QUANDLE による SHADOW COLORING と PSL(2,C) 表現の体積と CHERN-SIMONS 不変量

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1. 概要

このノートではカンドルのホモロジーの理論を用いて双曲体積と Chern-Simons 不変量が計算できる事を報告する．講演では具体的な計算を中心に話したので、ここではあまり説明できなかった理論的な部分について記述する．ほとんどの内容は体積と Chern-Simons 不変量の計算のためだけでなくカンドルに関する一般的な結果である．

K を結び目、D をそのダイアグラムとする．カンドル X による arc coloring は K の補空間の基本群から X の associated group G_X への表現を与える事がわかる．次に shadow coloring $\mathcal{S}$ を定義し、カンドルホモロジー $H_2^Q(X;\mathbb{Z}[X])$ に値を取る不変量 $[C(\mathcal{S})]$ を定義する．このホモロジー類 $[C(\mathcal{S})]$ は K の補空間の表現の共役類のみによる量である事がわかる．よってこのホモロジー群から別の群に準同型が存在すればそれは明らかに G_X への表現の不変量になる．つぎに単体的カンドルホモロジー $H_n^\Delta(X)$ を定義しラックホモロジー $H_n^R(X;\mathbb{Z}[X])$ から単体的カンドルホモロジー $H_{n+1}^\Delta(X)$ への写像を定義する．特に３次元の場合にはカンドルホモロジーからの写像を導き、[Wee] や [Ino] で与えられた三角形分割の代数的な構成を与えている．

これらの結果を $\mathrm{PSL}(2,\mathbb{C})$ の放物的元のなすカンドル $\mathcal{P}$ に対して適用する．このカンドルによる彩色は結び目 K の補空間のメリディアンを放物的元に移す表現の全体と１対１に対応する．また $\mathcal{P}$ は $(\mathbb{C}^2\setminus\{0\})/\pm$ と同一視できる事を示す．この事実から $H_3^\Delta(\mathcal{P})$ は Dupont-Zickert [DZ] で研究されたホモロジーとほぼ同じである事がわかる．彼らはそのホモロジーから extended Bloch 群への写像を構成した．同様の方法で準同型 $H_2^Q(\mathcal{P};\mathbb{Z}[\mathcal{P}])\to\widehat{\mathcal{B}}(\mathbb{C})$ を構成する．結果として結び目の $\mathcal{P}$ による arc coloring から extended Bloch 群の元を作る事ができる．これは Neumann により定義された $\mathrm{PSL}(2,\mathbb{C})$ 表現の不変量と一致する事がわかる．Neumann [Neu] の結果により Rogers dilogarithm 関数 R を用いて体積と Chern-Simons 不変量を計算する事ができる．

我々の作ったホモロジー $H_n^\Delta(X)$ は群の相対ホモロジーと密接に関係している．もっと簡単な形での群のホモロジーとの関連があるが他であまり触れられていないようなので群のホモロジーとの関連についても３節で解説する．

2. Quandle and Quandle homology

A quandle is a set X with a binary operation $*$ satisfying the following axioms:

(1) $x*x=x$,
(2) the map $*y: X\to X$ defined by $x\mapsto x*y$ is a bijection,
(3) $(x*y)*z=(x*z)*(y*z)$,

for any $x,y,z\in X$. We denote the inverse of $*y$ by $*^{-1}y$. For a quandle X, we define the associated group G_X by $\langle x\in X \mid y^{-1}xy=x*y \quad (x,y\in X)\rangle$. A quandle X has a right G_X-action in the following way. Let $g=x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n}$ be an element of G_X where $x_i\in X$ and $\varepsilon_i=\pm1$. Define $x*g=(\cdots((x*^{\varepsilon_1}x_1)*^{\varepsilon_2}x_2)\cdots)*^{\varepsilon_n}x_n$. One can easily check that this is a right action of G_X on X. So the free abelian group $\mathbb{Z}[X]$ generated by X is a right $\mathbb{Z}[G_X]$-module.

