The volume and the Chern-Simons invariant of a PSL(2,C)-representation and quandle homology

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We use *quandles* to study representations of knot groups into some group. (e.g. into $\mathrm{PSL}(2,\mathbb{C})$)

Quandle homology is a useful tool to construct a conjugacy invariant of the representations.

We apply these to $\mathrm{PSL}(2,\mathbb{C})$-representations of a knot complement. As a result we obtain a diagrammatic description of extended Bloch invariants.

Strategy

1.

$\rho : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$ parabolic representations	$\overset{1:1}{\longleftrightarrow}$	*Arc colorings* $\mathcal{A}$ by a quandle $\mathcal{P}$ (*Shadow colorings* $\mathcal{S}$)

2. Construct an invariant $[C(\mathcal{S})]$ with values in the quandle homology $H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}])$.

3.

Quandle homology	general theory $\downarrow$	Simplicial quandle homology	Extended Bloch group
$H_2^Q(\mathcal{P}; \mathbb{Z}[\mathcal{P}])$	$\xrightarrow{\varphi_*}$	$H_3^{\Delta}(\mathcal{P})$	$\widehat{\mathcal{B}}(\mathbb{C})$
$\Cup$			
$[C(\mathcal{S})]$			

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$\Cup$		'$\wr\Vert$'	$\nearrow$	Dupont-Zickert
$[C(\mathcal{S})]$		$H_3(C_*^{h\neq}(\mathbb{C}^2)_{\mathrm{PSL}(2,\mathbb{C})})$		

Quandle

The definition of quandles was given by Joyce in 1982.

A quandle X is a set with a binary operation $*$ satisfying

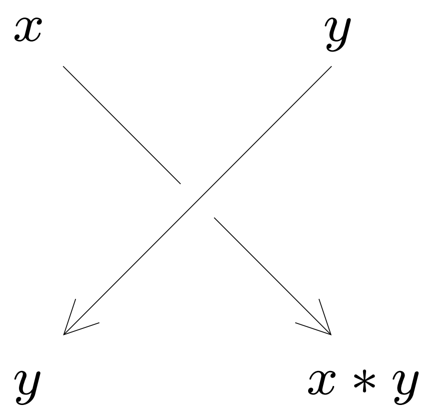
1. $x * x = x$ for any $x \in X$,
2. there exists an inverse of $*y : X \to X$, (denote it by $*^{-1}y$,)
3. $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

Example

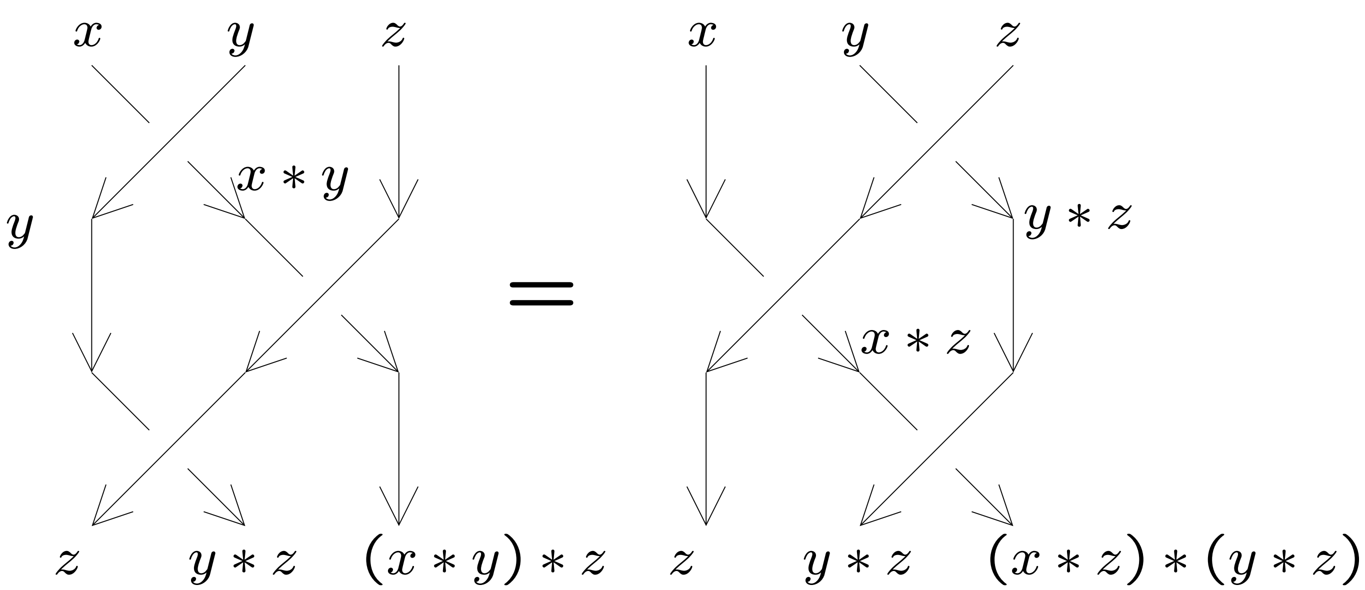
A group G has a quandle structure by conjugation $x * y = y^{-1}xy$.

Relation with quantum invariants

The map $c : X \times X \to X \times X$ defined by $(x, y) \mapsto (y, x * y)$ is a (set theoretic) Yang-Baxter solution, i.e. satisfying the following relation:



$$(c \times \mathrm{id})(\mathrm{id} \times c)(c \times \mathrm{id}) = (\mathrm{id} \times c)(c \times \mathrm{id})(\mathrm{id} \times c)$$



$$\begin{array}{c} X \times X \times X \\ \downarrow \\ X \times X \times X \end{array}$$

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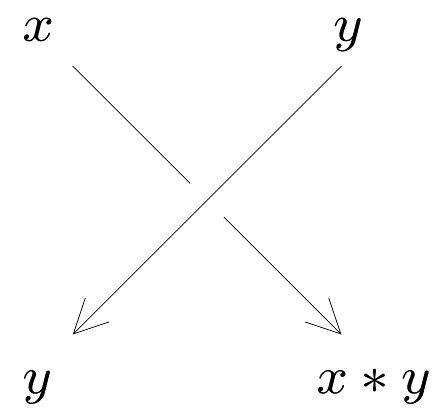

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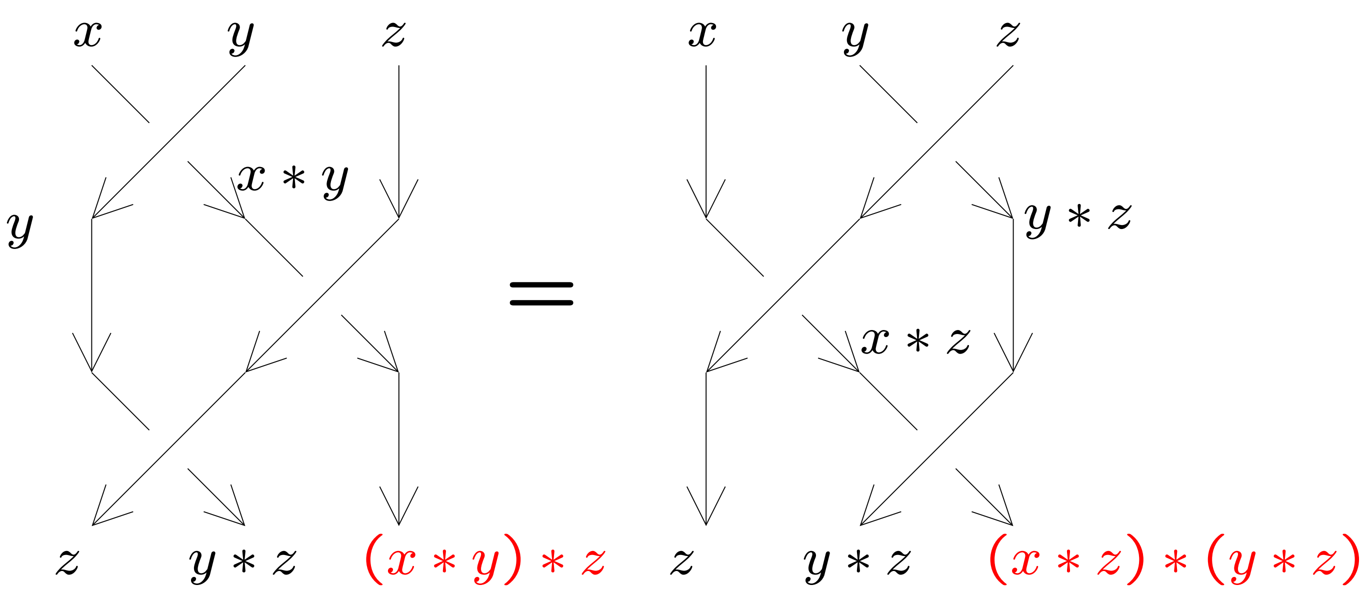
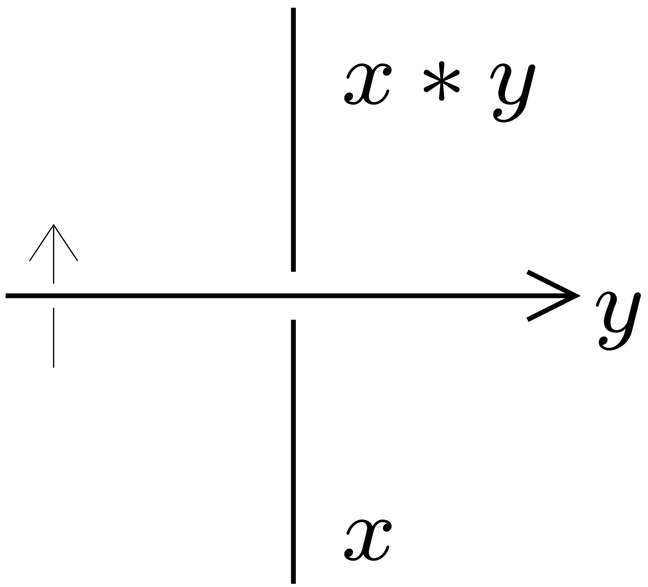


$$\begin{array}{c} X \times X \times X \\ \downarrow \\ X \times X \times X \end{array}$$

Arc coloring

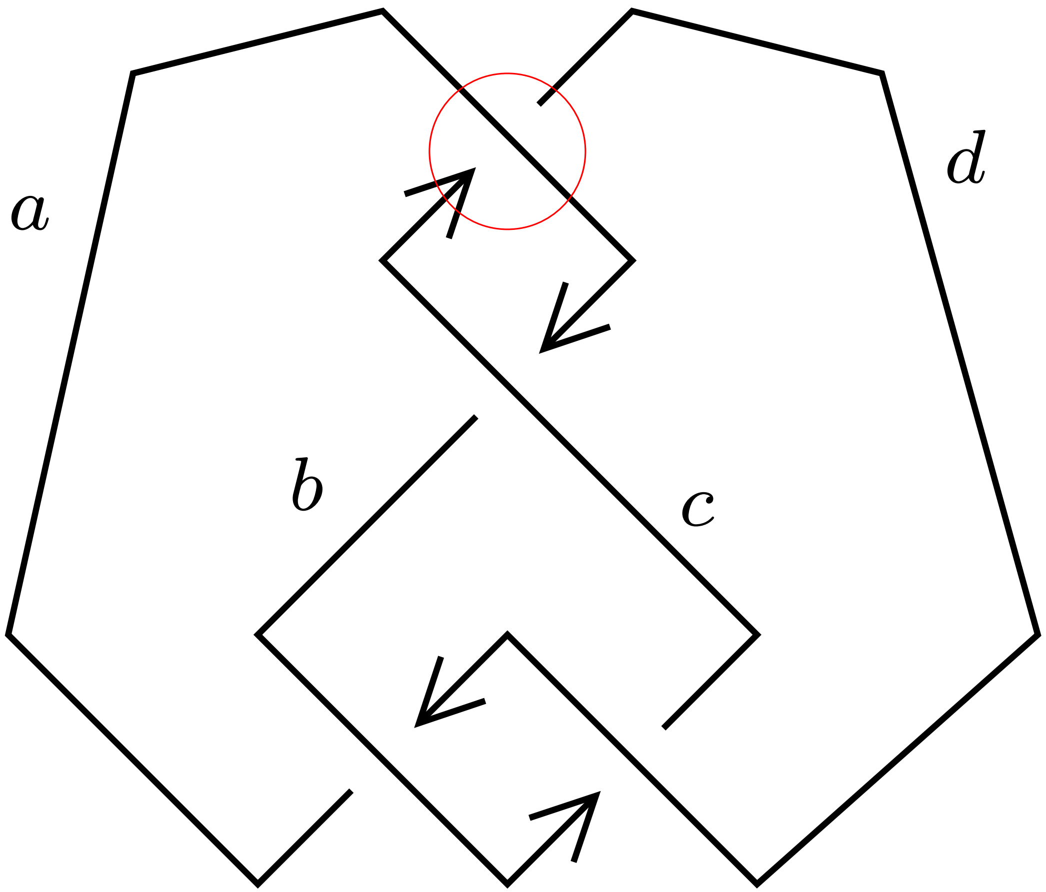
Let D be a diagram of a knot K.

