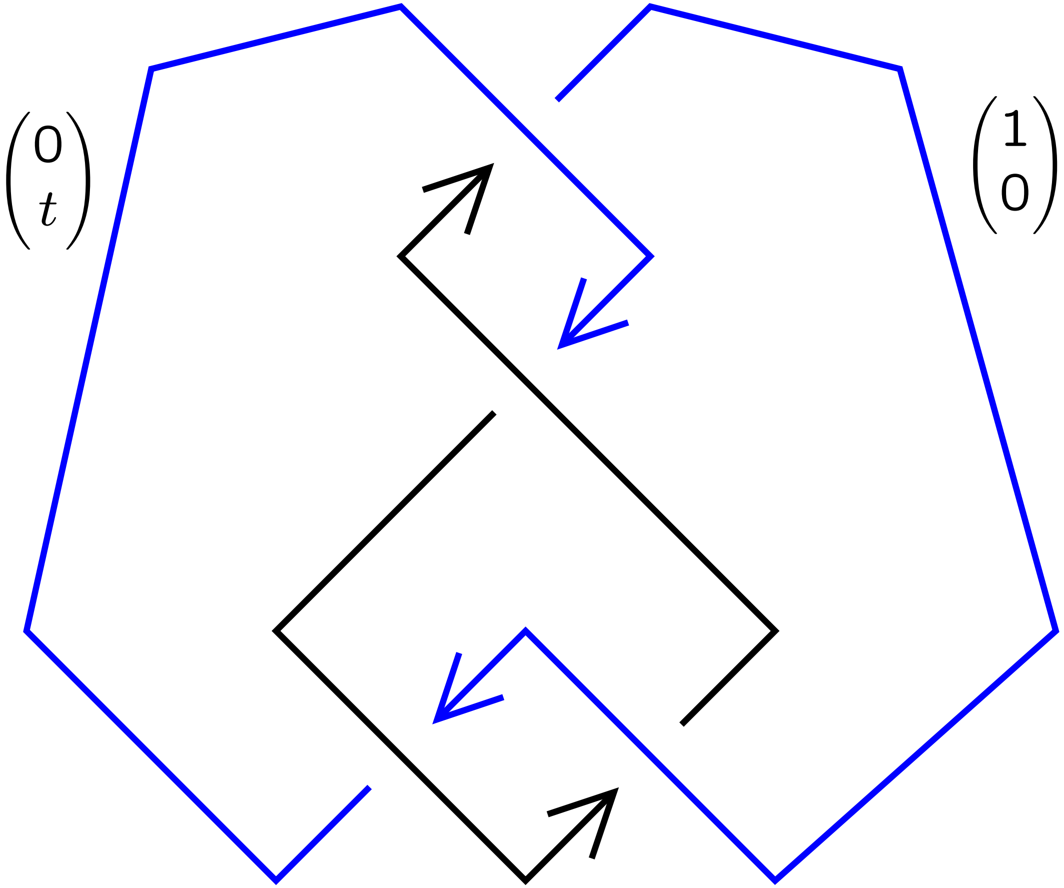
x, y and $x * y \in (\mathbb{C}^2 \setminus \{0\})/\pm$

Arc coloring of the figure eight knot



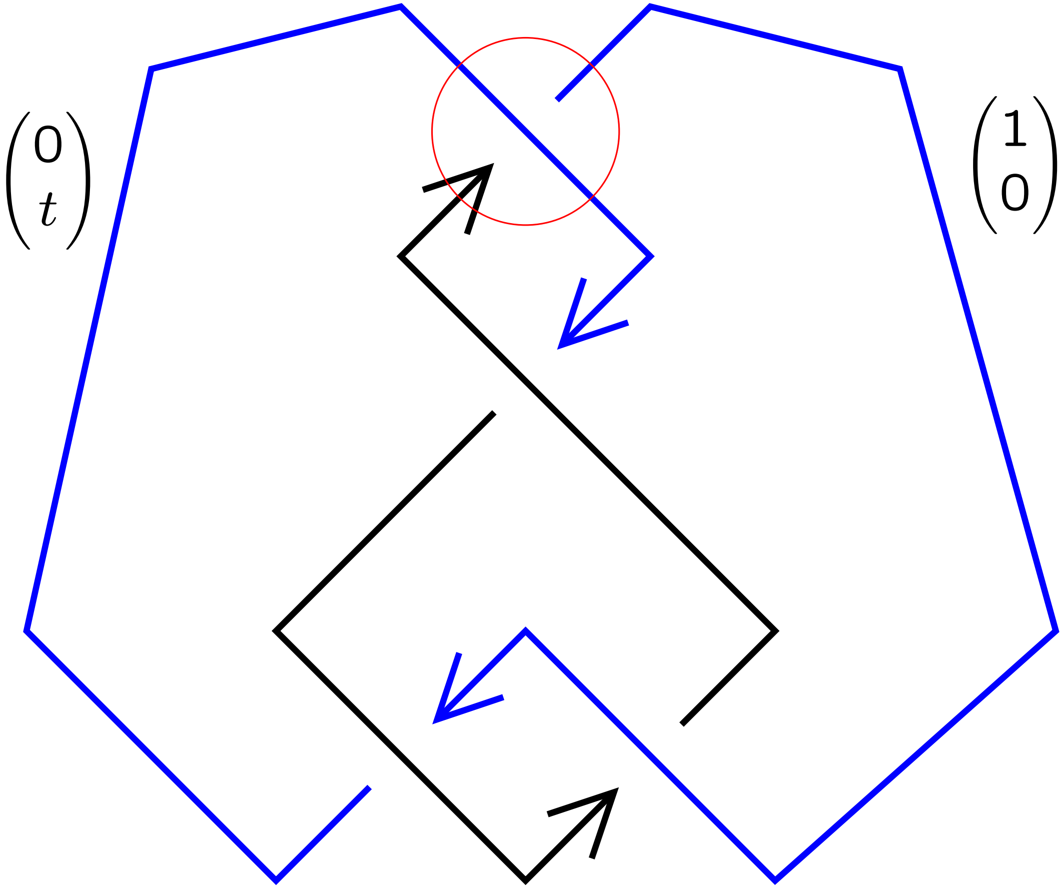
This is the figure eight knot.

Arc coloring of the figure eight knot



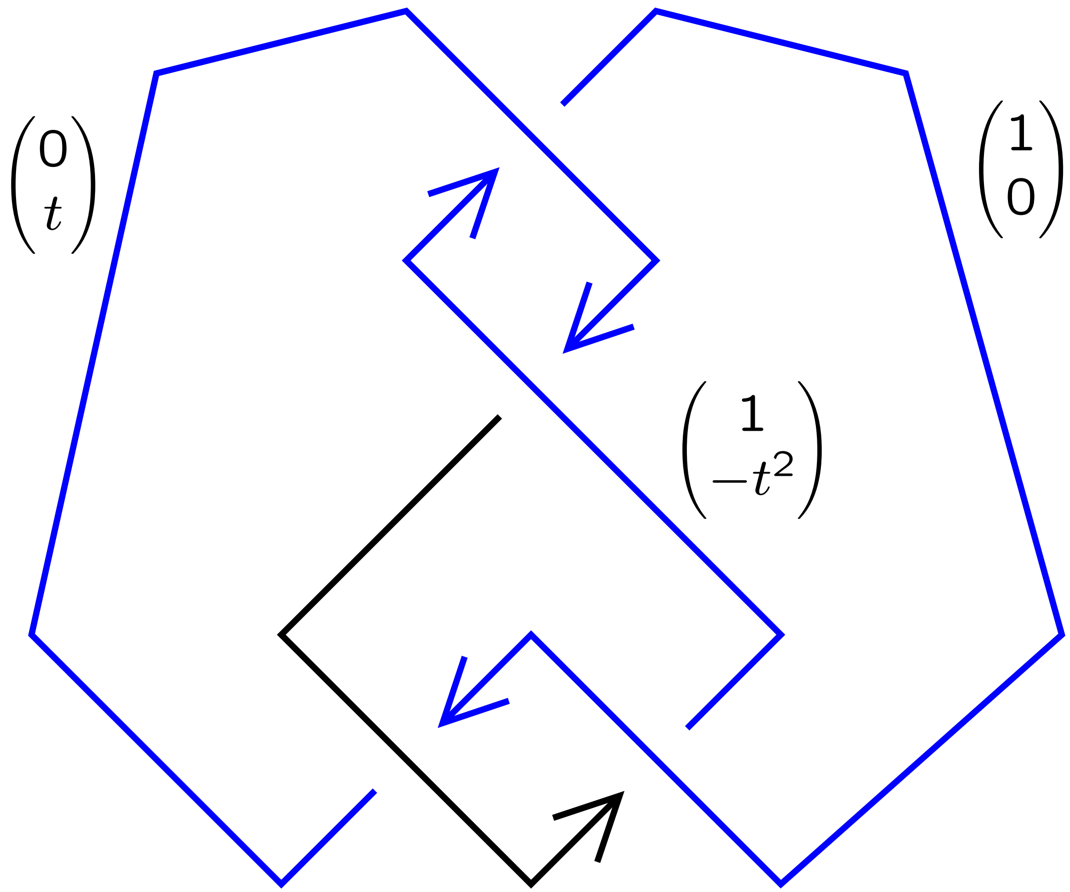
Color two arcs.