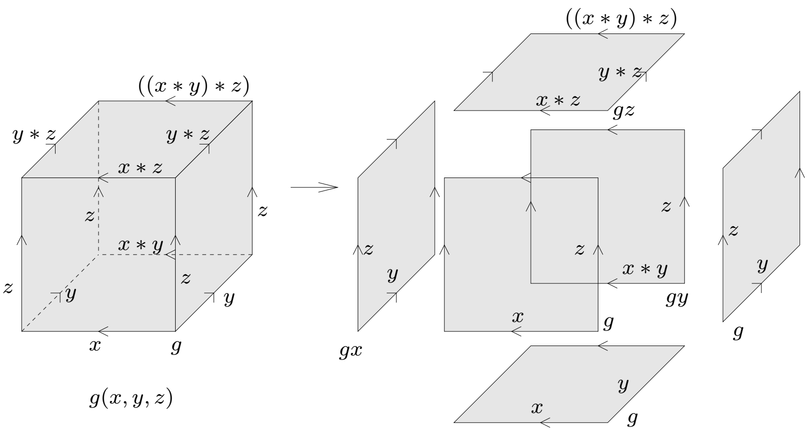


FIGURE 1. $\partial(g(x, y, z)) = -(g(y, z) - gx(y, z)) + (g(x, z) - gy(x * y, z)) - (g(x, y) - gz(x * z, y * z))$. Here $x, y, z \in X$ and $g \in G_X$. Edges are labeled by elements of X and vertices are labeled by elements of G_X.

Let $C_n^R(X)$ be the free (left) $\mathbb{Z}[G_X]$-module generated by X^n. We define the boundary map $C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i ((x_1, \ldots, \widehat{x_i}, \ldots, x_n) - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)).$$

A graphical picture of the boundary map is given in Figure 1. Let $C_n^D(X) = \mathrm{span}_{\mathbb{Z}[G_X]}\{(x_1, x_2, \ldots, x_n) | x_i = x_{i+1} \text{for some } i)\}$ and $C_n^Q(X) = C_n^R(X)/C_n^D(X)$. Let M be a right $\mathbb{Z}[G_X]$-module. We define the *rack homology* of M by the homology of $M \otimes_{\mathbb{Z}[G_X]} C_*^R(X)$ and denote it by $H_n^R(X, M)$. We also define the *quandle homology* of M by the homology of $M \otimes_{\mathbb{Z}[G_X]} C_*^Q(X)$ and denote it by $H_n^Q(X; M)$. The homology $H_n^Q(X; \mathbb{Z})$, here $\mathbb{Z}$ is the trivial $\mathbb{Z}[G_X]$-module, is equal to the usual quandle homology $H_n^Q(X)$. Let Y be a set with a right G_X-action. Then the free abelian group $\mathbb{Z}[Y]$ generated by Y is a right $\mathbb{Z}[G_X]$-module. In this note, we will mainly study the quandle homology $H_2^Q(X; \mathbb{Z}[X])$. (This is usually denoted by $H_2^Q(X)_X$.)

3. Group homology

Let G be a group. Let $C_n(G) = \mathrm{span}_{\mathbb{Z}[G]}\{[g_1|g_2| \ldots |g_n] | g_i \in G\}$ and define the boundary map $\partial : C_n(G) \to C_{n-1}(G)$ by

$$\partial([g_1| \ldots |g_n]) = g_1[g_2| \ldots |g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1| \ldots |g_i g_{i+1}|g_n] + (-1)^n [g_1| \ldots |g_{n-1}].$$

Let $C_0(G) \cong \mathbb{Z}[G] \to \mathbb{Z} \to 0$ be the augmentation map. We remark that the chain complex $\{\cdots \to C_1(G) \to C_0(G) \to \mathbb{Z} \to 0\}$ is acyclic. So the chain complex $C_*(G)$ gives a free resolution of $\mathbb{Z}$. Let M be a right $\mathbb{Z}[G]$-module. The homology of $M \otimes_{\mathbb{Z}[G]} C_*(G)$ is called the *group homology* of M and denoted by $H_n(G; M)$. In other words, $H_n(G; M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(M, \mathbb{Z})$.

We can construct a map from the rack homology $H_n^R(X; M)$ to the group homology $H_n(G_X; M)$. The following lemma is well-known.