A map $\mathcal{A}$: {arcs of D} $\longrightarrow X$ is called an *arc coloring* if it satisfies the following relation at each crossing.



x, y and $x * y \in X$

Arc coloring of the figure eight knot



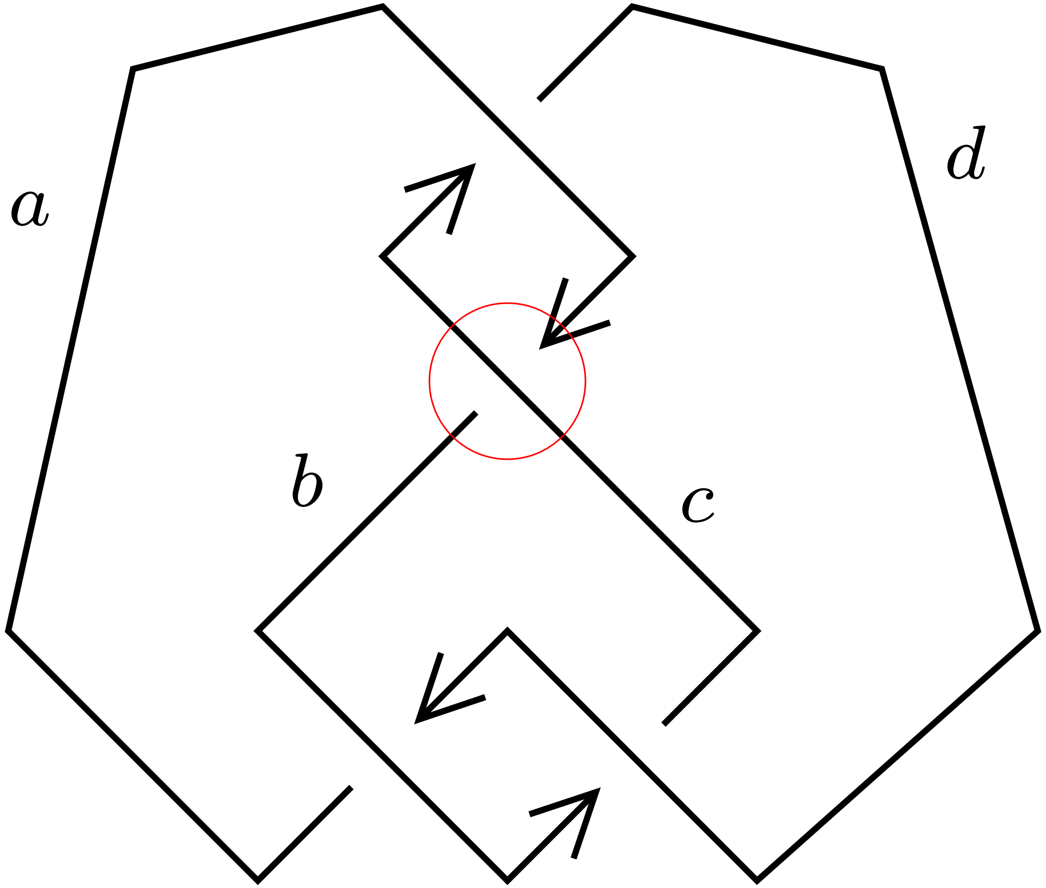
$$c * a = d,$$

$$a * c = b,$$

$$a * b = d,$$

$$c * d = b.$$

Arc coloring of the figure eight knot



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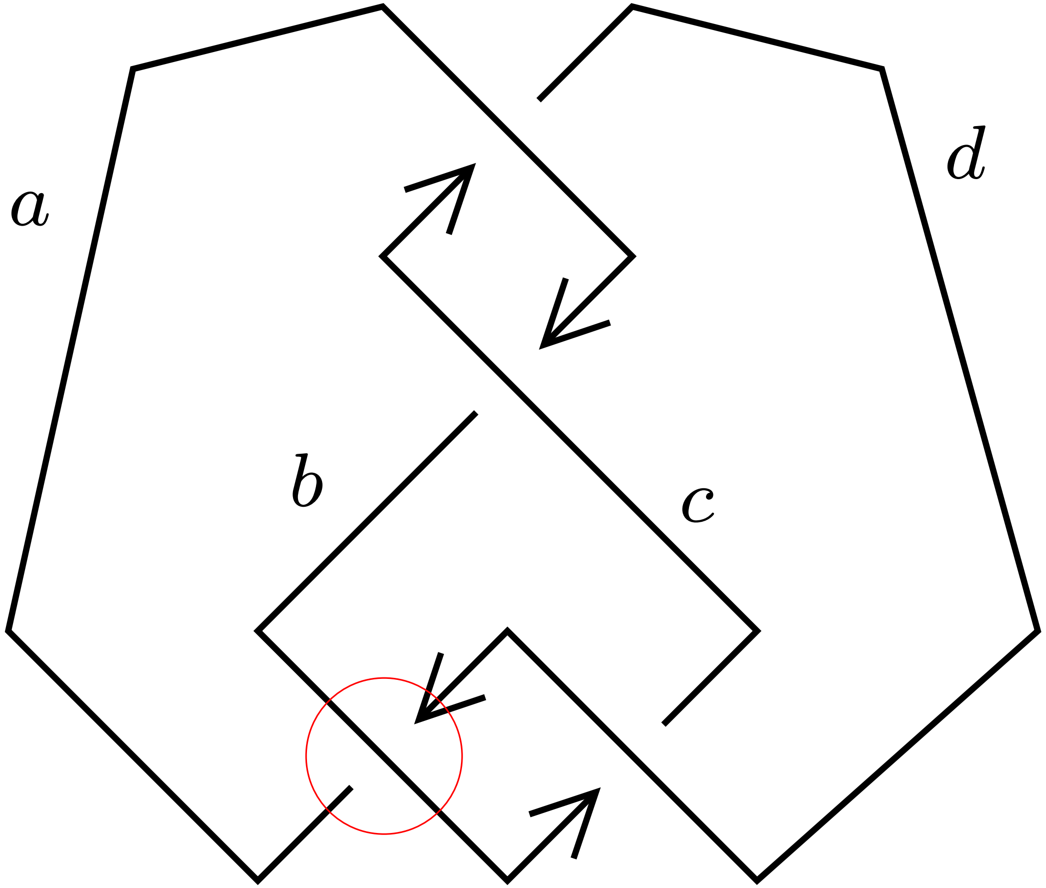

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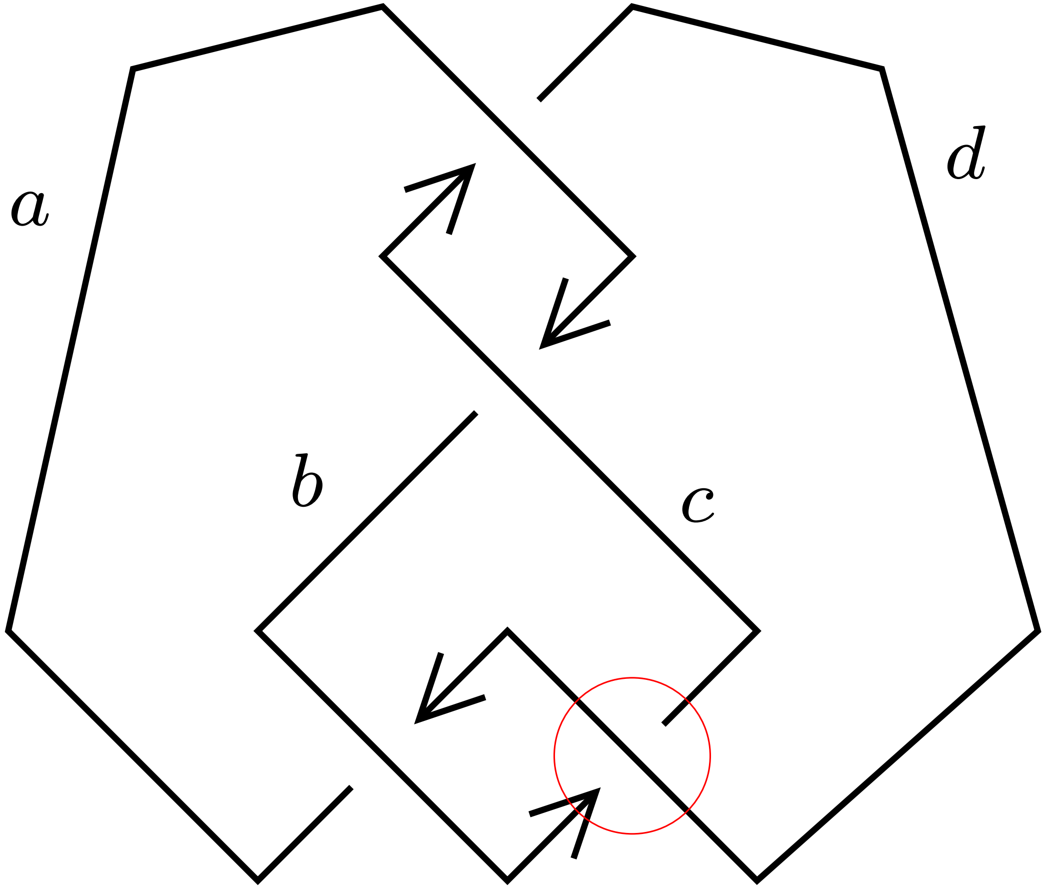

$c * a = d,$

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$a * b = d,$

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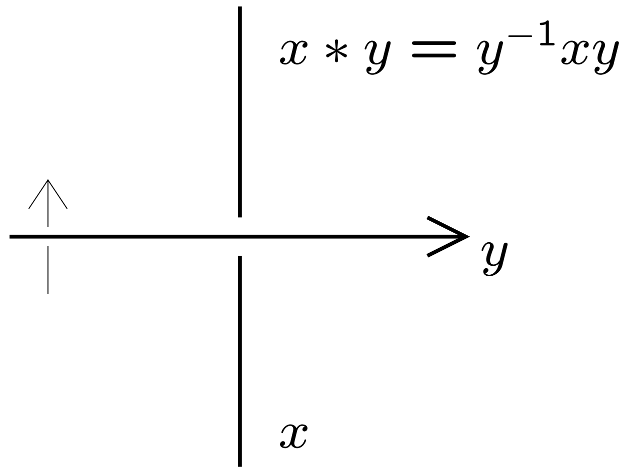
Associated group

For a quandle X, define the group G_X by $\langle x \in X | x * y = y^{-1}xy \rangle$. This is called the associated group of X.

An arc coloring by X gives a representation $\pi_1(S^3 \setminus K) \to G_X$ which sends each meridian to its color. This is a consequence of the Wirtinger presentation of a knot group.



Let $\mathcal{P}$ be the set of the parabolic elements of $\mathrm{PSL}(2,\mathbb{C})$. This has a quandle structure by conjugation $x * y = y^{-1}xy$. As we have seen, an arc coloring gives a representation $\pi_1(S^3 \setminus K) \to G_{\mathcal{P}}$.

Since there is a natural surjection $G_{\mathcal{P}} \to \mathrm{PSL}(2,\mathbb{C})$, this induces a representation $\pi_1(S^3 \setminus K) \to \mathrm{PSL}(2,\mathbb{C})$ which sends each peripheral subgroup to a parabolic subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We call such a representation *parabolic representation*. A typical example is a discrete faithful representation of a hyperbolic knot complement.

Quandle homology

Let $C_n^R(X) = \text{span}_{\mathbb{Z}[G_X]}\{(x_1,\dots,x_n)|x_i \in X\}$. Define the boundary operator $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\begin{aligned}\partial(x_1,\dots x_n) = \sum_{i=1}^{n} (-1)^i\{&(x_1,\dots,\widehat{x_i},\dots,x_n)\\ &- x_i(x_1 * x_i,\dots,x_{i-1} * x_i, x_{i+1},\dots,x_n)\}\end{aligned}$$

Let M be a right $\mathbb{Z}[G_X]$-module. The homology group of $M \otimes_{\mathbb{Z}[G_X]} C_n^R(X)$ is called the *rack homology* $H_n^R(X; M)$.

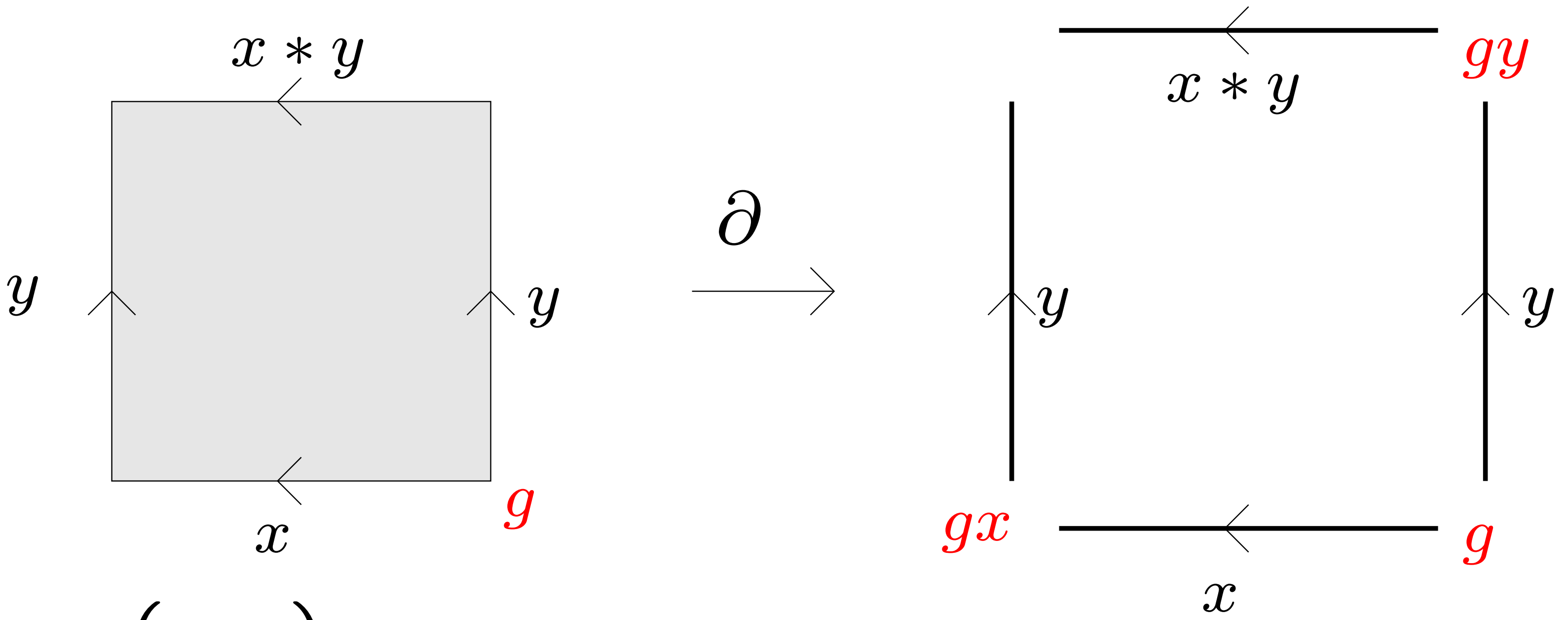
Factoring degenerate chains, we also define the quandle homology $H_n^Q(X;M)$.