Arc coloring of the figure eight knot



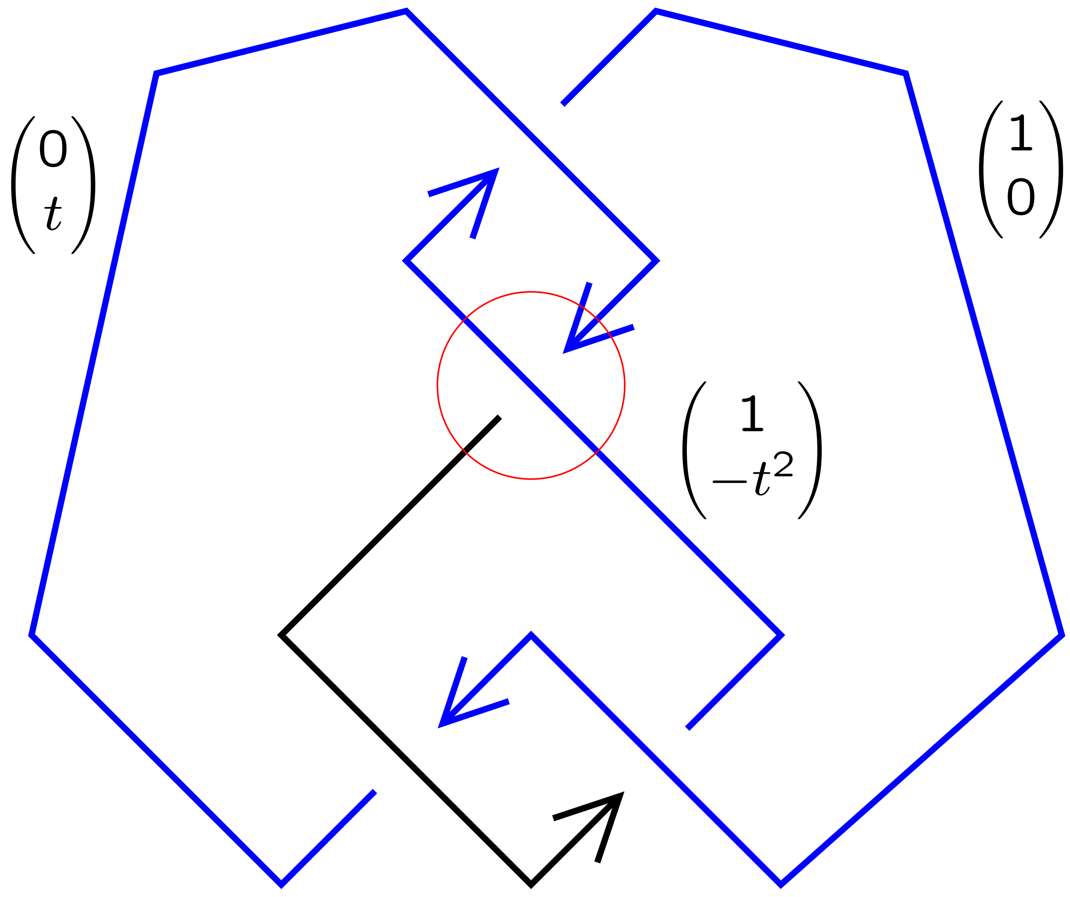
Consider the relation at a crossing.

Arc coloring of the figure eight knot



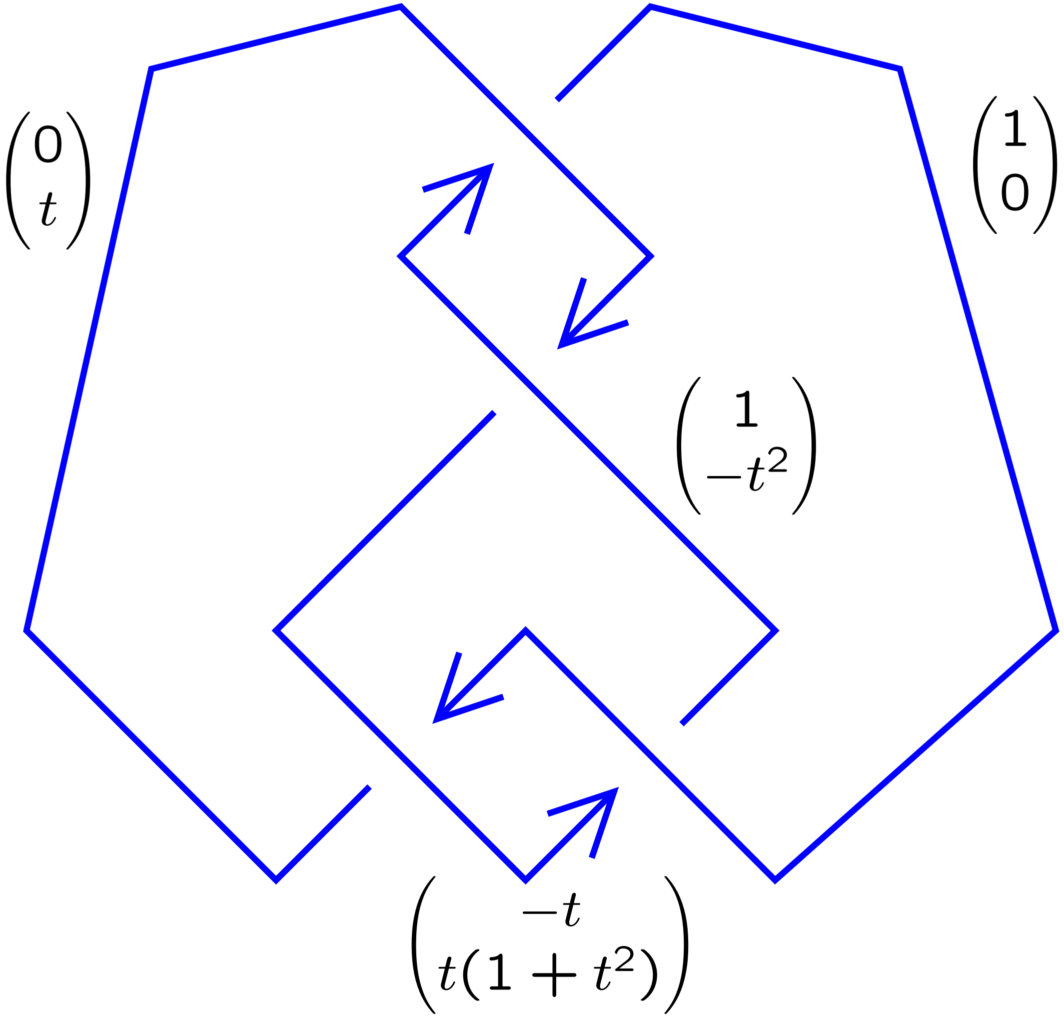
$$\begin{pmatrix}1\\0\end{pmatrix} *^{-1} \begin{pmatrix}0\\t\end{pmatrix} = \begin{pmatrix}1\\-t^2\end{pmatrix}$$