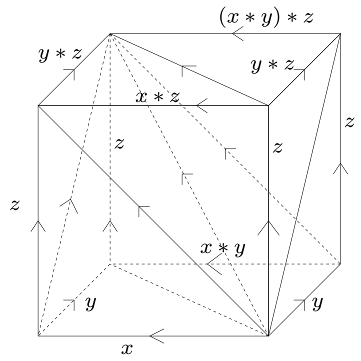


FIGURE 2

Lemma 3.1. *Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a chain complex where P_i are projective (e.g. free). Let $\cdots \to C_1 \to C_0 \to N \to 0$ be an acyclic complex. Any homomorphism $M \to N$ can be extended to a chain map from $\{P_*\}$ to $\{C_*\}$. Moreover such a chain map is unique up to chain homotopy.*

So there exists a unique chain map from $C^R_*(X)$ to $C_*(G_X)$ up to homotopy. This map induces $M \otimes_{\mathbb{Z}[G_X]} C^R_*(X) \to M \otimes_{\mathbb{Z}[G_X]} C_*(G_X)$ and then $H^R_n(X; M) \to H_n(G_X; M)$. We give an explicit chain map f. Let $(x_1, \ldots, x_n)$ be a generator of $C^R_n(X)$. We define the map f by

$$f((x_1, \ldots, x_n)) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)[y_{\sigma,1}| \cdots |y_{\sigma,i}| \cdots |y_{\sigma,n}]$$

where $y_{\sigma,i} \in X$ is defined for a permutation σ and $i \in \{1, \ldots, n\}$ as follows: Let $j_1, \ldots, j_{k_i} < i$ be the numbers satisfying $\sigma(i) < \sigma(j_1) < \sigma(j_2) < \cdots < \sigma(j_{k_i})$. Then define

$$y_{\sigma,i} = x_{\sigma(i)} * (x_{\sigma(j_1)} x_{\sigma(j_2)} \cdots x_{\sigma(j_{k_i})}).$$

The graphical picture of this map is given in Figure 2.

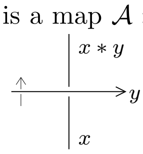
Example 3.2. Let $(x, y, z) \in C^R_3(X)$. Then the explicit chain map $f : C^R_3(X) \to C_3(G_X)$ constructed above is given by

$$\begin{aligned} \partial((x,y,z)) =& [x|y|z] - [x|z|y*z] + [y|z|(x*y)*z] - [y|x*y|z] \\ &+ [z|x*z|y*z] - [z|y*z|(x*y)*z] \end{aligned}$$

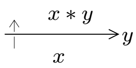
Remark 3.3. Fenn, Rourke and Sanderson defined the *Rack space* BX. Since $\pi_1(BX)$ is isomorphic to G_X, there exists a unique map, up to homotopy, from BX to the Eilenberg-MacLane space $K(G_X, 1)$ which induces an isomorphism between their fundamental groups. The map we have constructed is essentially same as this map.

As we have seen, there exists a relation between quandle homology and group homology. We shall give another relation which seems to reflect more geometric feature.

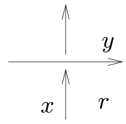
4. SHADOW COLORING AND FUNDAMENTAL CYCLE

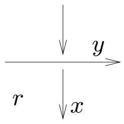
Let X be a quandle. Let K be a knot in S^3 and D be a diagram of K. An *arc coloring* is a map $\mathcal{A} : \{\text{arcs of } D\} \to X$ if it satisfies the following relation at each crossing: 

, where x, y and $x * y \in X$. By the Wirtinger presentation of a knot complement, an arc coloring determines a representation $\pi_1(S^3 \setminus K) \to G_X$. This is obtained by sending each meridian to its color.

A map $\mathcal{D}$: {regions of D} $\to X$ is called a *region coloring* if it satisfies the following relation for a pair of adjacent regions: 

, where x, y and $x*y \in X$. The notion of region coloring is generalized for any set Y with G_X-action, but we only study this special case. A pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ is called a *shadow coloring.*

We define a cycle $[C(\mathcal{S})]$ of $H_2^Q(X; \mathbb{Z}[X])$ for a shadow coloring $\mathcal{S}$ by X. Assign $+r \otimes (x, y)$ for a positive crossing colored by 

and $-r \otimes (x, y)$ for a negative crossing 

. Then we define

$$C(\mathcal{S}) = \sum_{c:\text{crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[X]),$$

here $\varepsilon_c = \pm 1$. We can easily check that this is a cycle. The homology class $[C(\mathcal{S})]$ in $H_2^Q(X; \mathbb{Z}[X])$. is invariant under Reidemeister moves and does not depend on the choice of region coloring. Moreover we have

Proposition 4.1. *The homology class* $[C(\mathcal{S})]$ *only depends on the conjugacy class of the representation* $\pi_1(S^3 \setminus K) \to G_X$ *induced by the arc coloring* $\mathcal{A}$.