Let

$$C_n^D(X) = \text{span}_{\mathbb{Z}[G_X]}\{(x_1,\ldots,x_n)|x_i \in X, \\ x_i = x_{i+1} \text{(for some } i)\}.$$

This is a subcomplex of $C_n^R(X)$. Let $C_n^Q(X)$ be the quotient $C_n^R(X)/C_n^D(X)$. The homology of $M \otimes_{\mathbb{Z}[G_X]} C_n^Q(X)$ is called the *quandle homology* $H_n^Q(X;M)$

Geometric interpretation

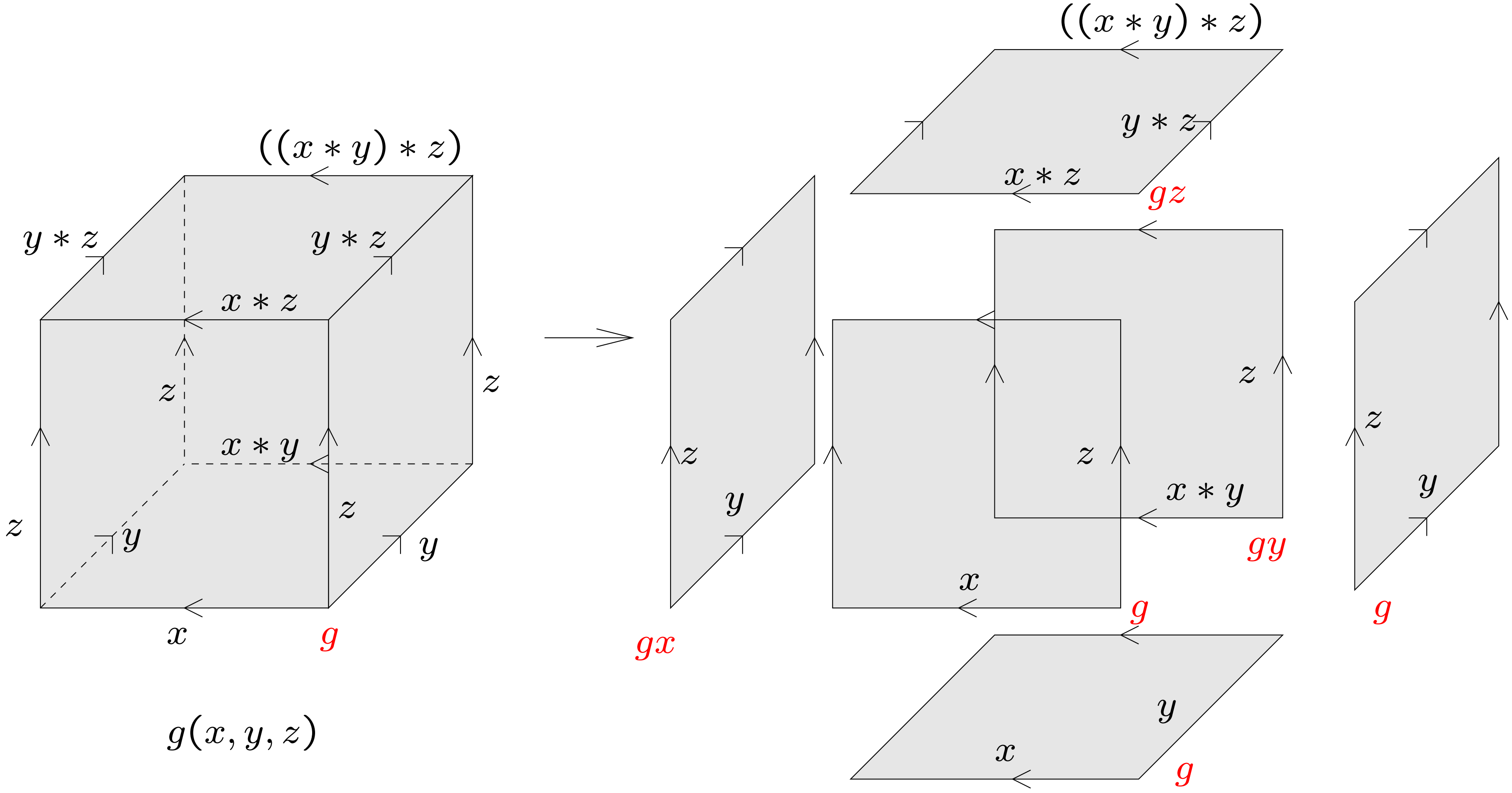


$$g(x, y)$$

$$\begin{aligned} &-g(y) + gx(y) \\ &+g(x) - gy(x * y) \end{aligned}$$

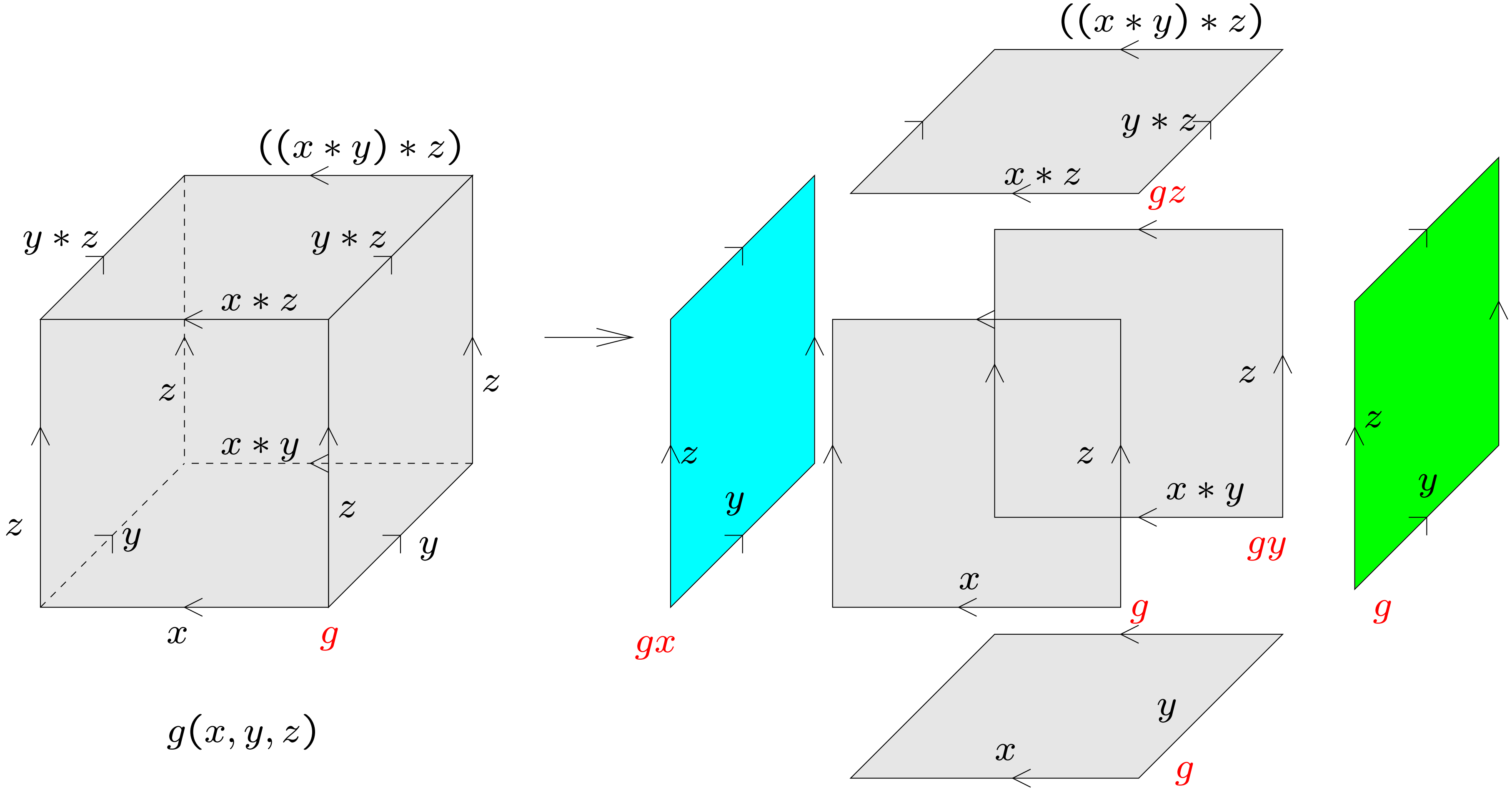
$$\sum_{i=1}^{n} (-1)^i \{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\ - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

Geometric interpretation



$$g(x,y,z) \mapsto -g(y,z) + gx(y,z) + g(x,z) - gy(x*y,z)$$
$$-g(x,y) + gz(x*z,y*z)$$

Geometric interpretation



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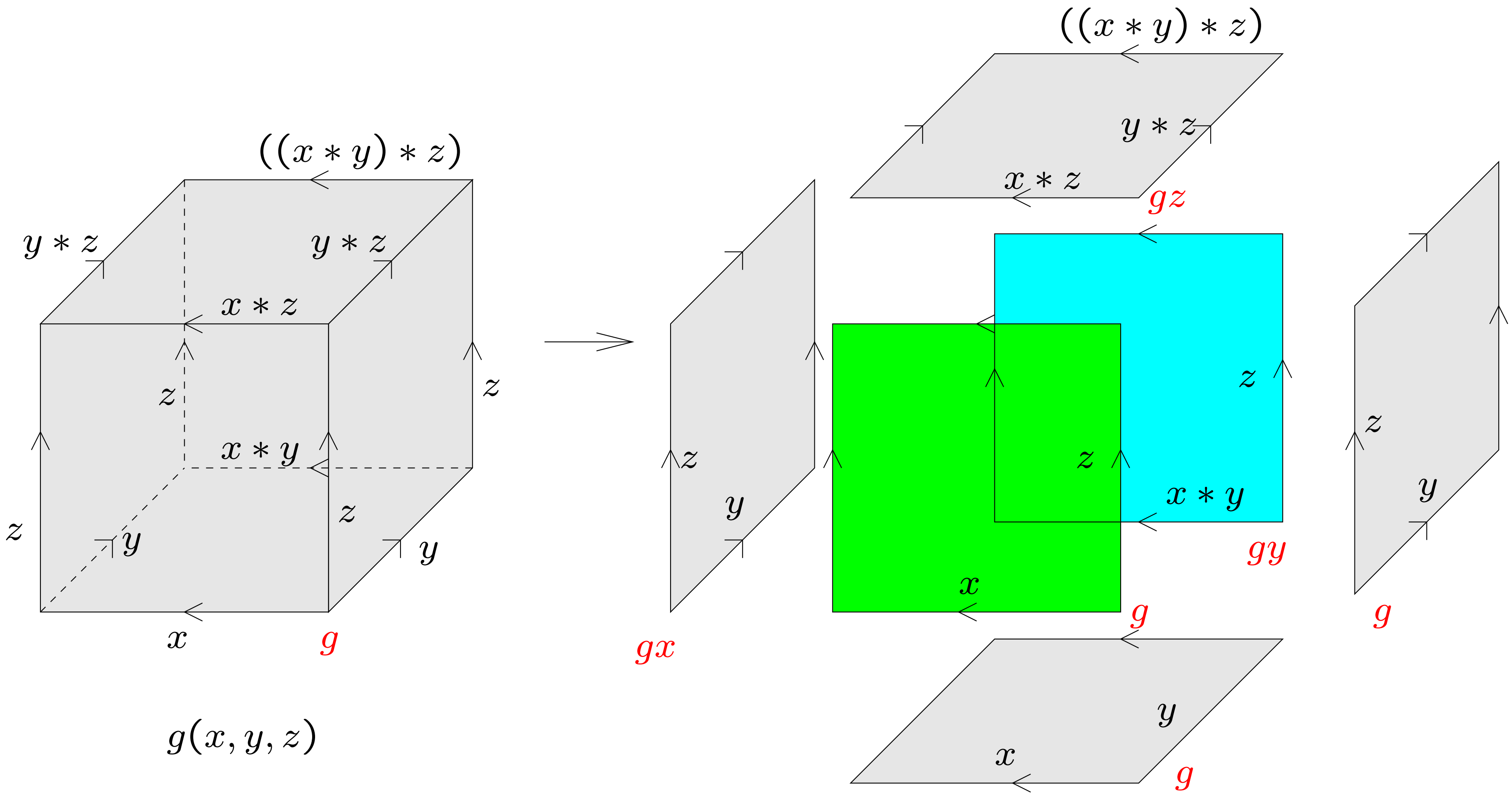

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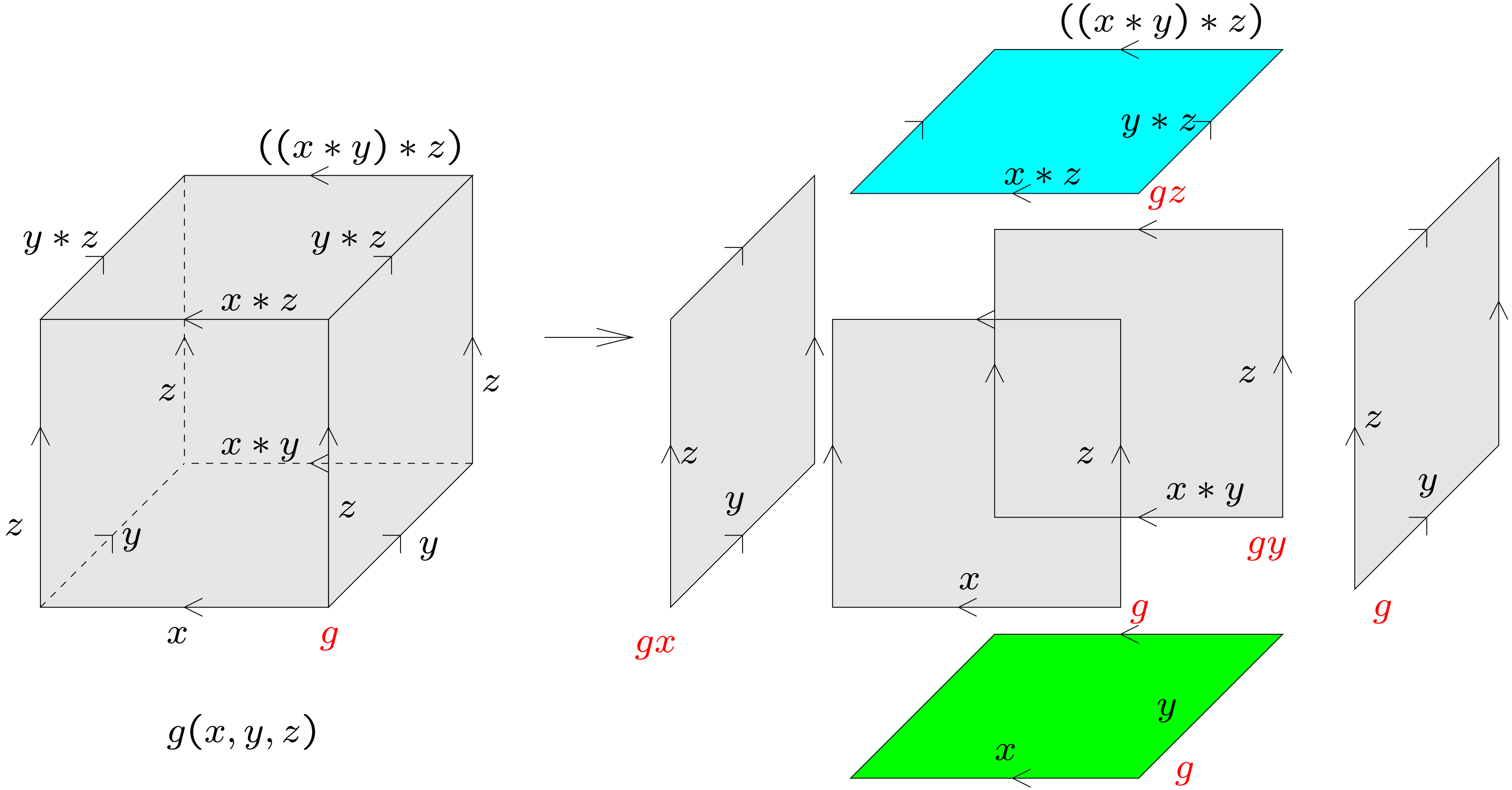