Arc coloring of the figure eight knot



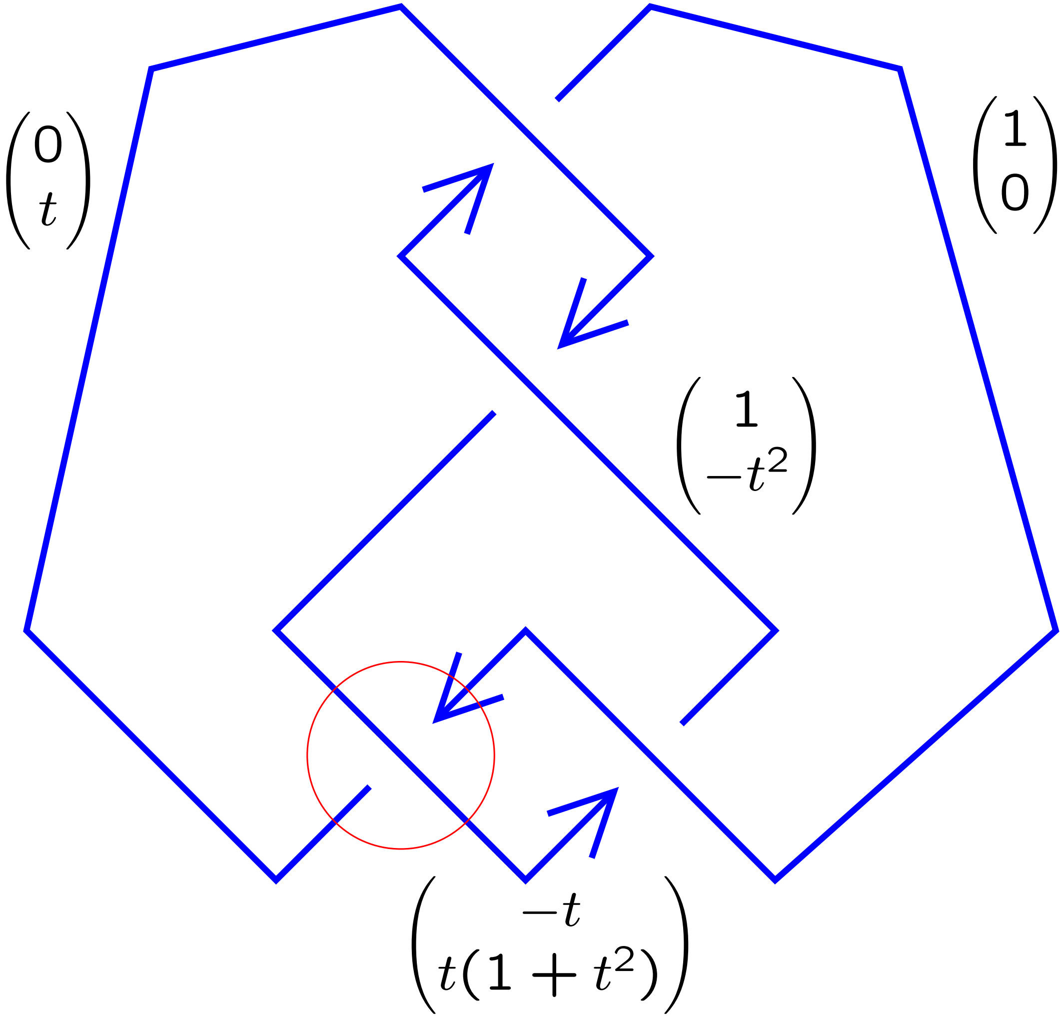
Consider the relation at another crossing.

Arc coloring of the figure eight knot



$$\begin{pmatrix} 0 \\ t \end{pmatrix} * \begin{pmatrix} 1 \\ -t^2 \end{pmatrix} = \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix}$$

Arc coloring of the figure eight knot



The relation at this crossing is

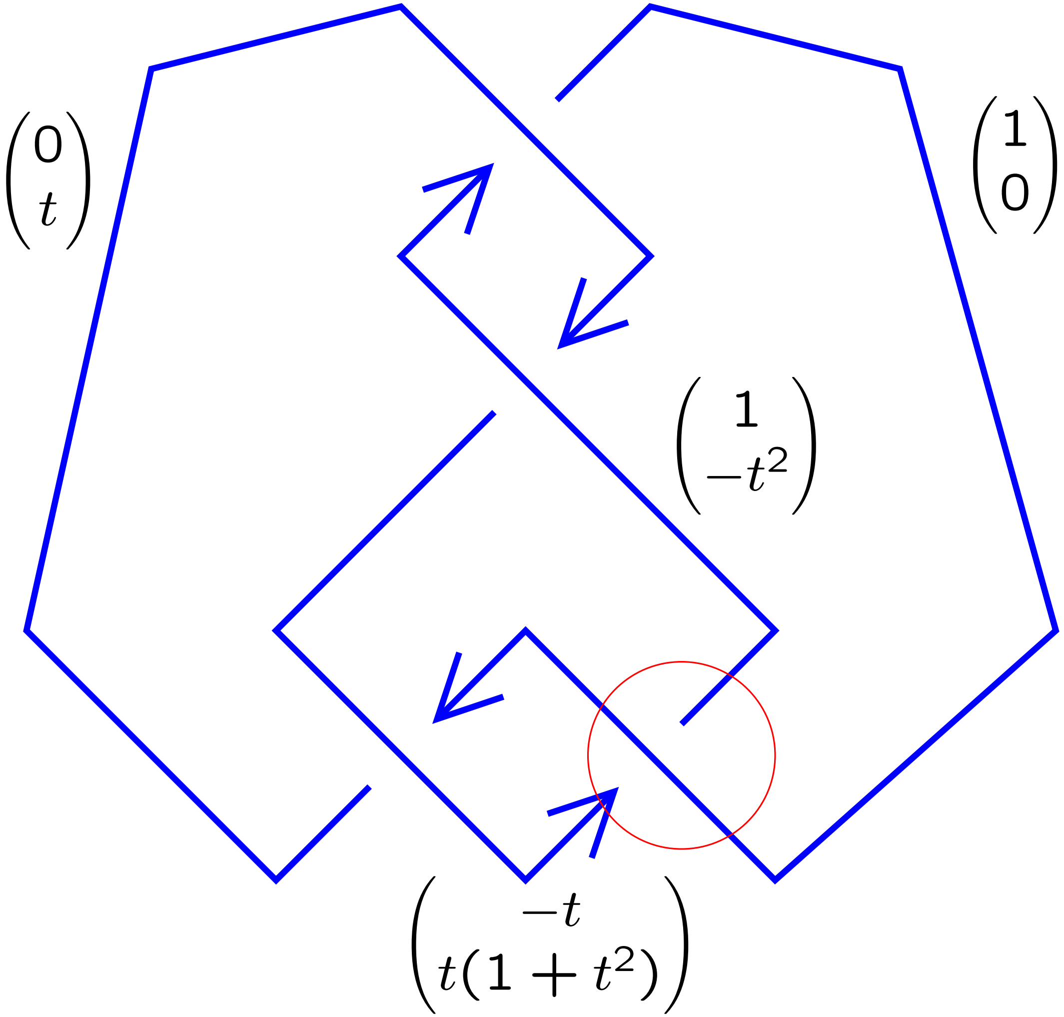
$$\left(\begin{pmatrix} 0 \\ t \end{pmatrix} * \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix} = \right)$$

$$\begin{pmatrix} -t^3 \\ t(1+t^2+t^4) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} (t+1)(t^2-t+1) = 0 \\ t(t^2+t+1)(t^2-t+1) = 0 \end{cases}$$

$$\therefore t^2 - t + 1 = 0$$

Arc coloring of the figure eight knot



The relation at this crossing is

$$\left(\begin{pmatrix} 1 \\ -t^2 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \right)$$

$$\begin{pmatrix} 1+t^2 \\ -t^2 \end{pmatrix} = \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix}$$