5. Simplicial quandle homology $H_n^\Delta(X)$ and the map $H_n^R(X; \mathbb{Z}[X]) \to H_{n+1}^\Delta(X)$

Let $C_n^\Delta(X) = \text{span}_\mathbb{Z}\{(x_0, \dots, x_n) | x_i \in X\}$. We define the boundary operator of $C_n^\Delta(X)$ by

$$\partial(x_0, \dots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \dots, \widehat{x_i}, \dots, x_n).$$

Since X has a right action of G_X, the chain complex $C_n^\Delta(X)$ has a right action of G_X by $(x_0, \dots, x_n)*g = (x_0*g, \dots, x_n*g)$. We denote the homology of $C_n^\Delta(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by $H_n^\Delta(X)$ and call it a *simplicial quandle homology* of X.

We define a set I_n consisting of maps $\iota : \{1, 2, \cdots, n\} \to \{0, 1\}$. We let $|\iota|$ denote the cardinality of the set $\{i \mid \iota(i) = 1, 1 \le i \le n\}$. For each generator $r \otimes (x_1, x_2, \cdots, x_n)$ of $C_n^R(X; \mathbb{Z}[X])$, here $r, x_1, \dots, x_n \in X$, we define

$$r(\iota) = r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)}) \in X,$$

$$x(\iota, i) = x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)}) \in X,$$

for any $\iota \in I_n$. Fix an element $p \in X$. For each $n \ge 1$, we define a homomorphism

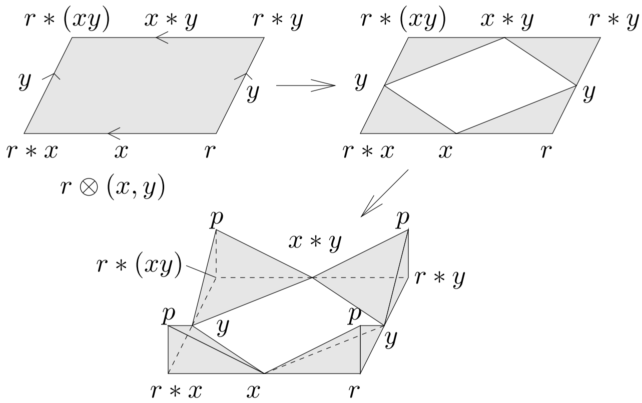
$$\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^\Delta(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$$

by

$$\varphi(r \otimes (x_1, x_2, \cdots, x_n)) = \sum_{\iota \in I_n} (-1)^{|\iota|} (p, r(\iota), x(\iota, 1), x(\iota, 2), \cdots, x(\iota, n)). \tag{5.1}$$

For example, in the case $n = 2$ (see Figure 3),

$$\varphi(r \otimes (x, y)) = (p, r, x, y) - (p, r*x, x, y) - (p, r*y, x*y, y) + (p, (r*x)*y, x*y, y),$$



$$(p, r, x, y) - (p, r * x, x, y)$$
$$-(p, r * y, x * y, y) + (p, r * (xy), x * y, y)$$

FIGURE 3

and in the case $n = 3$,

$$\begin{aligned}
\varphi(r \otimes (x, y, z)) = &\\
&(p, r, x, y, z) - (p, r * x, x, y, z)\\
&- (p, r * y, x * y, y, z) - (p, r * z, x * z, y * z, z)\\
&+ (p, (r * x) * y, x * y, y, z) + (p, (r * x) * z, x * z, y * z, z)\\
&+ (p, (r * y) * z, (x * y) * z, y * z, z) - (p, ((r * x) * y) * z, (x * y) * z, y * z, z).
\end{aligned}$$

Theorem 5.1. *The map $\varphi : C^R_n(X; \mathbb{Z}[X]) \longrightarrow C^\Delta_{n+1}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ is a chain map.*

So φ induces a homomorphism $\varphi_* : H^R_n(X; \mathbb{Z}[X]) \to H^\Delta_{n+1}(X)$. We remark that the induced map $\varphi_* : H^R_n(X; \mathbb{Z}[X]) \to H^\Delta_{n+1}(X)$ does not depend on the choice of $p \in X$. When $n = 2$, the map reduces to the map $\varphi_* : H^Q_2(X; \mathbb{Z}[X]) \to H^\Delta_3(X)$

5.1. **Relative group homology.** Let G be a group. Let H be a subgroup of G. We define the relative group homology $H_n(G, H; \mathbb{Z})$ by the homology of the mapping cone of the map $C_n(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to C_n(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. We can compute $H_n(G, H; \mathbb{Z})$ as follows (see [Zic]).