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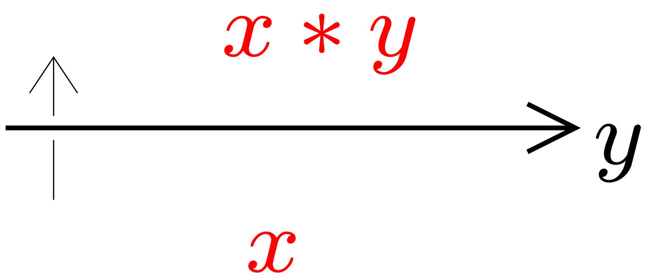
Region coloring

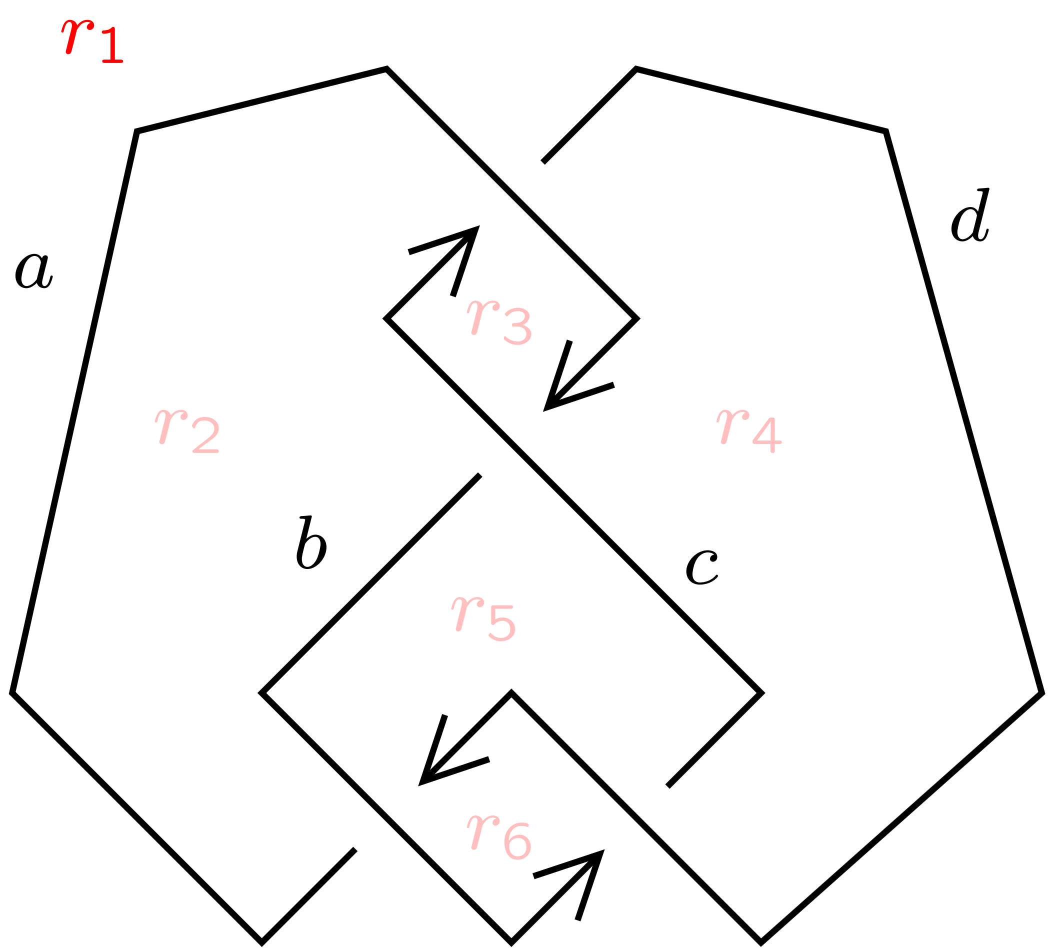
Let D be a diagram and $\mathcal{A}$ be an arc coloring by X. A map $\mathcal{D}$: {regions of D} $\to X$ is called an *region coloring* if it satisfies the following relation:



x, y and $x * y \in X$

A pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ ($\mathcal{A}$: arc coloring, $\mathcal{R}$: region coloring) is called a *shadow coloring*.

Shadow coloring of the figure eight knot

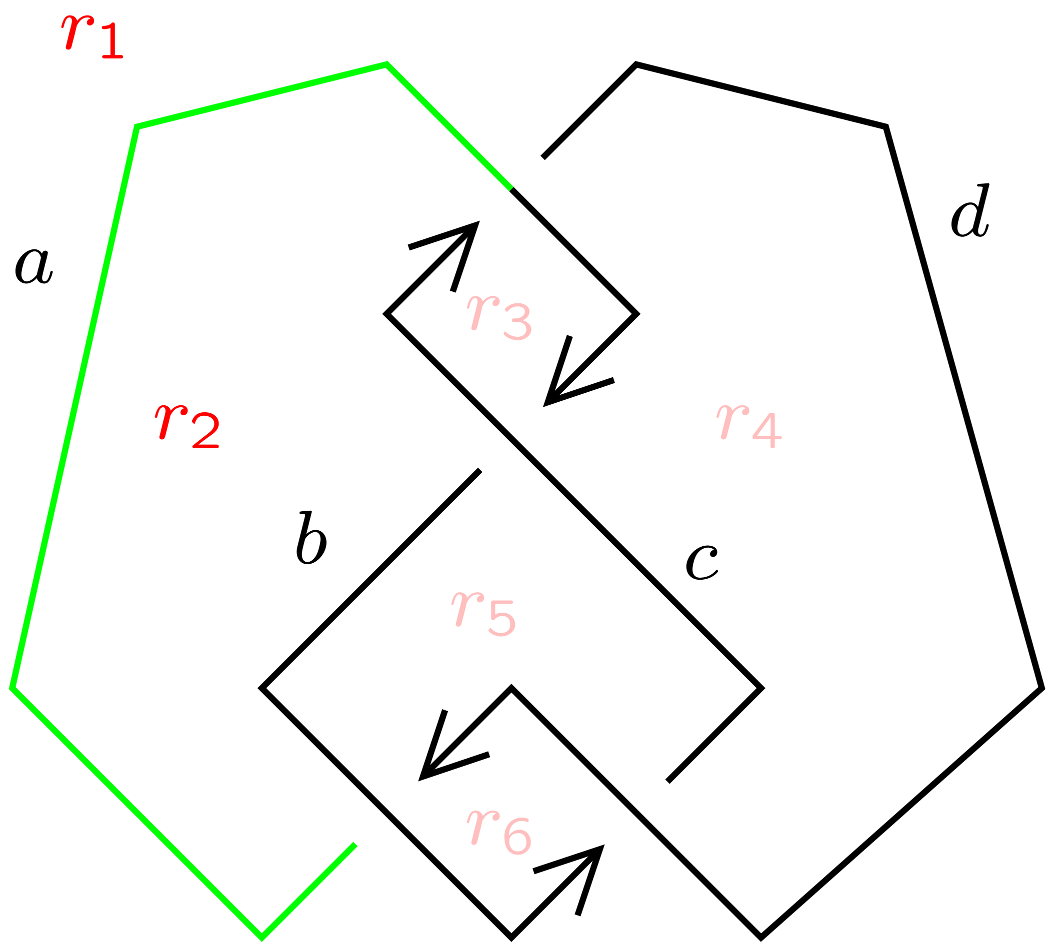


$$r_2 * a = r_1, \qquad r_3 * c = r_2,$$

$$r_3 * a = r_4, \qquad r_2 * b = r_5,$$

$$r_5 * d = r_6,$$

Shadow coloring of the figure eight knot



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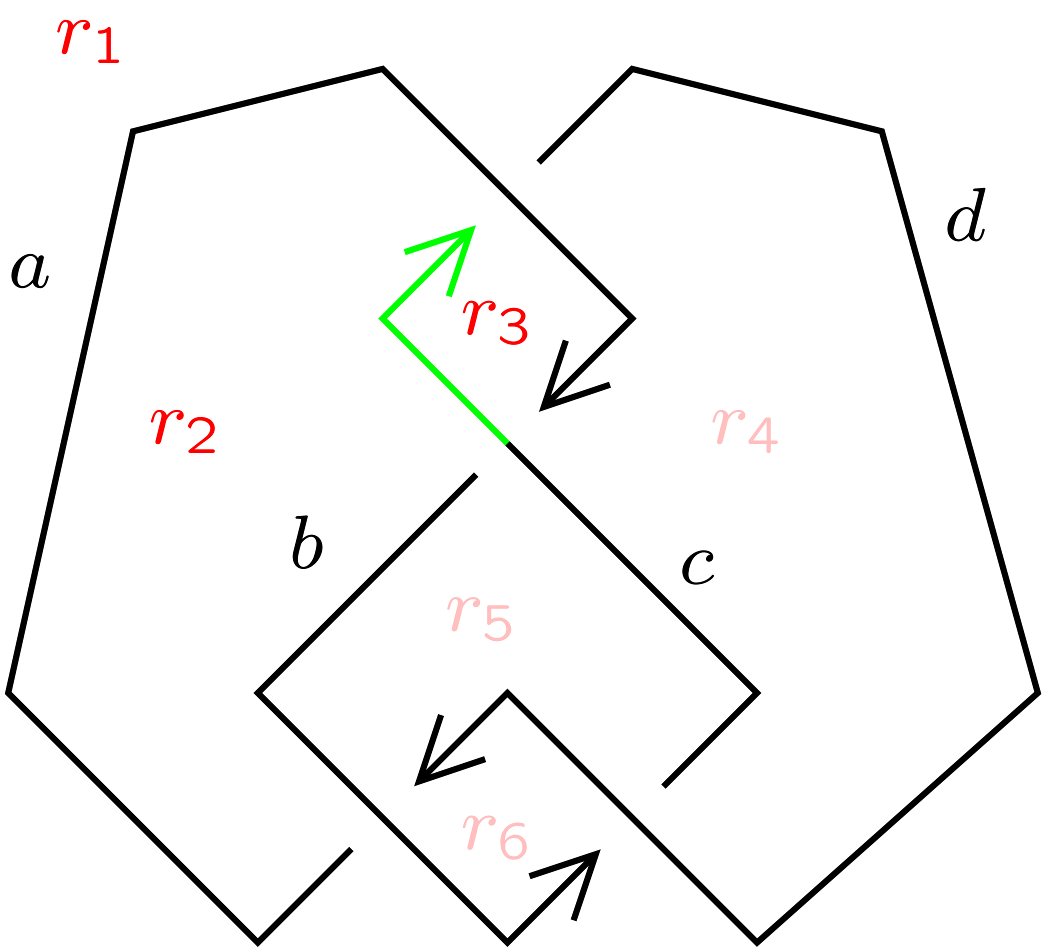

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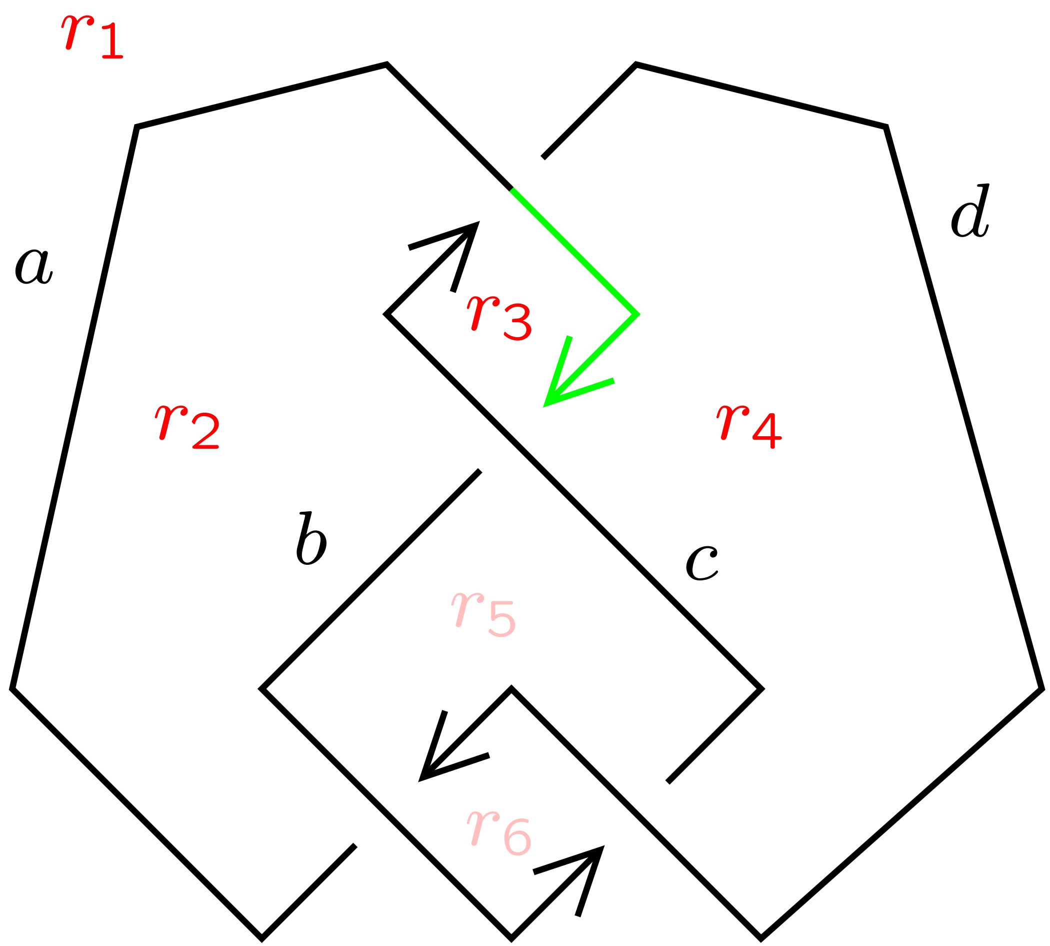