$$\begin{cases} t^2+t+1=0 \\ t(t^2+t+1)=0 \end{cases}$$

$$\therefore t^2+t+1=0$$

Arc coloring of the figure eight knot

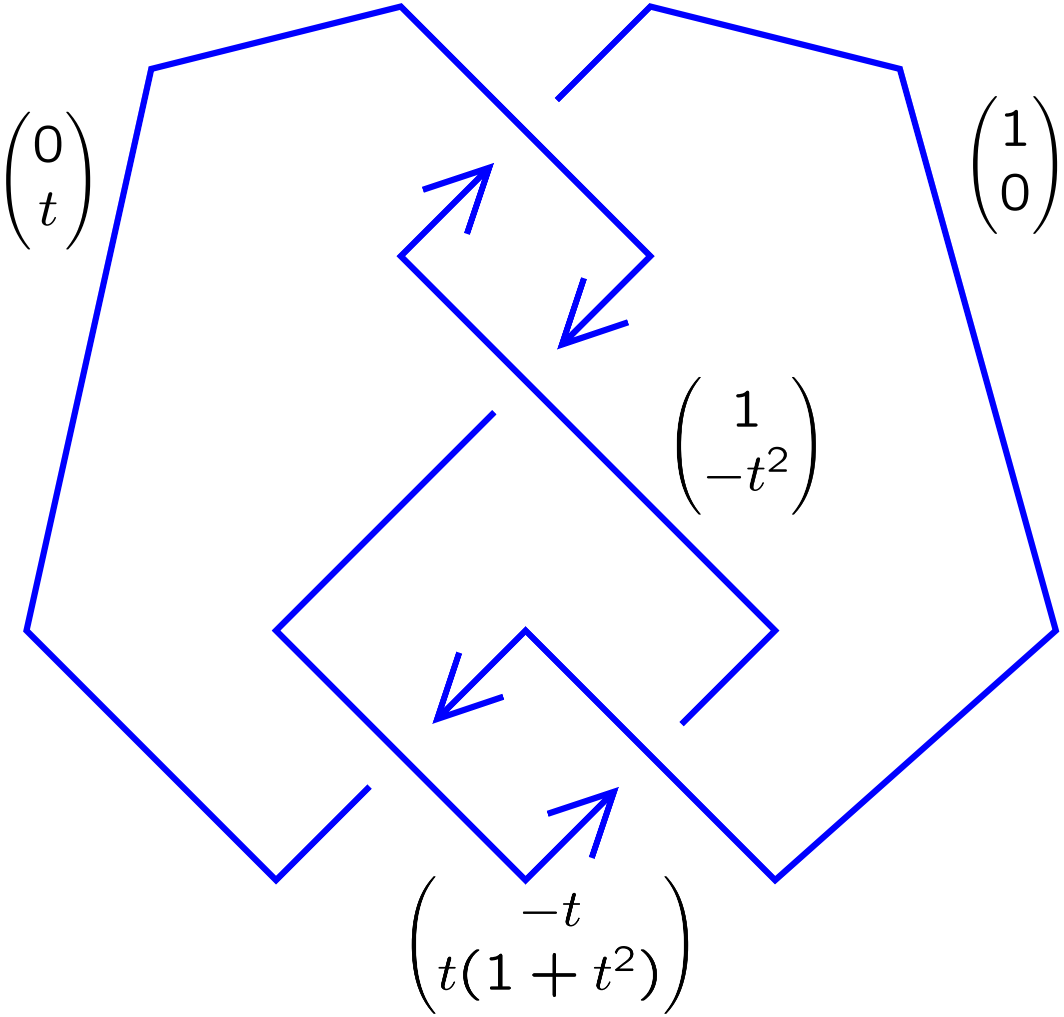
There are two relations

$$t^2 + t + 1 = 0, \quad t^2 - t + 1 = 0$$

which do not have any common solution. But we have a coloring by $(\mathbb{C}^2 \setminus \{0\})/\pm \cong \mathcal{P}$ ($t = \pm\frac{1+\sqrt{3}i}{2}$ or $\pm\frac{1-\sqrt{3}i}{2}$).

Because the trace of the longitude is -2, the coloring by $\mathcal{P}$ does not lift to a coloring by $\mathbb{C}^2 \setminus \{0\}$. But we can color the *long knot* by $\mathbb{C}^2 \setminus \{0\}$.

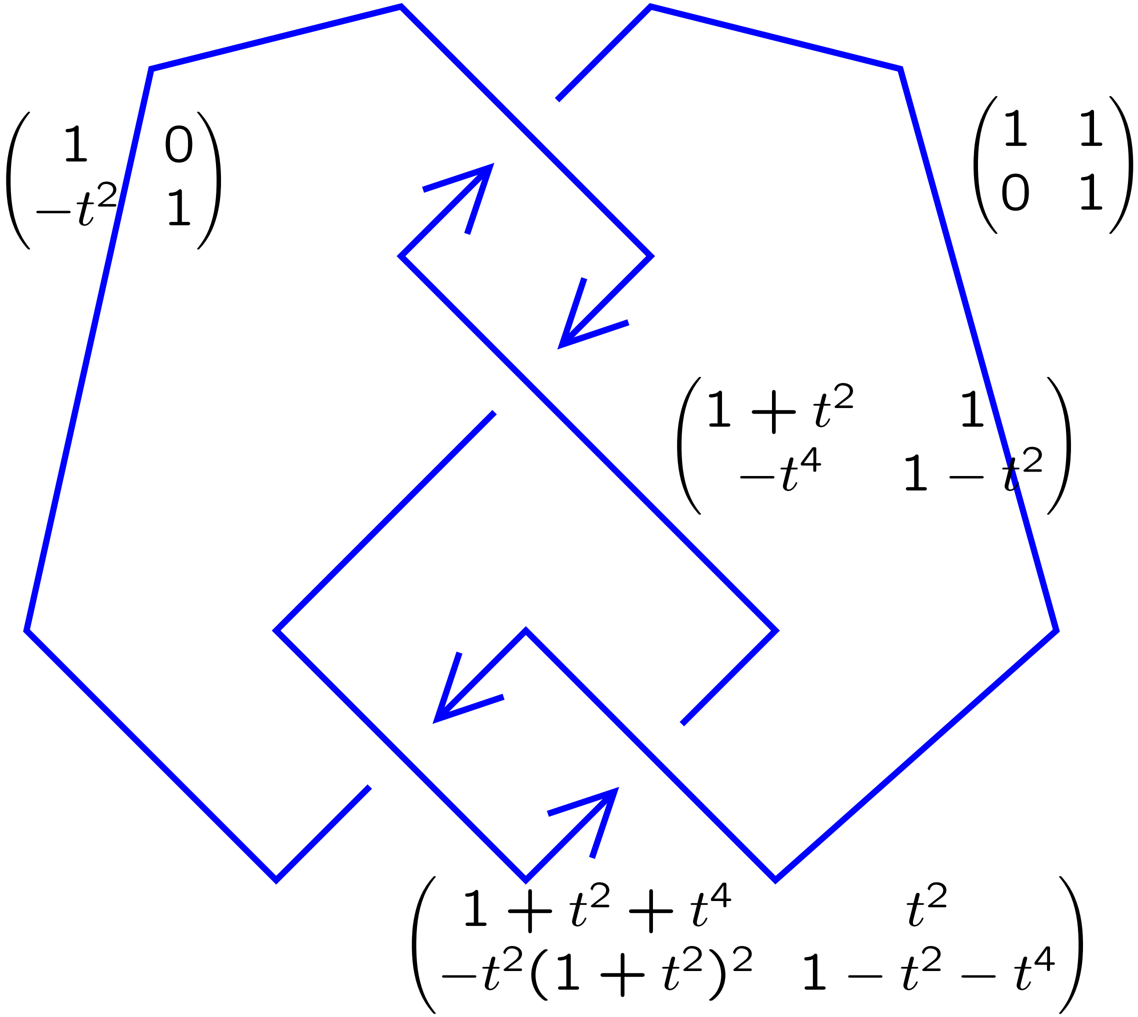
Arc coloring of the figure eight knot



A parabolic representation can be obtained by the map

$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}1-xy & x^2\\ -y^2 & 1+xy\end{pmatrix}$$

Arc coloring of the figure eight knot



A parabolic representation can be obtained by

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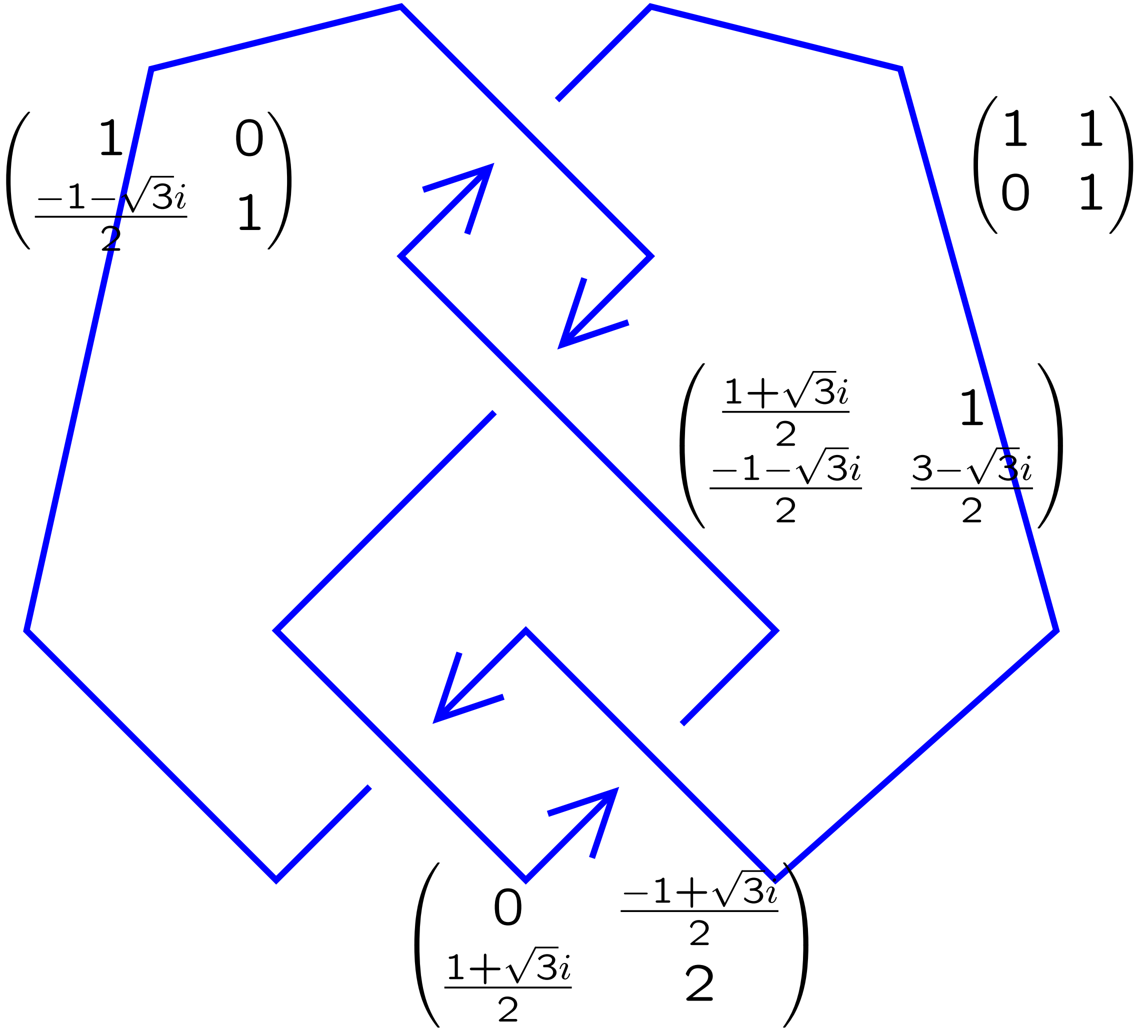
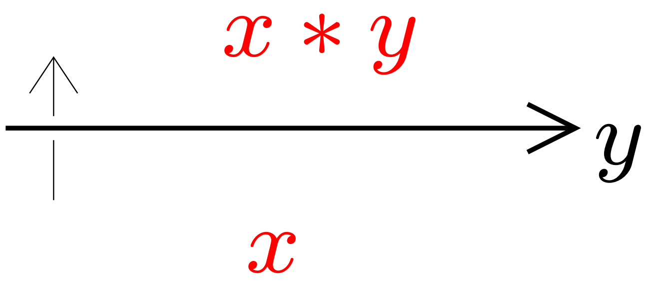
Arc coloring of the figure eight knot