Lemma 5.2. *Let K be the kernel of $C_0(H \backslash G) \to \mathbb{Z}$. Let $\cdots \to F_2 \to F_1 \to K \to 0$ be a free resolution of K as $\mathbb{Z}[G]$-module. Then $H_n(G, H; \mathbb{Z}) \cong H_n(F_* \otimes_{\mathbb{Z}[G]} \mathbb{Z})$ for $n \geq 1$.*

Most of important quandles have a homogeneous presentation, in other words it can be presented in the form $H \backslash G$ with some group G and a subgroup H of G [Joy]. Since the complex $C^\Delta_*(X)$ is acyclic and have a $\mathbb{Z}[G]$-module structure, so if

$$\cdots \to C^\Delta_2(X) \to C^\Delta_1(X) \to \mathrm{Ker}(C^\Delta_0(X) \to \mathbb{Z}) \to 0$$

is a projective resolution, $H^\Delta_n(X)$ is isomorphic to $H_n(G, H; \mathbb{Z})$. This is another relationship with group homology.

6. Quandle structure on $\mathbb{C}^2 \setminus \{0\}$

Define a binary operation $*$ on $\mathbb{C}^2 \setminus \{0\}$ by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} * \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} 1 - x_2 y_2 & -x_2^2 \\ y_2^2 & 1 + x_2 y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This satisfies the quandle axioms. Let $\mathcal{P}$ be the set of the parabolic elements of $\mathrm{PSL}(2,\mathbb{C})$. This has a quandle structure by conjugation $x * y = y^{-1}xy$. This is isomorphic to the quandle formed by the parabolic elements of $\mathrm{SL}(2,\mathbb{C})$ with trace 2 (or -2). Define a map $\mathbb{C}^2 \setminus \{0\} \to \mathcal{P}$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1-xy & -x^2 \\ y^2 & 1+xy \end{pmatrix}.$$

This is a quandle homomorphism and induces a quandle isomorphism $\mathcal{P} \cong (\mathbb{C}^2 \setminus \{0\})/\pm$. The quandle $(\mathbb{C}^2 \setminus \{0\})/\pm$ and therefore $\mathcal{P}$ has a homogeneous presentation $\mathrm{PSL}(2,\mathbb{C})/P$ where P is the parabolic subgroup. So $H_3^{\Delta}(\mathcal{P})$ is closely related to the relative homology $H_3(\mathrm{PSL}(2,\mathbb{C}), P; \mathbb{Z})$.

7. Extended Bloch group

The *pre-Bloch group* $\mathcal{P}(\mathbb{C})$ is the quotient of the free abelian group generated by symbols $[z]$, $z \in \mathbb{C} \setminus \{0,1\}$ and the relation given by

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0$$

for each $x, y \in \mathbb{C} \setminus \{0,1\}$ with $x \neq y$. This relation is called the *five term relation*. The *Bloch group* $\mathcal{B}(\mathbb{C})$ is the kernel of the homomorphism $\lambda : \mathcal{P}(\mathbb{C}) \to \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$ defined by $\lambda([z]) = z \wedge (1-z)$.

The *extended pre-Bloch group* $\widehat{\mathcal{P}}(\mathbb{C})$ is the quotient of the free abelian group generated by $[z; p, q]$ where $z \in \mathbb{C} \setminus \{0,1\}$ and $p, q \in \mathbb{Z}$ with relation given by the *lifted five term relation*, which is something like a lifting of the five term relation. In some sense, the pre-Bloch group is a lift of pre-Bloch group to the universal abelian cover of $\mathbb{C} \setminus \{0,1\}$, and p and q represent the branches at 0 and 1 respectively. The *extended Bloch group* is the kernel of the map $\widehat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C} \wedge_{\mathbb{Z}} \mathbb{C}$ defined by $[z; p, q] \mapsto (\mathrm{Log}(z) + p\pi i) \wedge (-\mathrm{Log}(1-z) + q\pi i)$. See [Neu] for details.