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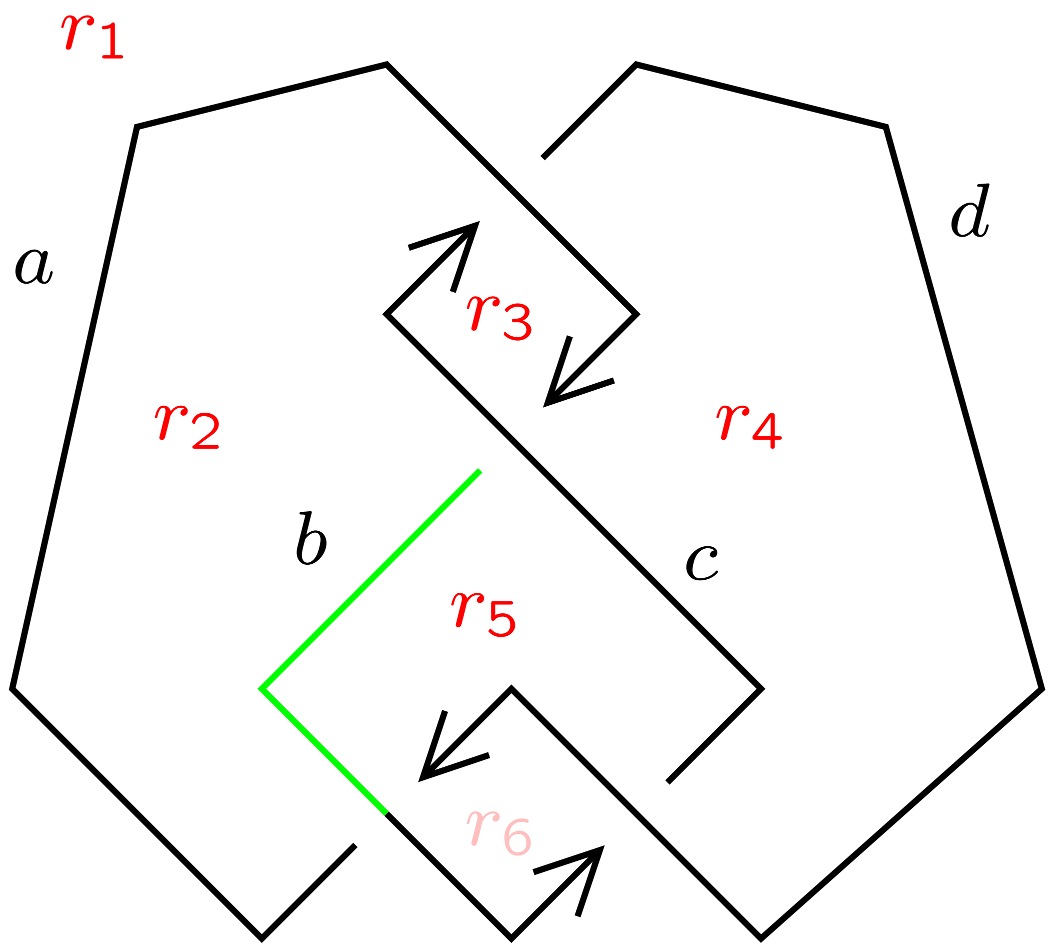

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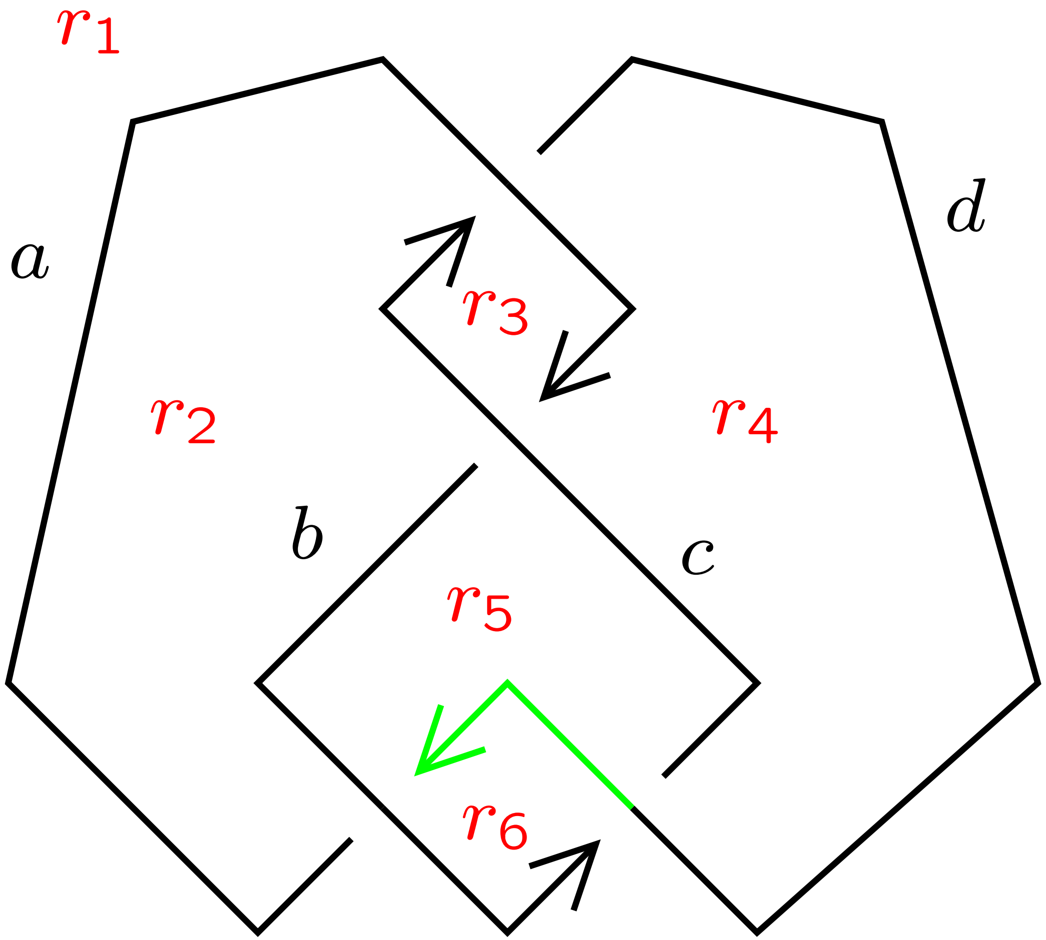

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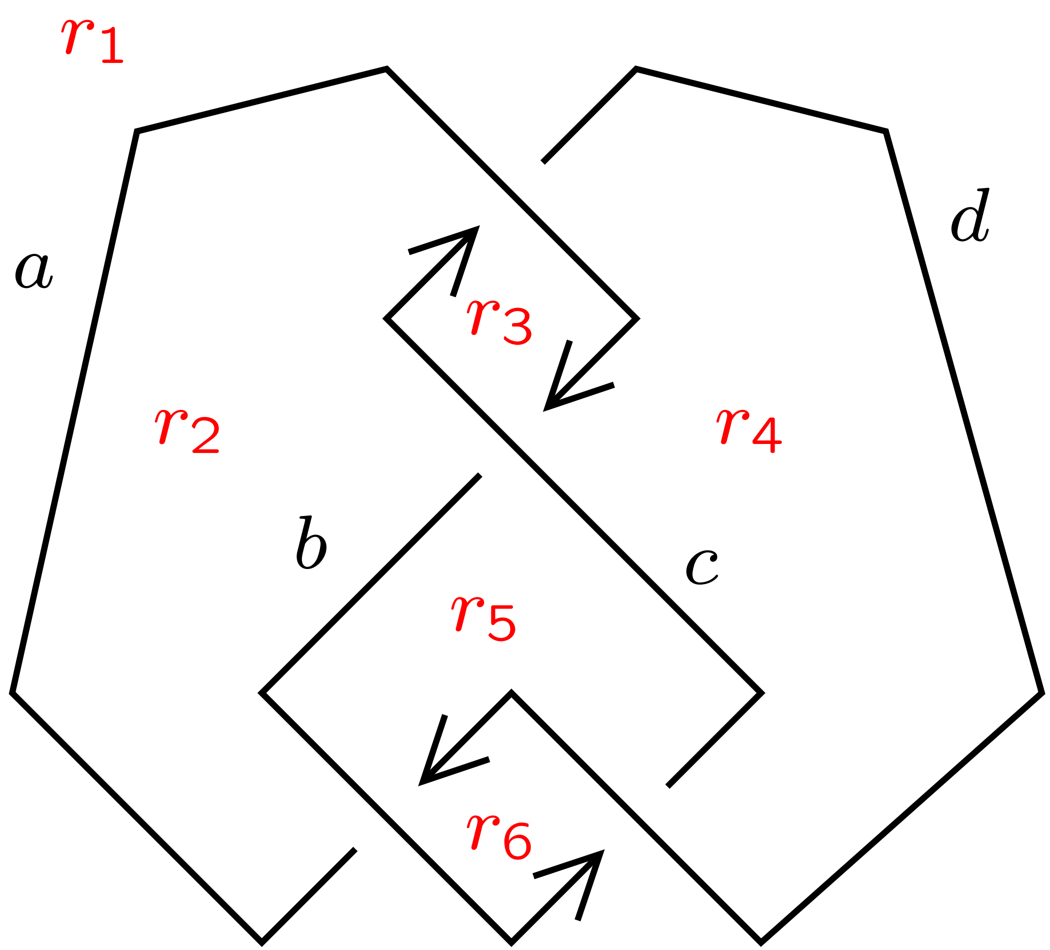


$$r_2 * a = r_1, \quad r_3 * c = r_2,$$

$$r_3 * a = r_4, \quad r_2 * b = r_5,$$

$$r_5 * d = r_6,$$

Other relations are automatically satisfied by the relations of arc colorings:

$$r_4 * d = r_1, \quad r_4 * c = r_5,$$

$$r_1 * b = r_6$$

Remark

If we fix a color of one region, then the colors of other regions are uniquely determined.

Region colorings give no information on the representation of knot group, but it plays an important role to compute volume and Chern-Simons.

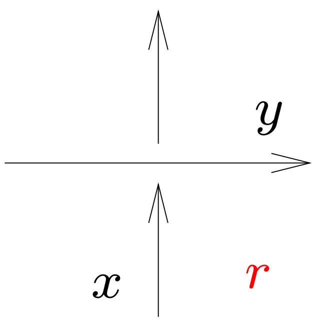
Cycle $[C(\mathcal{S})]$ associated with a shadow coloring

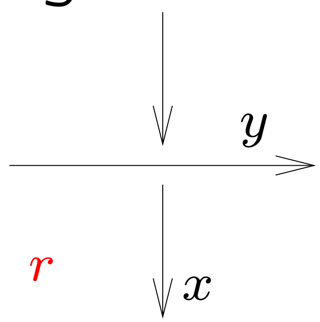
A quandle X itself has a right G_X-action defined by

$$x * (x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}) = (\dots((x *^{\varepsilon_1} x_1) *^{\varepsilon_2} x_2) \dots) *^{\varepsilon_n} x_n.$$

So the free abelian group $\mathbb{Z}[X]$ is a right $\mathbb{Z}[G_X]$-module.

Let $\mathcal{S}$ be a shadow coloring by a quandle X. Assign

$+r \otimes (x, y)$ for 

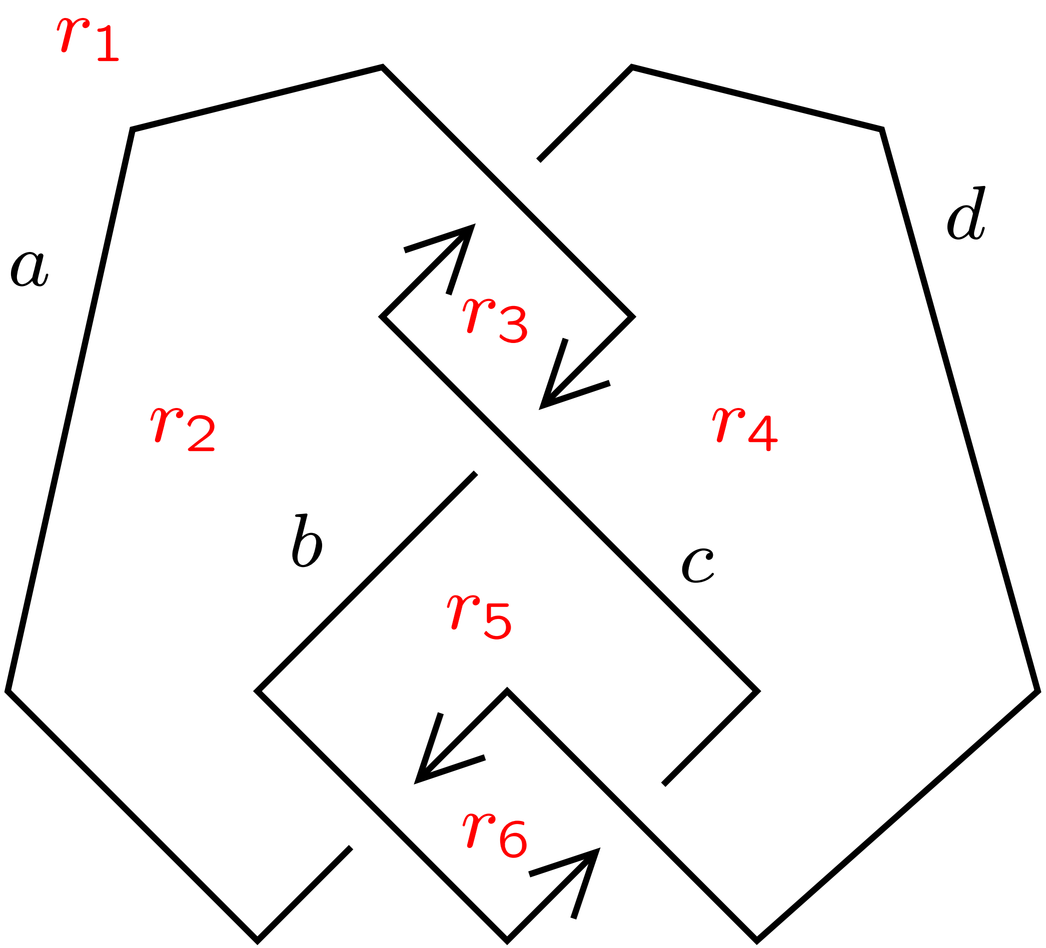
and $-r \otimes (x, y)$ for 

.