When $t^2 = \frac{-1+\sqrt{3}i}{2}$:

Region coloring

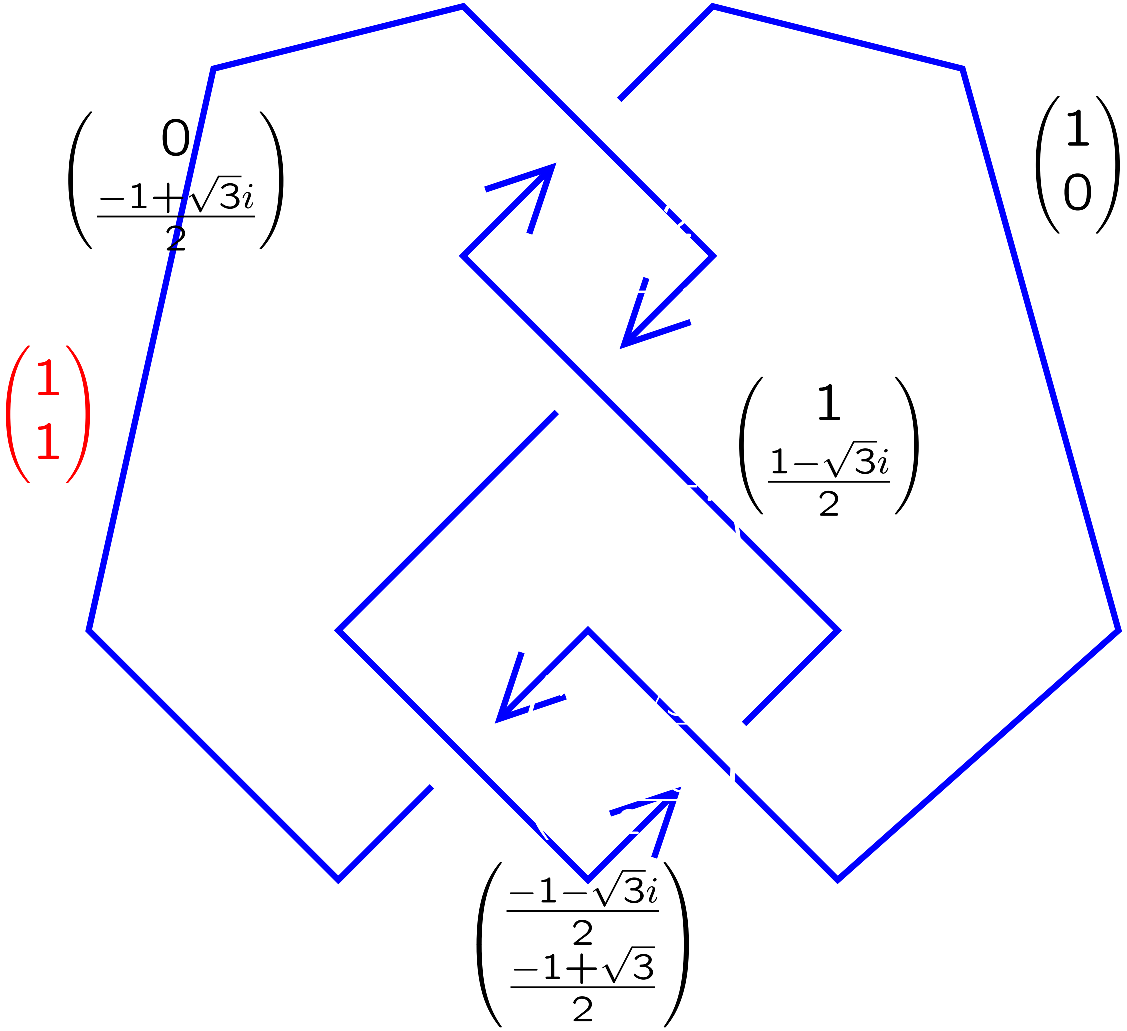
Let D be a diagram and $\mathcal{A}$ be an arc coloring by $(\mathbb{C}^2 \setminus \{0\})/\pm$. A map $\mathcal{D} : \{\text{regions of } D\} \to (\mathbb{C}^2 \setminus \{0\})/\pm$ is called an *region coloring* if it satisfies the following relation at each arc of D.



x, y and $x * y \in (\mathbb{C}^2 \setminus \{0\})/\pm$

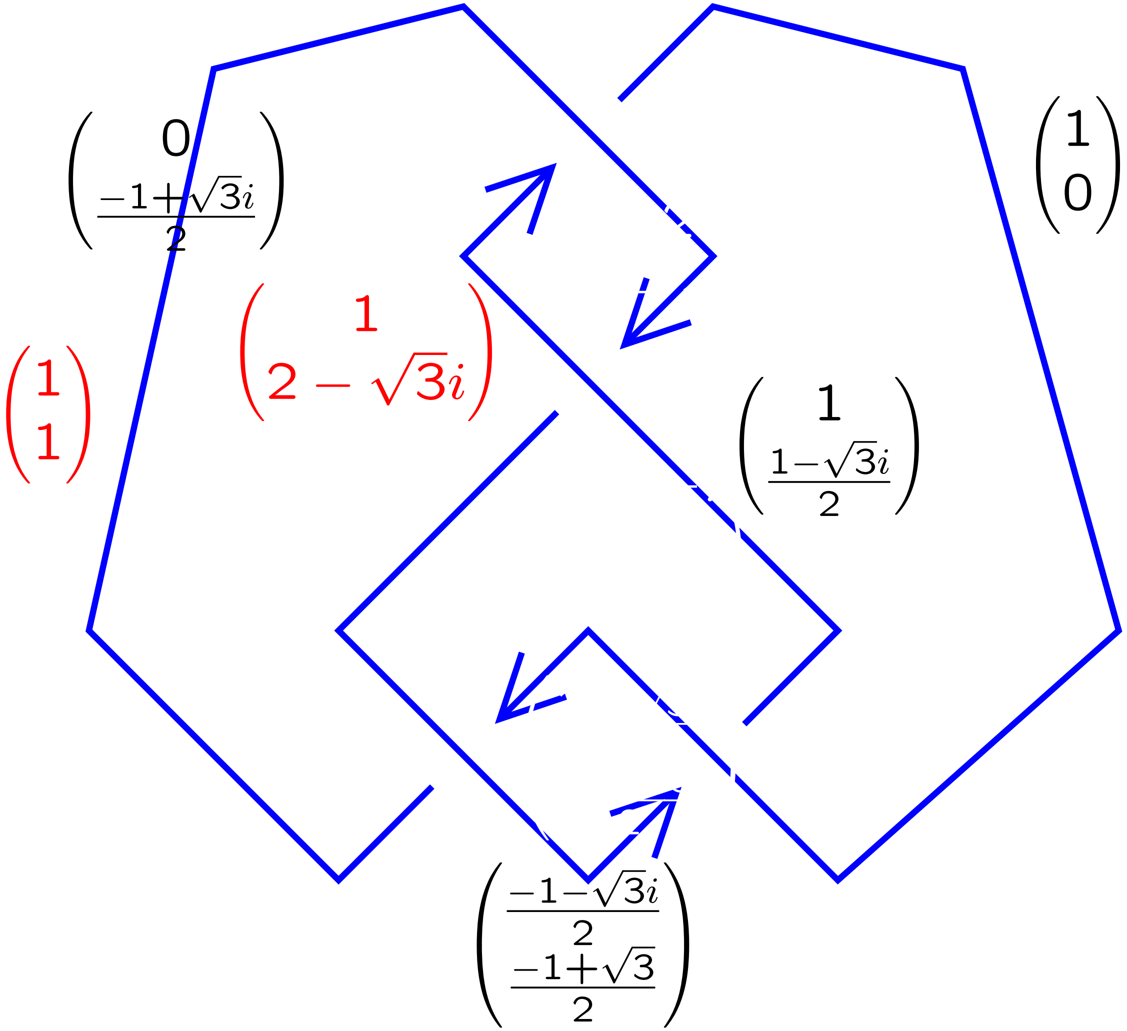
A pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ ($\mathcal{A}$: arc coloring, $\mathcal{R}$: region coloring) is called a *shadow coloring*.