We construct a map from $C_3^{\Delta}(\mathcal{P}) \to \widehat{\mathcal{P}}(\mathbb{C})$ along with the work of Dupont and Zickert [DZ]. In this note we omit the discussion on the treatment of degenerate simplices because it makes the argument more complicated. Let $(x_0, \ldots, x_3)$ be an element of $C_3^{\Delta}(\mathcal{P})$. Since we have $\mathcal{P} \cong (\mathbb{C}^2 \setminus \{0\})/\pm$, we regarded $x_0, \ldots, x_3$ as 2-dimensional column vectors. We define three complex numbers by

$$\begin{aligned} w_0 &= \mathrm{Log}\det(x_0,x_3) + \mathrm{Log}\det(x_1,x_2) - \mathrm{Log}\det(x_0,x_2) - \mathrm{Log}\det(x_1,x_3), \\ w_1 &= \mathrm{Log}\det(x_0,x_2) + \mathrm{Log}\det(x_1,x_3) - \mathrm{Log}\det(x_0,x_1) - \mathrm{Log}\det(x_2,x_3), \\ w_2 &= \mathrm{Log}\det(x_0,x_1) + \mathrm{Log}\det(x_2,x_3) - \mathrm{Log}\det(x_0,x_3) - \mathrm{Log}\det(x_1,x_2). \end{aligned} \tag{7.1}$$

Here $\det(x_i, x_j)$ is the determinant of $\begin{pmatrix} x_i^1 & x_j^1 \\ x_i^2 & x_j^2 \end{pmatrix}$ for $x_i = \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix}$ and $x_j = \begin{pmatrix} x_j^1 \\ x_j^2 \end{pmatrix}$, and the Log is defined by $\mathrm{Log}(z) = \log|z| + i\arg(z)$ $(-\pi < \arg(z) \leq \pi)$. Since x_i are well-defined only up to sign, the value $\mathrm{Log}\det(x_i, x_j)$ has an ambiguity of $\pm\pi i$. So we fix the value of $\det(x_i, x_j)$ so that $0 \leq \arg(\det(x_i, x_j)) < \pi$. (We can show that another choice of sign does not change the image. This can be shown by using the *cycle relation* [Neu, Section 6].) Let z be the complex number defined by

$$z = \frac{(x_0^1/x_0^2 - x_3^1/x_3^2)\,(x_1^1/x_1^2 - x_2^1/x_2^2)}{(x_0^1/x_0^2 - x_2^1/x_2^2)\,(x_1^1/x_1^2 - x_3^1/x_3^2)},$$

then w_0, w_1, w_2 have the following form:

$$\begin{aligned} w_0 &= \mathrm{Log}(z) + p\pi i, \quad w_1 = -\mathrm{Log}(1-z) + q\pi i, \\ w_2 &= -\mathrm{Log}(z) + \mathrm{Log}(1-z) - (p+q)\pi i. \end{aligned}$$

here p and q are some integers. We assign $[z;p,q] \in \widehat{\mathcal{P}}(\mathbb{C})$ for $(x_0, x_1, x_2, x_3) \in C_3^\Delta(\mathcal{P})$. This defines a map $C_3^\Delta(\mathcal{P}) \to \widehat{\mathcal{P}}(\mathbb{C})$ and induces the map $H_3^\Delta(\mathcal{P}) \to \widehat{\mathcal{B}}(\mathbb{C})$ (here this is not precise statement, see Remark 7.2). Composing with the map defined in Section 5, we have the following theorem.

Theorem 7.1. *There exists a homomorphism*

$$H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}]) \to \widehat{\mathcal{B}}(\mathbb{C}). \tag{7.2}$$

The image of the cycle $[C(\mathcal{S})]$ by this map gives the extended Bloch invariant of the parabolic representation.

Remark 7.2. We could not construct a map $C_3^\Delta(\mathcal{P}) \to \widehat{\mathcal{B}}(\mathbb{C})$ directory because we could not remove the degenerate simplices. But we can deform any cycle of $H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}])$ by adding boundary term so that the image by the map $\varphi_* : H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}]) \to H_3^\Delta(\mathcal{P})$ consists of non-degenerate simplices without changing the homology class. So we can actually construct the map (7.2).

Neumann showed in [Neu] that $\widehat{\mathcal{B}}(\mathbb{C}) \cong H_3(\mathrm{PSL}(2,\mathbb{C}); \mathbb{Z})$. He also defined the Rogers dilogarithm function $R : \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}$ which gives a combinatorial formula of the Cheeger-Chern-Simons class via the isomorphism. Applying the function R to the image of $[C(\mathcal{S})]$ by the map (7.2), we obtain a diagrammatic description of the volume and the Chern-Simons invariant.

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