Let

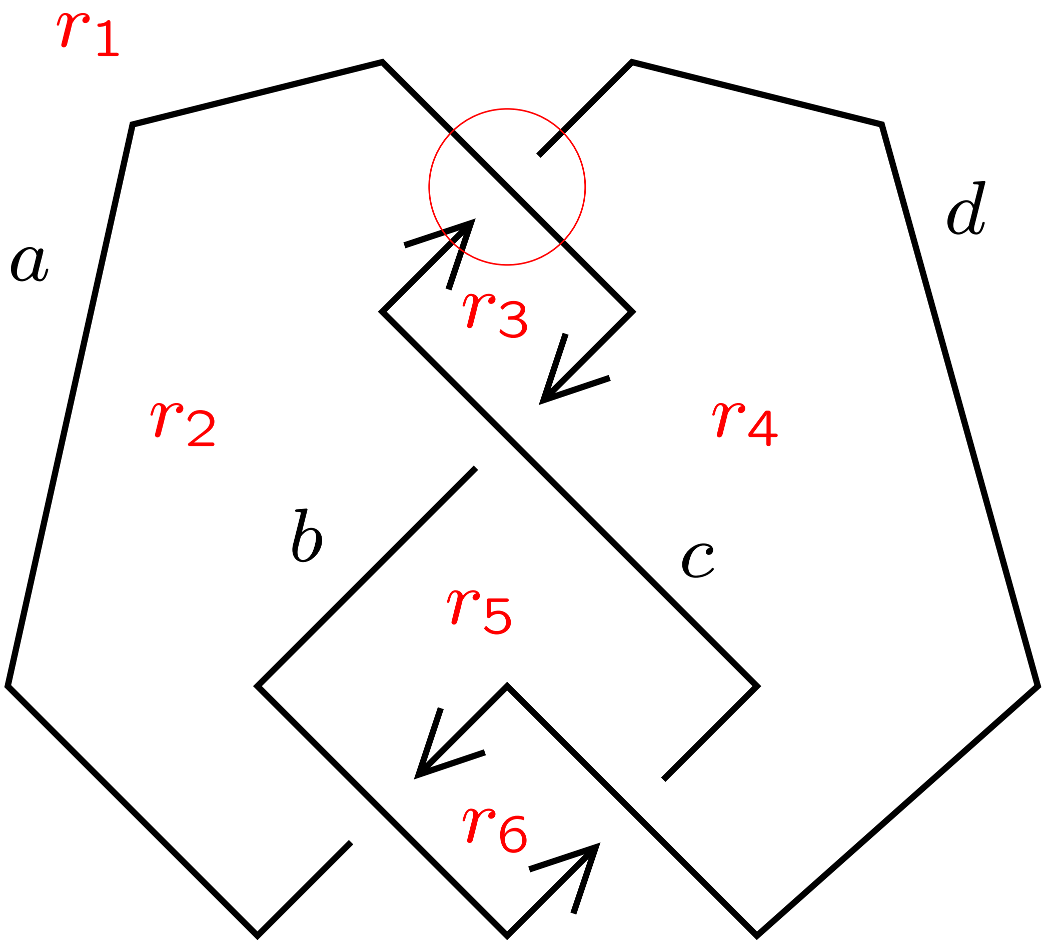
$$C(\mathcal{S}) = \sum_{c:\text{crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[X]).$$

Example: $C(\mathcal{S})$ for the figure eight knot



$$
\begin{aligned}
C(\mathcal{S}) = \\
& r_3 \otimes (c, a) + r_3 \otimes (b, c) \\
& - r_2 \otimes (a, b) - r_4 \otimes (c, d)
\end{aligned}
$$

Example: $C(\mathcal{S})$ for the figure eight knot



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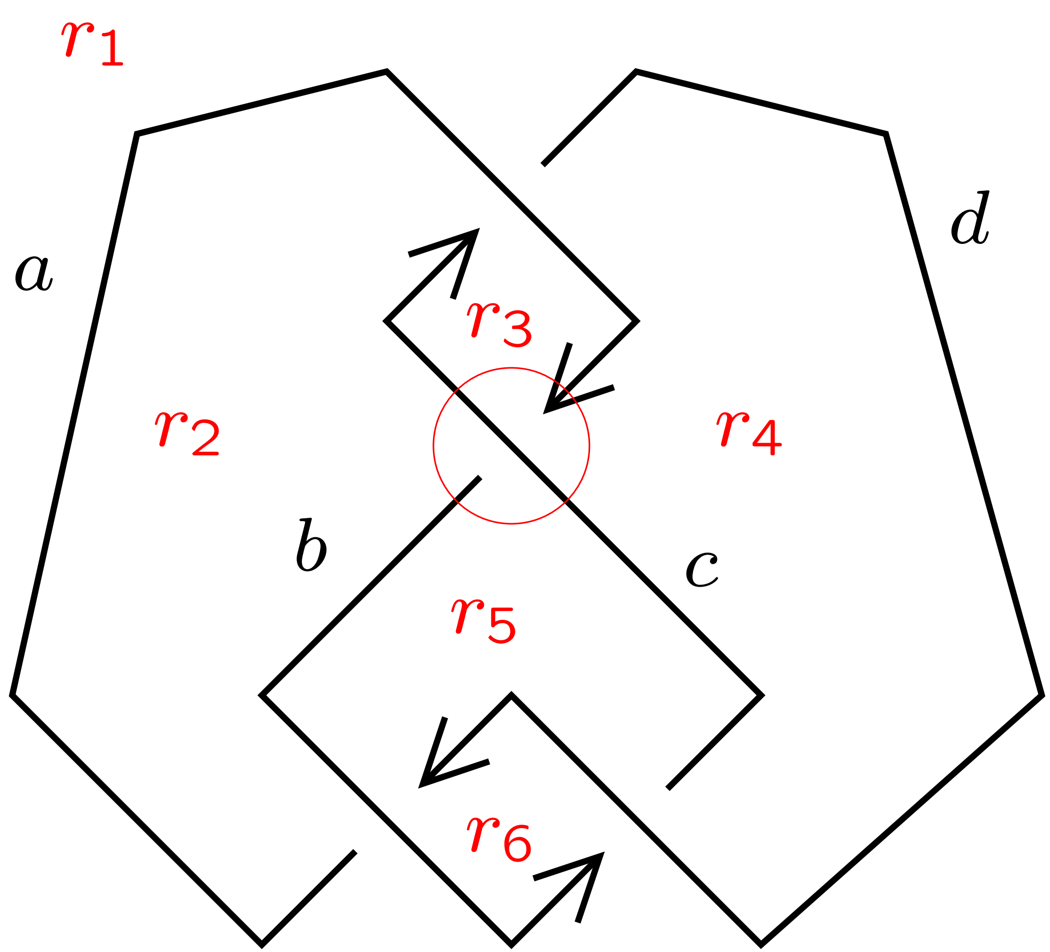

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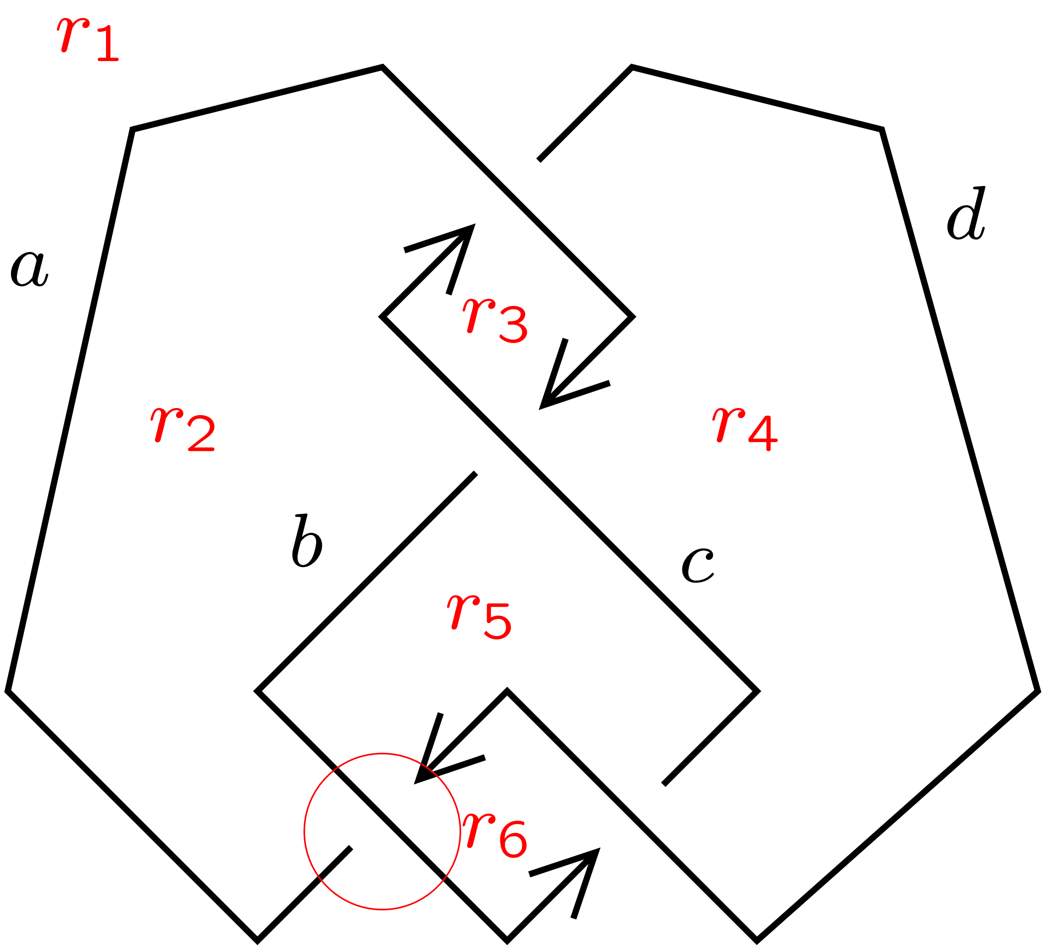

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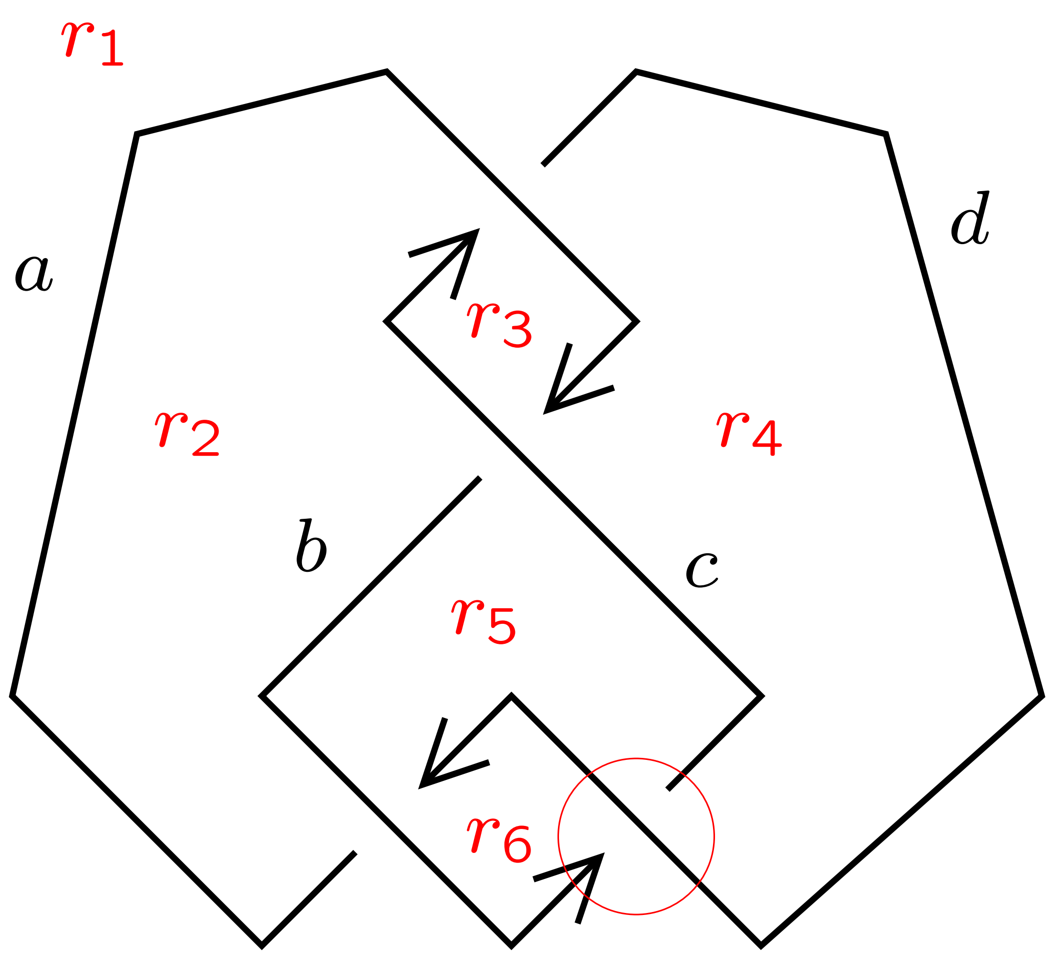

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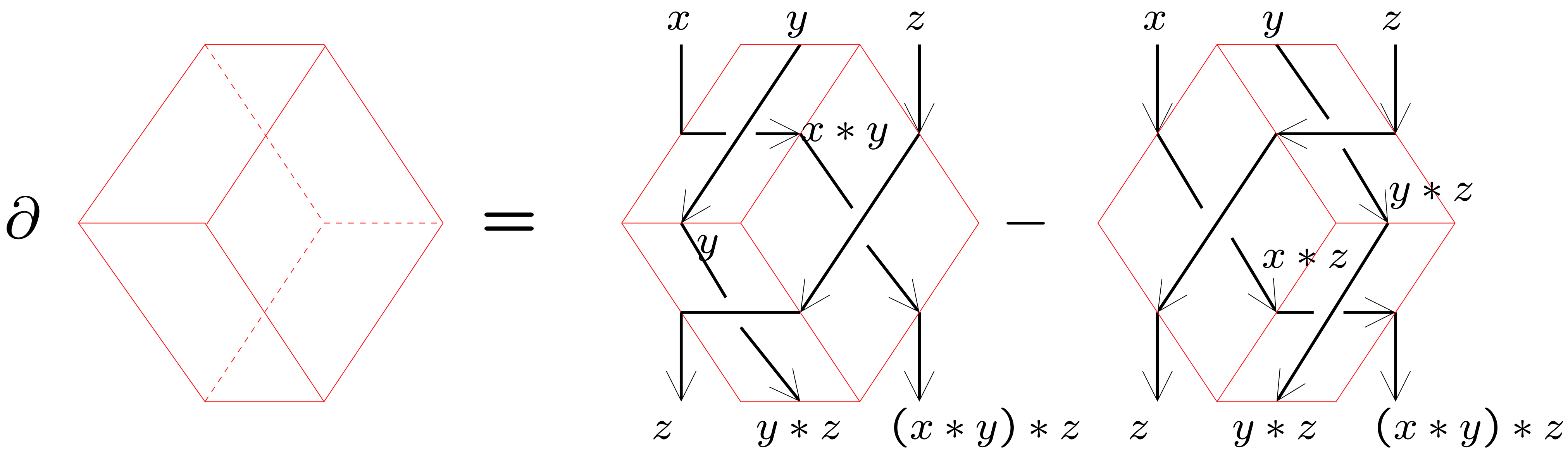
$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$C(\mathcal{S})$ is a cycle. The homology class $[C(\mathcal{S})]$ in $H_2^Q(X;\mathbb{Z}[X])$ is invariant under the Reidemeister moves. The invariance under the Reidemeister III moves is shown in the following figure.



$$\partial(r \otimes (x,y,z)) = (r \otimes (x,y) + r * y \otimes (x * y, z) + r \otimes (y,z))$$
$$- (r \otimes (x,z) + r * x \otimes (y,z) + r * z \otimes (x * z, y * z))$$

We can show that the homology class $[C(\mathcal{S})]$ does not depend on the region coloring. Moreover it only depends on the conjugacy class of the representation $\pi_1(S^3 \setminus K) \to G_X$ induced by the arc coloring. When $X = \mathcal{P}$,

Proposition *the homology class* $[C(\mathcal{S})]$ *in* $H_2^Q(\mathcal{P}, \mathbb{Z}[\mathcal{P}])$ *only depends on the conjugacy class of the parabolic representation* $\pi_1(S^3 \setminus K) \to \mathsf{PSL}(2, \mathbb{C})$ *induced by the arc coloring* $\mathcal{A}$.