Region coloring of the figure eight knot



Put a region color at a region e.g. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

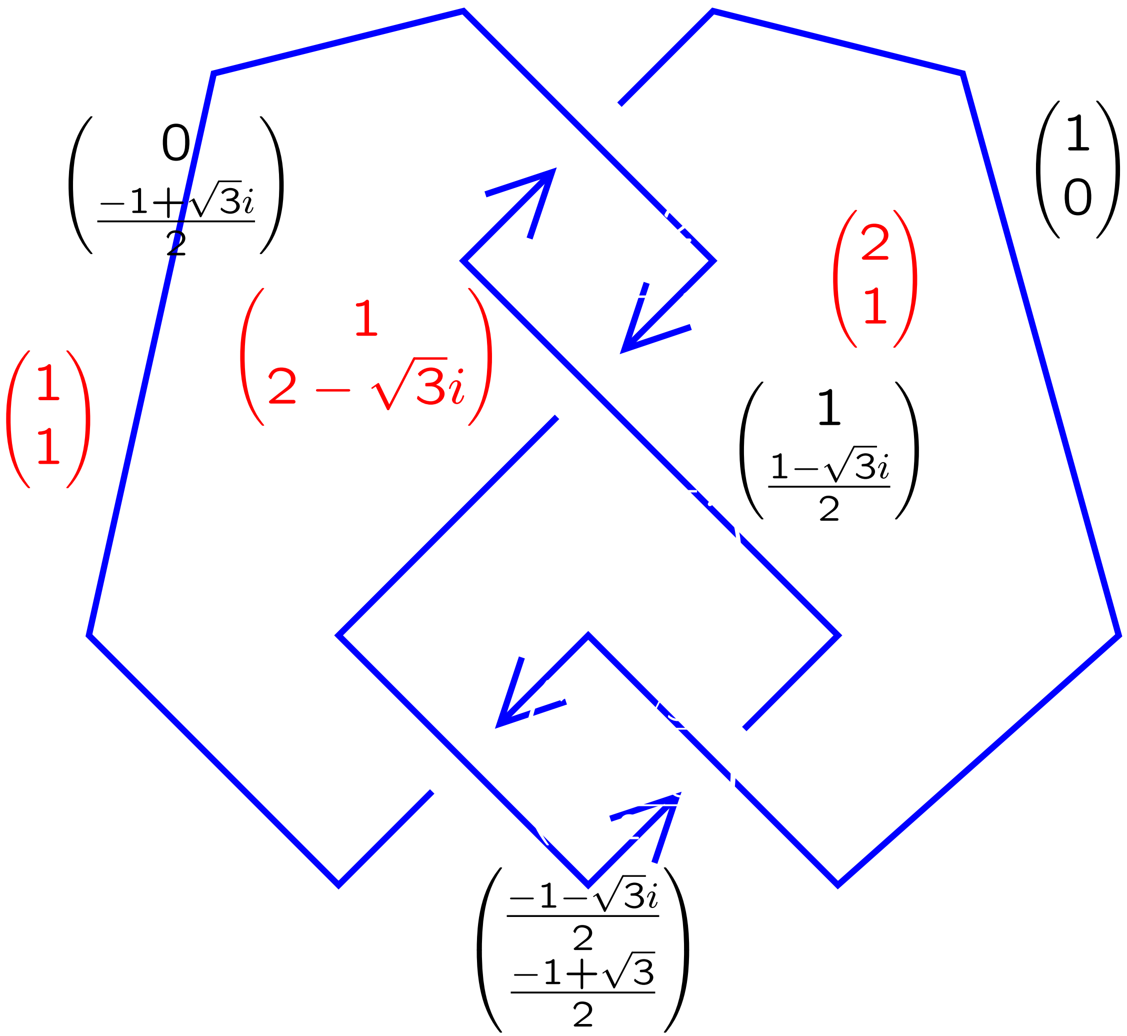
Region coloring of the figure eight knot



The color of an adjacent region is determined by the relation.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *^{-1} \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}i}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\sqrt{3}i \end{pmatrix}$$

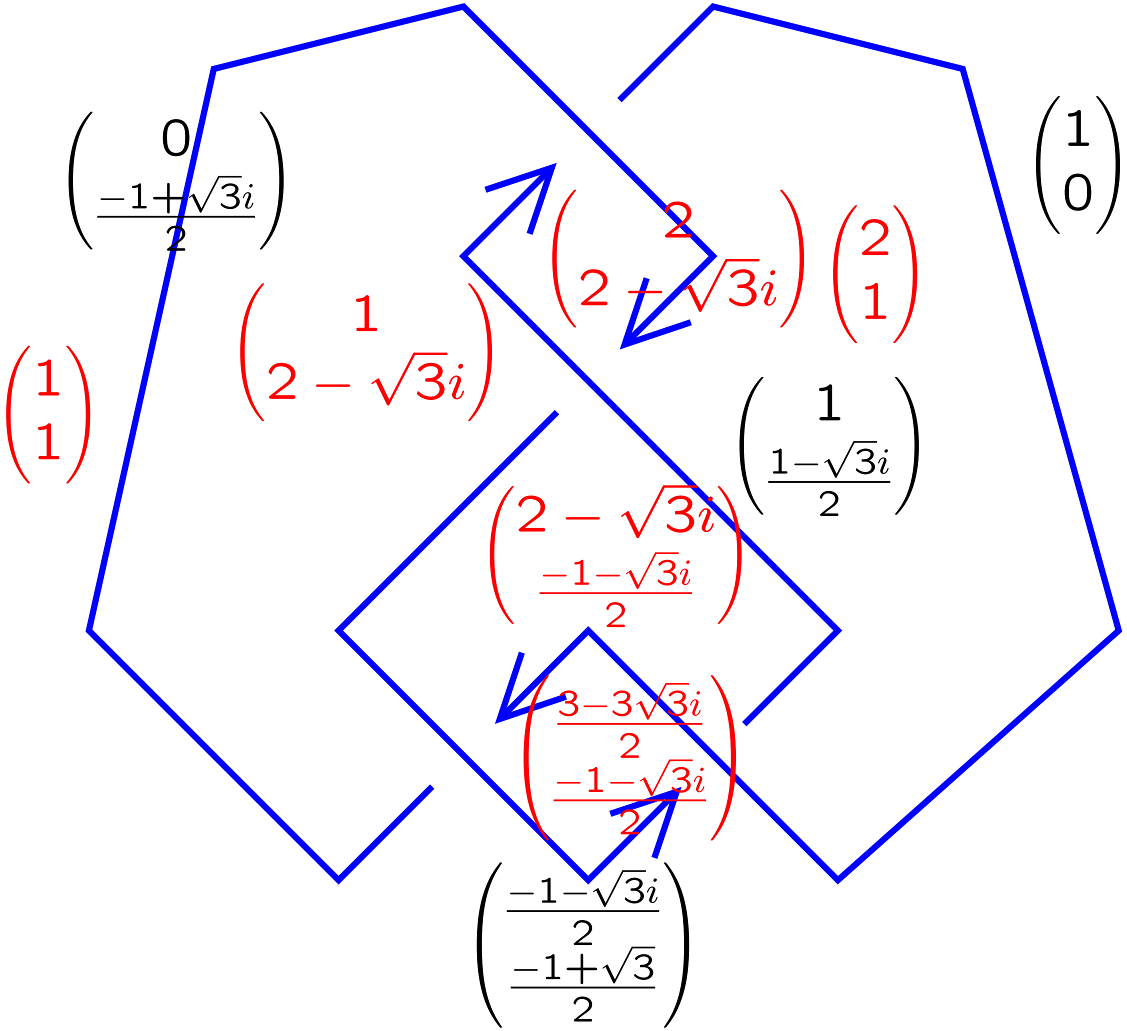
Region coloring of the figure eight knot



The color of an adjacent region is determined by the relation.

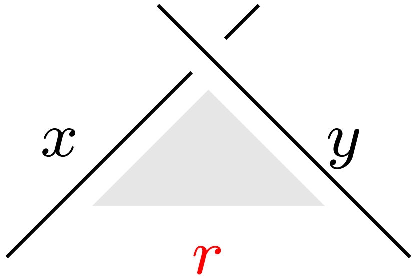
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Region coloring of the figure eight knot



The color of an adjacent region is determined by the relation.