Simplicial quandle homology $H_n^{\Delta}(X)$

Let $C_n^{\Delta}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ by

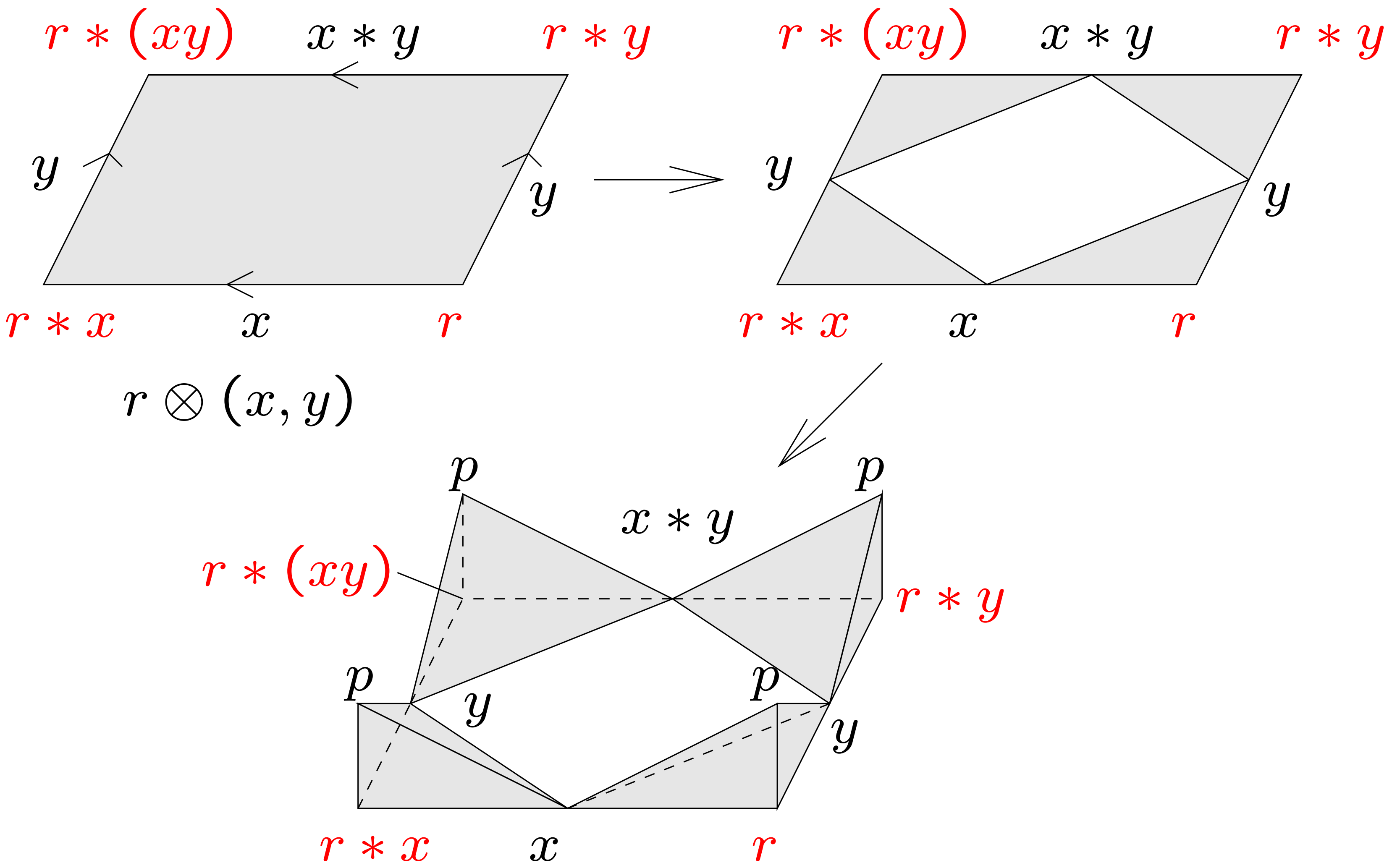
$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

$C_n^{\Delta}(X)$ has a natural right action by $\mathbb{Z}[G_X]$. Denote the homology of $C_n^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by $H_n^{\Delta}(X)$. We can construct a map

$$H_n^R(X; \mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$$

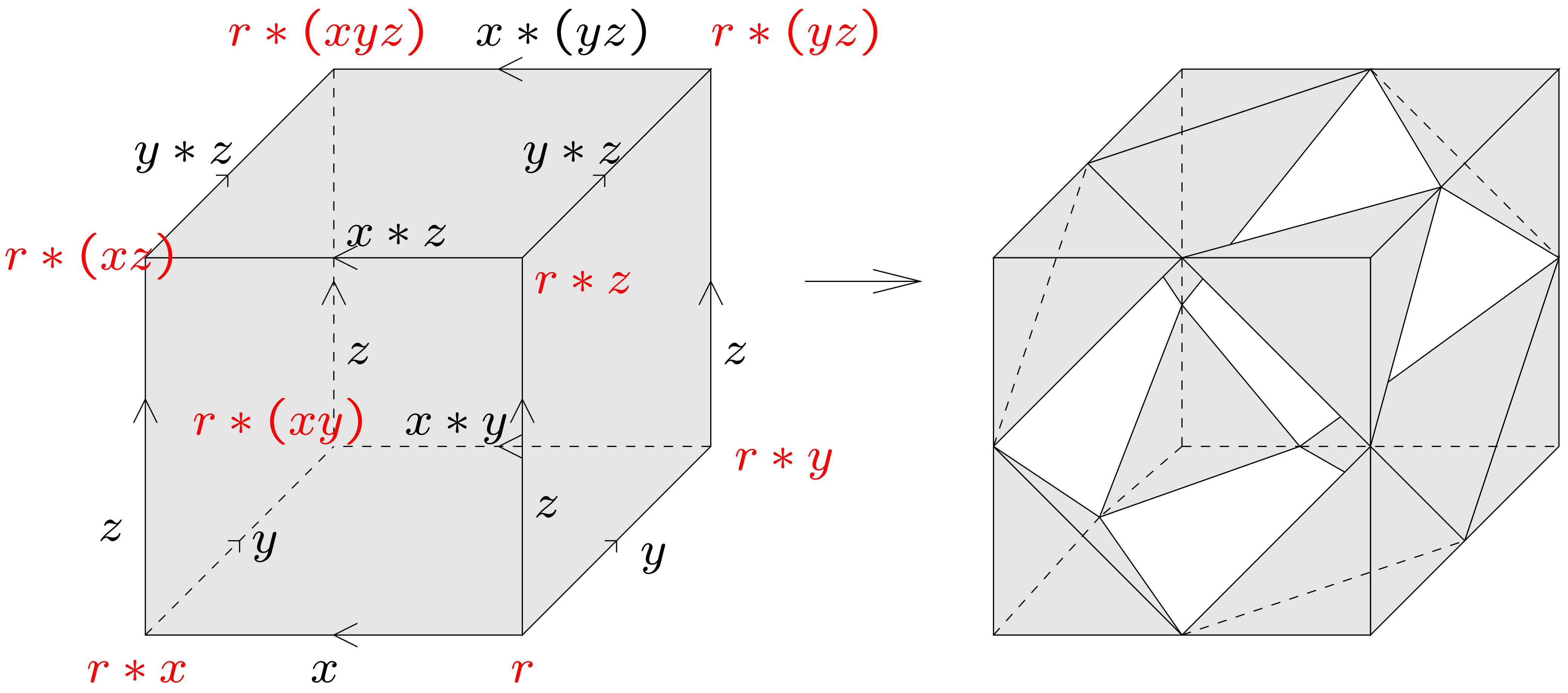
in the following way:

$n = 2$



$$(p, r, x, y) - (p, r * x, x, y)$$
$$-(p, r * y, x * y, y) + (p, r * (xy), x * y, y)$$

$n = 3$



$$
\begin{aligned}
r \otimes (x, y, z) \mapsto (p, r, x, y, z) - (p, r * x, x, y, z) - (p, r * y, x, x * y, z) \\
-(p, r * z, x * z, y * z, z) + (p, r * (xy), x * y, y, z) \\
+(p, r * (xz), x * z, y * z, z) + (p, r * (yz), x * (yz), y * z, z) \\
-(p, r * (xyz), x * (yz), y * z, z)
\end{aligned}
$$

For general case, let I_n be the set of maps $\iota : \{1, 2, \cdots, n\} \to \{0, 1\}$. Let $|\iota|$ denote the cardinality of the set $\{k \mid \iota(k) = 1,\ 1 \leq k \leq n\}$. For $r \otimes (x_1, x_2, \cdots, x_n) \in C_n^R(X; \mathbb{Z}[X])$ and $\iota \in I_n$, define

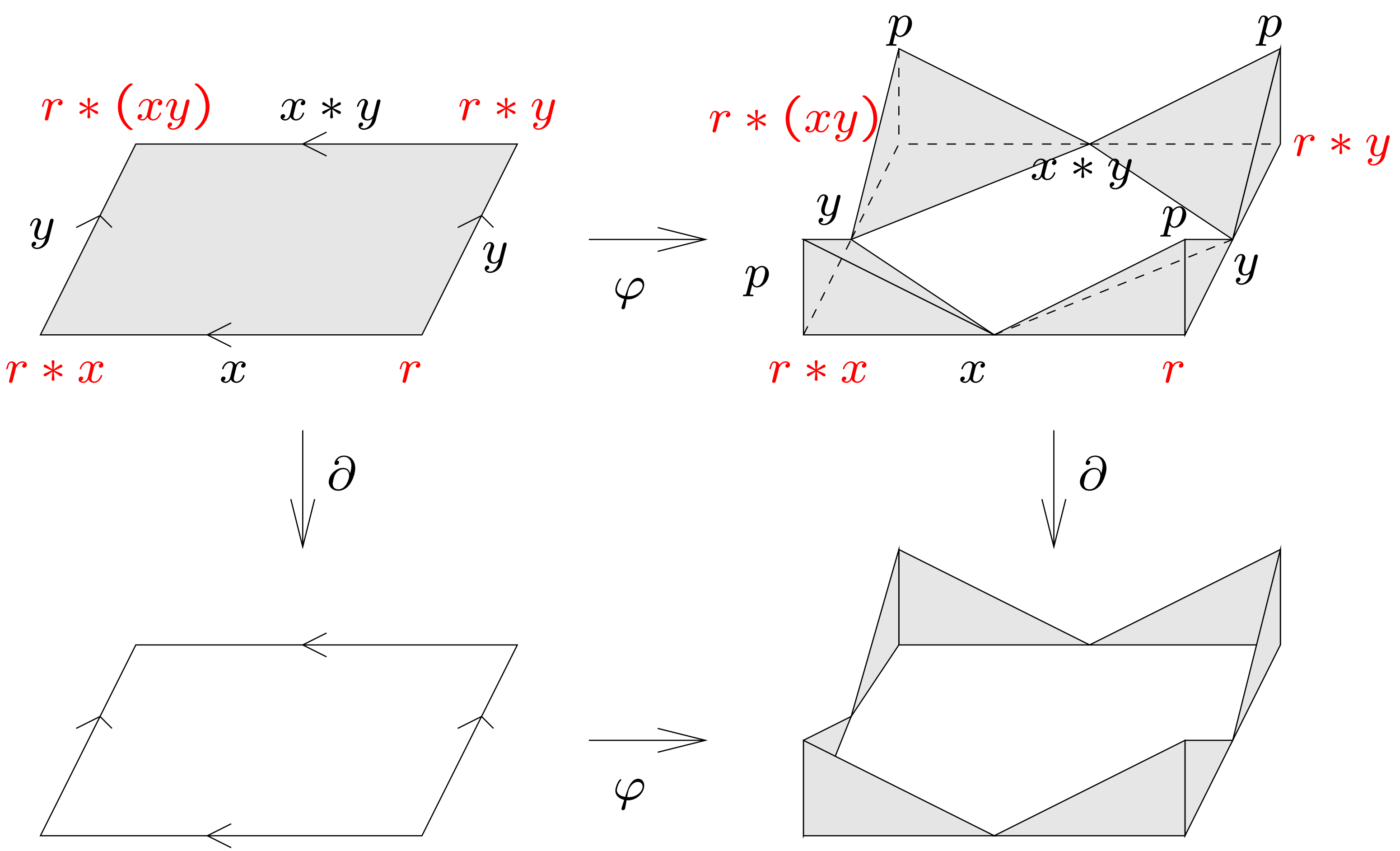
$$\begin{aligned} r(\iota) &= r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)}) \\ x(\iota, i) &= x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)}). \end{aligned}$$

Fix $p \in X$. Define $\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X)_{G_X}$ by

$$\begin{aligned} &\varphi(r \otimes (x_1, x_2, \cdots, x_n)) \\ &\qquad = \sum_{\iota \in I_n} (-1)^{|\iota|} (p, r(\iota), x(\iota, 1), x(\iota, 2), \cdots, x(\iota, n)). \end{aligned}$$

Theorem $\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X)_{G_X}$ *is a chain map.*

Proof.



■

Theorem $\varphi : C_n^R(X;\mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X)_{G_X}$ *is a chain map.*

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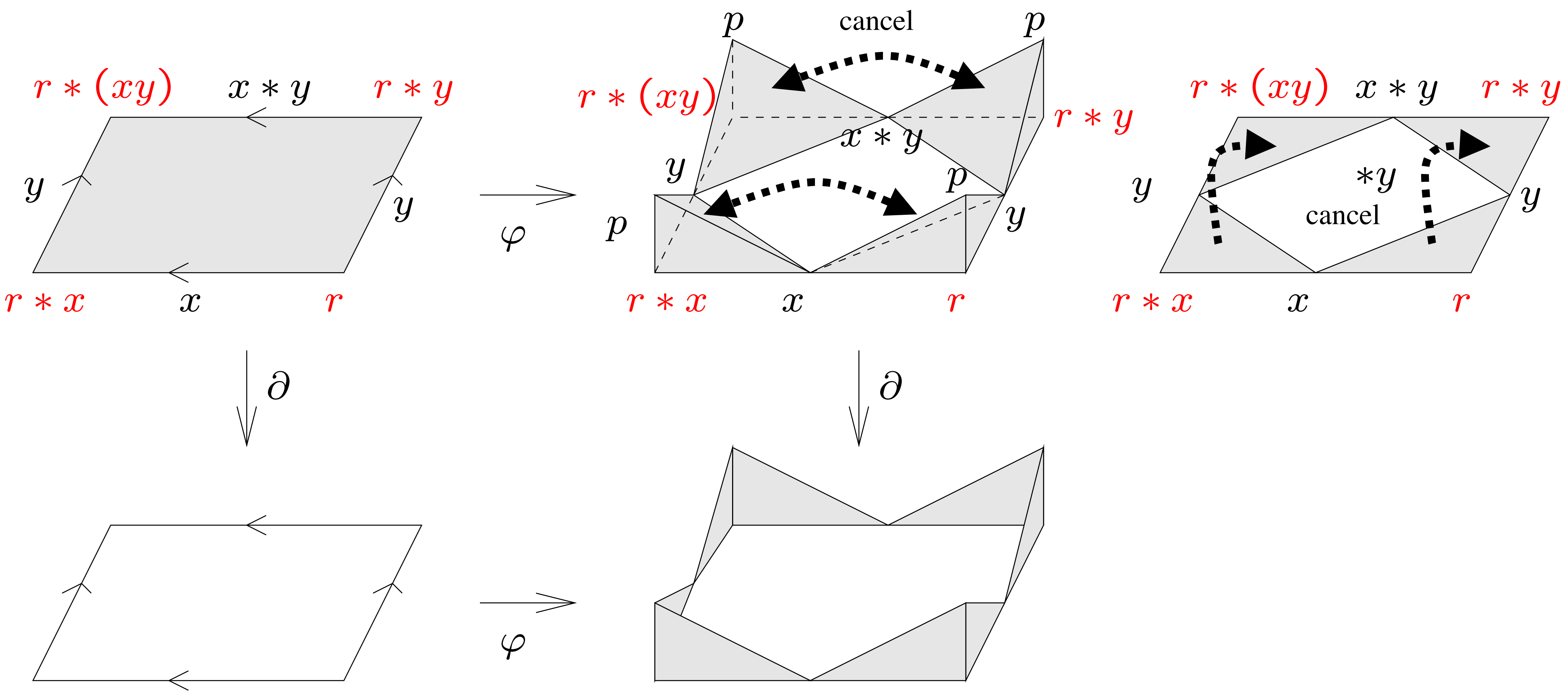


■

The map φ induces a homomorphism

$$H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X).$$

So we can construct a quandle cocycle from a cocycle of $H_{n+1}^{\Delta}(X)$. If we have a function f from X^{k+1} to some abelian group A satifying

1. $\sum_i (-1)^i f(x_0, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = 0$ and
2. $f(x_0 g, \ldots, x_k g) = f(x_0, \ldots, x_k)$ and
3. $f(x_0, \ldots, x_k) = 0$ if $x_i = x_{i+1}$ for some i,

then f gives a cocycle of $H_k^{\Delta}(X)$ and a cocycle of $H_{k-1}^{Q}(X;\mathbb{Z}[X])$.