Fix an element p_0 of $\mathbb{C}^2 \setminus \{0\}$ e.g. $p_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

At a corner colored by 

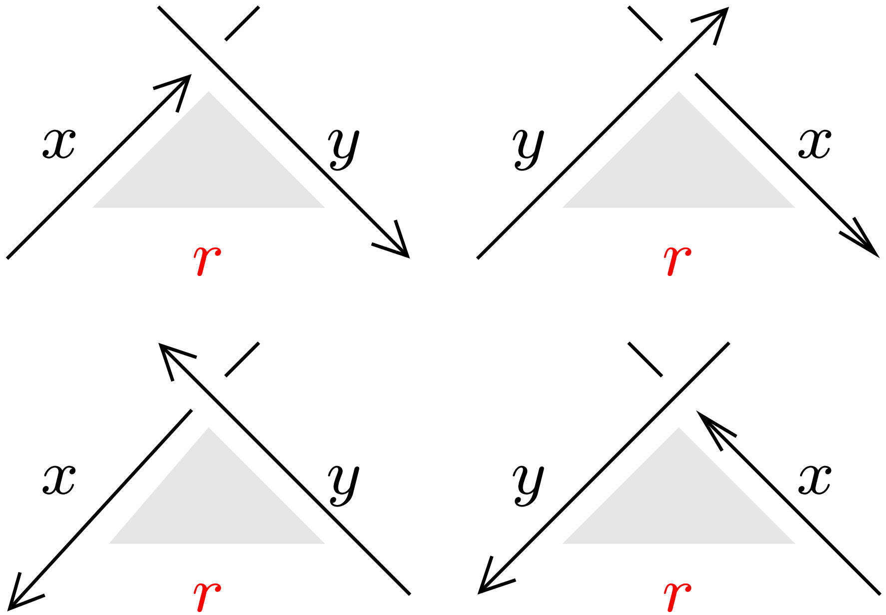
($x \leftrightarrow$ under arc, $y \leftrightarrow$ over arc), we let

$$
\begin{aligned}
z =& \frac{\det(p_0, y)\det(r, x)}{\det(r, y)\det(p_0, x)} \\
p\pi i =& \mathrm{Log}(\det(p_0, y)) + \mathrm{Log}(\det(r, x)) \\
& - \mathrm{Log}(\det(r, y)) - \mathrm{Log}(\det(p_0, x)) - \mathrm{Log}(z) \\
q\pi i =& \mathrm{Log}(\det(p_0, x)) + \mathrm{Log}(\det(r, y)) \\
& - \mathrm{Log}(\det(p_0, r)) - \mathrm{Log}(\det(x, y)) - \mathrm{Log}(\frac{1}{1-z})
\end{aligned}
$$

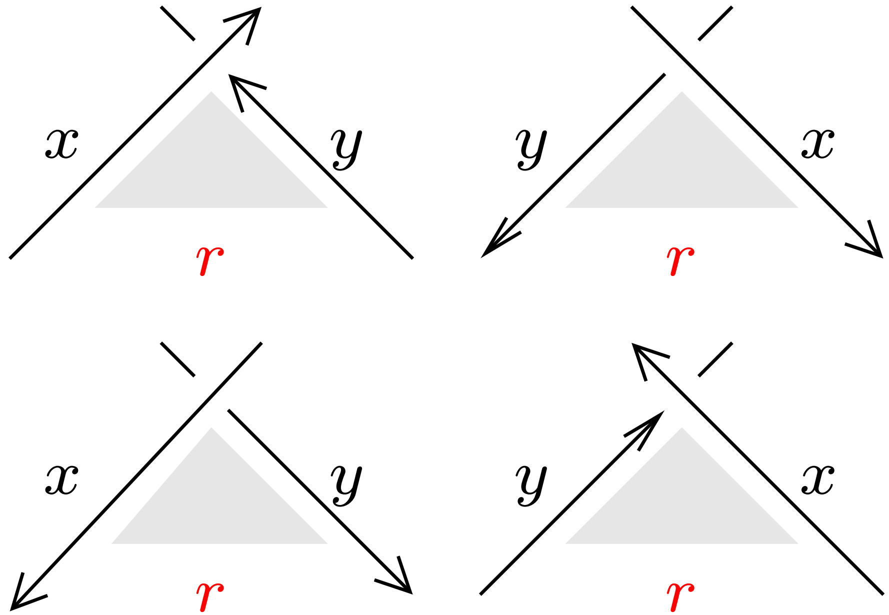
where $\mathrm{Log}(z) = \log|z| + i \arg(z)$ $(-\pi < \arg(z) \leq \pi)$

We remark that $p, q \in \mathbb{Z}$.

Then define the sign in the following rule:



and



$+[z; p, q]$

(in-out or out-in)

$-[z; p, q]$

(in-in or out-out)

Let

$$R(z;p,q) = \mathcal{R}(z) + \frac{\pi i}{2}\left(q\mathrm{Log}(z) - p\mathrm{Log}\left(\frac{1}{1-z}\right)\right) - \frac{\pi^2}{6}.$$

where $\mathcal{R}(z)$ is given by

$$\mathcal{R}(z) = -\int_0^z \frac{\mathrm{Log}(1-t)}{t}dt + \frac{1}{2}\mathrm{Log}(z)\mathrm{Log}(1-z)$$

Theorem 1

$$\sum_{c:\text{corners}} \varepsilon_c R(z_c;p_c,q_c) = i(\mathrm{Vol}(S^3 \setminus K,\rho) + i\mathrm{CS}(S^3 \setminus K,\rho))$$

where ρ is the parabolic representation determined by the arc coloring.

Background materials

X : a quandle

$G_X = \langle x \in X | y^{-1}xy = x * y \rangle$: the associated group

X has a right G_X-action defined by

$$x * (x_1^{\varepsilon_1} x_1^{\varepsilon_2} \dots x_n^{\varepsilon_n}) = (\dots((x *^{\varepsilon_1} x_1) *^{\varepsilon_2} x_2) \dots) * x_n^{\varepsilon_n}$$

So $\mathbb{Z}[X]$ is a right $\mathbb{Z}[G_X]$-module.

Quandle homology

Let $C_n^R(X) = \text{span}_{\mathbb{Z}[G_X]}\{(x_1, \ldots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

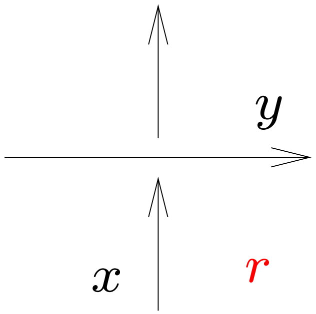
$$\partial(x_1, \ldots x_n) = \sum_{i=1}^{n} (-1)^i \{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\ - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

Let M be a right $\mathbb{Z}[G_X]$-module. The homology of $M \otimes_{\mathbb{Z}[G_X]} C_n^R(X)$ is the *rack homology* $H_n^R(X; M)$.

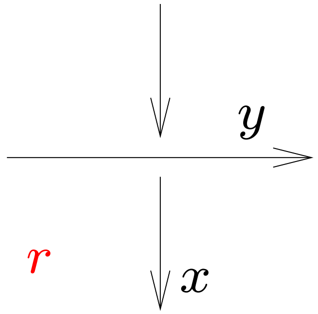
Considering non-degenerate chains, we also define the *quandle homology* $H_n^Q(X; M)$.

A cycle associated with a shadow coloring

Let $\mathcal{S}$ be a shadow coloring by a quandle X. Assign $+r \otimes (x, y)$ for



and $-r \otimes (x, y)$ for



. Let

$$C(\mathcal{S}) = \sum_{c:\text{crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[X]).$$

This is a cycle. The homology class $[C(\mathcal{S})]$ in $H_2^Q(X; \mathbb{Z}[X])$ (usually denoted by $H_2^Q(X)_X$) does not depend on the diagram and the region coloring. Moreover it only depends on the "conjugacy" class of the arc coloring. When $X = \mathcal{P}$, the cycle only depends on the conjugacy class of the corresponding parabolic representation.

Simplicial quandle homology $H_n^{\Delta}(X)$

Let $C_n^{\Delta}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ by

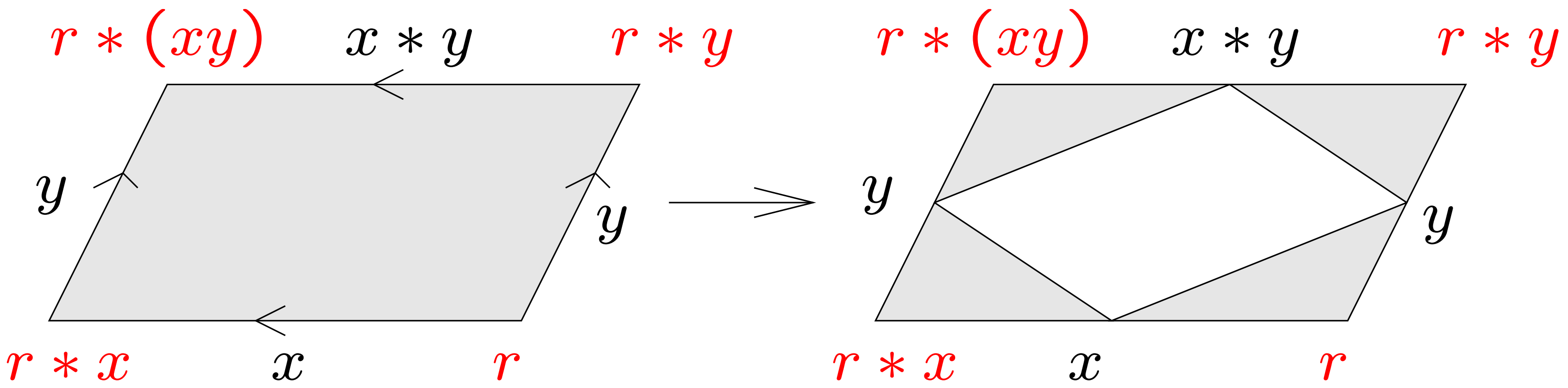
$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

$C_n^{\Delta}(X)$ has a natural right action by $\mathbb{Z}[G_X]$. Denote the homology of $C_n^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by $H_n^{\Delta}(X)$. We can construct a map

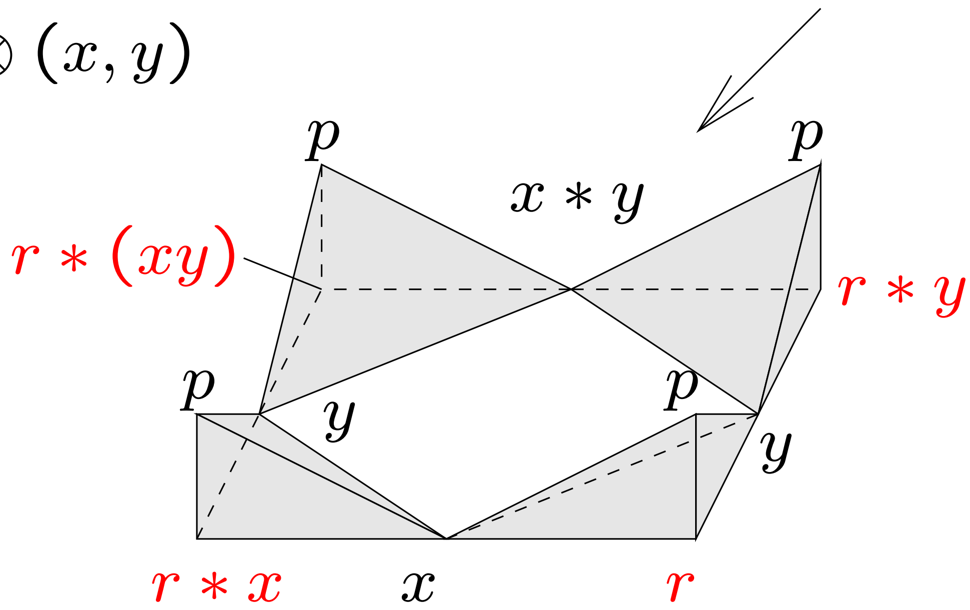
$$H_n^Q(X; \mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$$

in the following way:

$n = 2$

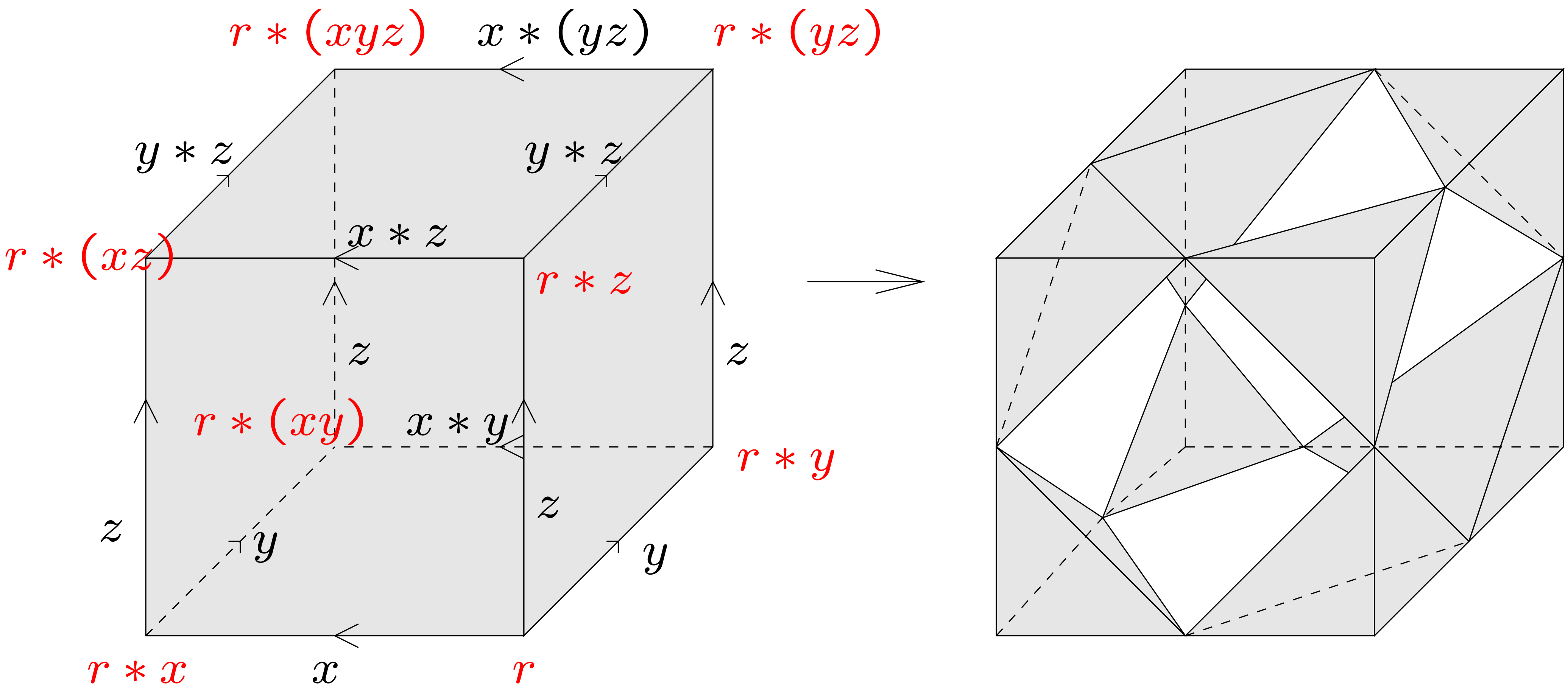


$r \otimes (x, y)$



$$(p, r, x, y) - (p, r*x, x, y)$$
$$-(p, r*y, x*y, y) + (p, r*(xy), x*y, y)$$

$n = 3$



$$
\begin{aligned}
r \otimes (x, y, z) \mapsto (p, r, x, y, z) - (p, r * x, x, y, z) - (p, r * y, x, x * y, z) \\
-(p, r * z, x * z, y * z, z) + (p, r * (xy), x * y, y, z) \\
+(p, r * (xz), x * z, y * z, z) + (p, r * (yz), x * (yz), y * z, z) \\
-(p, r * (xyz), x * (yz), y * z, z)
\end{aligned}
$$

Since we have a map

$$H_n^Q(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X),$$

we can construct a quandle cocycle from a cocycle of $H_{n+1}^{\Delta}(X)$. When X is given by a symmetric space $K\backslash G$, G-invariant closed k-form on $K\backslash G$ gives a k-cocycle by integrating the form.

When $X = \mathcal{P}$, $\mathcal{P} \cong (\mathbb{C}^2 \setminus \{0\})/\pm \cong P\backslash\mathrm{PSL}(2,\mathbb{C})$ (P is a parabolic subgroup), then $C_3^{\Delta}(X)$ is the complex studied by Dupont and Zickert. We can construct a map from $H_3^{\Delta}(X)$ to the *extended Bloch group* $\widehat{\mathcal{B}}(\mathbb{C})$ (defined by W. Neumann). This is the construction that we have seen before. But we need careful treatment on degenerate simplices.

Remark

Let $X = K\backslash G$ be a quandle. If the sequence $\cdots \to C_2^{\Delta}(X) \to C_1^{\Delta}(X) \to \mathrm{Ker}(C_0^{\Delta}(X) \to \mathbb{Z}) \to 0$ is a projective resolution, $H_n^{\Delta}(X)$ is isomorphic to the relative group homology $H_n(G, K)$.

When $X = \mathcal{P} \cong P\backslash \mathrm{PSL}(2, \mathbb{C})$, the image of $[C(\mathcal{S})]$ under the map $H_2^Q(X; \mathbb{Z}[X]) \to H_3^{\Delta}(X)$ gives a homology class in $H_3(\mathrm{PSL}(2, \mathbb{C}), P)$.

ありがとうございました。