Most of important quandles have *homogeneous* presentations $K\backslash G$ by some group G and a subgroup $K < G$. When X is given by a symmetric space $K\backslash G$, G-invariant closed k-form on $K\backslash G$ gives rise to a function satisfying the above conditions by integrating the form over geodesic k-simplices. For example the volume form on $\mathbb{H}^n$ is such a form.

Theorem *The n-dimensional hyperbolic volume is a quandle cocycle of the quandle formed by parabolic elements of* Isom$^+(\mathbb{H}^n)$.

We further study three dimensional case. In this case, Chern-Simons invariant is also a quandle cocycle.

We will construct a map from $H_3^{\Delta}(\mathcal{P})$ to the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ along with the work of Dupont and Zickert.

Quandle structure on $\mathbb{C}^2 \setminus \{0\}$

Define a binary operation $*$ on $\mathbb{C}^2 \setminus \{0\}$ by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} * \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} 1 - x_2 y_2 & -x_2^2 \\ y_2^2 & 1 + x_2 y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This satisfies the quandle axioms. Define a map $\mathbb{C}^2 \setminus \{0\} \xrightarrow{2:1} \mathcal{P}$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - xy & -x^2 \\ y^2 & 1 + xy \end{pmatrix}$$

This map induces a quandle isomorphism $(\mathbb{C}^2 \setminus \{0\})/\pm \cong \mathcal{P}$.

Dupont and Zickert's work

Let $C_n(\mathbb{C}^2) = \mathrm{span}_{\mathbb{Z}}\{(v_0, \dots, v_n) | v_i \in \mathbb{C}^2 \setminus \{0\}\}$. Define the boundary operator of $C_n(\mathbb{C}^2)$ by

$$\partial(v_0, \dots v_n) = \sum_{i=0}^{n} (-1)^i (v_0, \dots, \widehat{v_i}, \dots, v_n).$$

They defined a map from $C_3(\mathbb{C}^2)$ to the extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ by sending (v_0, v_1, v_2, v_3) to $(w_0, w_1, w_2) \in \widehat{\mathcal{P}}(\mathbb{C})$ (a combinatorial flattening) where

$$\begin{aligned}
w_0 &= \mathrm{Log}\det(v_0, v_3) + \mathrm{Log}\det(v_1, v_2) - \mathrm{Log}\det(v_0, v_2) - \mathrm{Log}\det(v_1, v_3),\\
w_1 &= \mathrm{Log}\det(v_0, v_2) + \mathrm{Log}\det(v_1, v_3) - \mathrm{Log}\det(v_0, v_1) - \mathrm{Log}\det(v_2, v_3),\\
w_2 &= \mathrm{Log}\det(v_0, v_1) + \mathrm{Log}\det(v_2, v_3) - \mathrm{Log}\det(v_0, v_3) - \mathrm{Log}\det(v_1, v_2).
\end{aligned}$$

Theorem (Dupont-Zickert) *The above map induces a homomorphism*

$$H_3(C_*(\mathbb{C}^2)_{\mathsf{PSL}(2,\mathbb{C})}) \to \widehat{\mathcal{B}}(\mathbb{C})$$

Remark *In this talk we do not discuss "degenerate" tetrahedra, for simplicity. In the original paper, they studied for* $\mathsf{SL}(2,\mathbb{C})$ *not* $\mathsf{PSL}(2,\mathbb{C})$.

Since $\mathcal{P} \cong (\mathbb{C}^2 \setminus \{0\})/\pm$, $C^{\Delta}_*(\mathcal{P})$ is nearly equal to $C_*(\mathbb{C}^2)$. So we can "construct" a map from $H^{\Delta}_3(\mathcal{P}) \to \widehat{\mathcal{B}}(\mathbb{C})$.

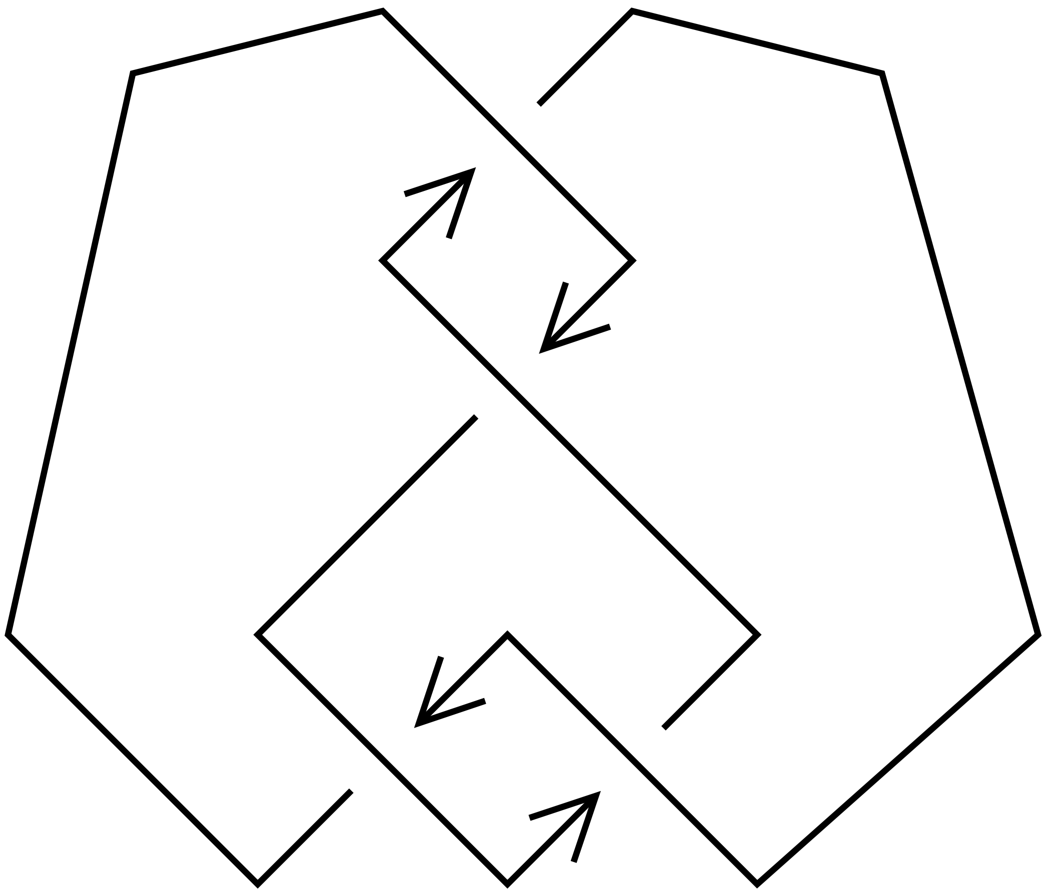
Theorem *There is a homomorphism*

$$H^Q_2(\mathcal{P}; \mathbb{Z}[\mathcal{P}]) \to \widehat{\mathcal{B}}(\mathbb{C}).$$

The image of $[C(\mathcal{S})]$ by this map gives the extended Bloch invariant of the parabolic representation.

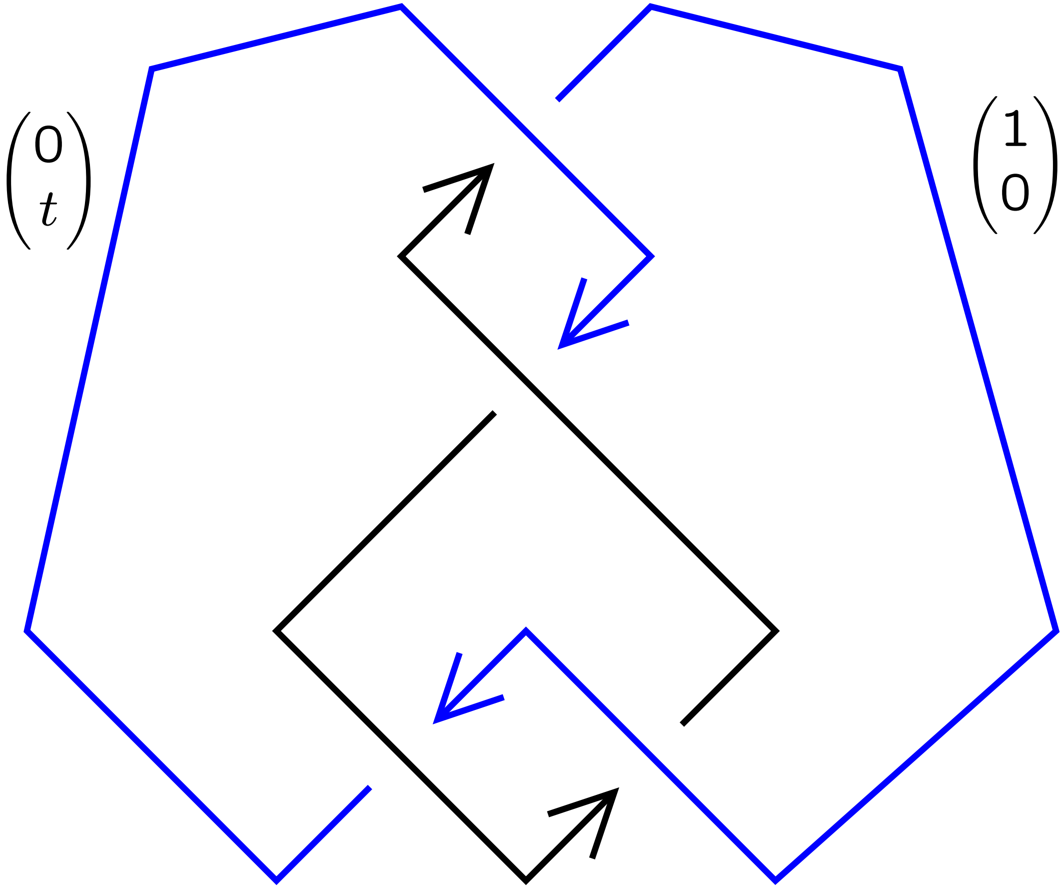
Our work is based on the quandle homology theory, but we do not use it for actual calculation.

Example: Arc coloring of the figure eight knot



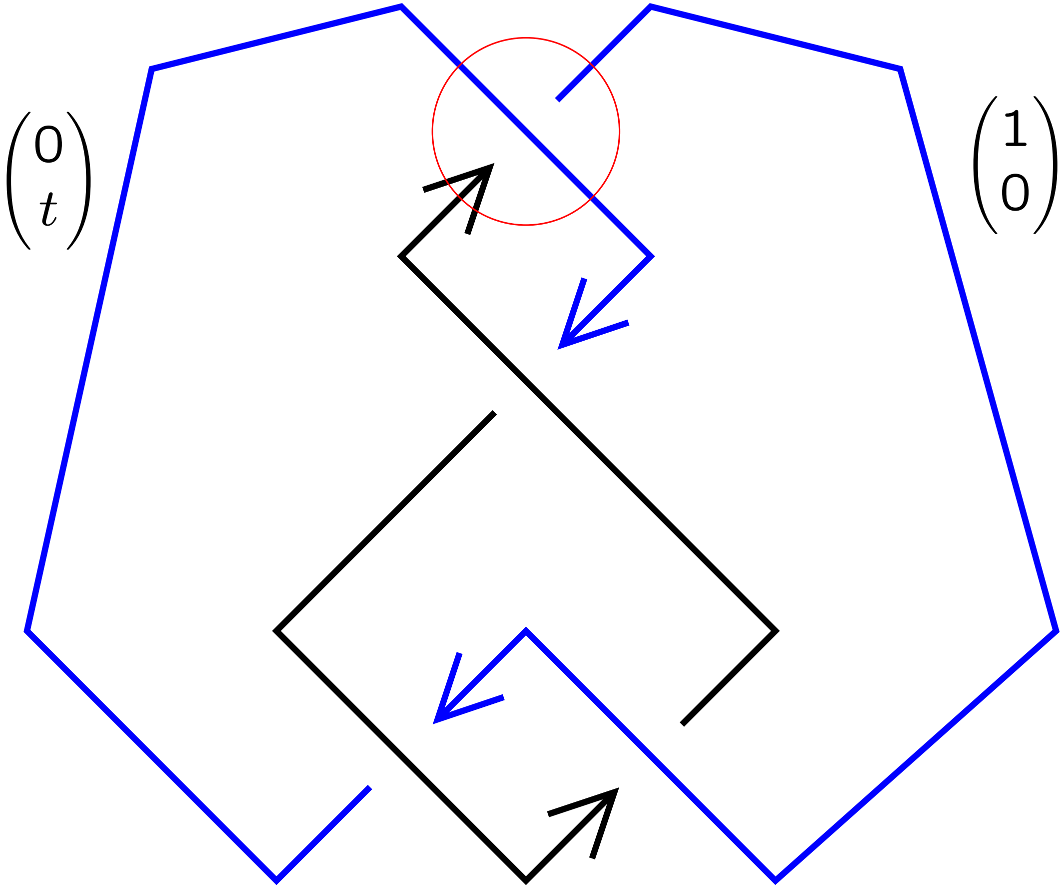
This is the figure eight knot.

Example: Arc coloring of the figure eight knot



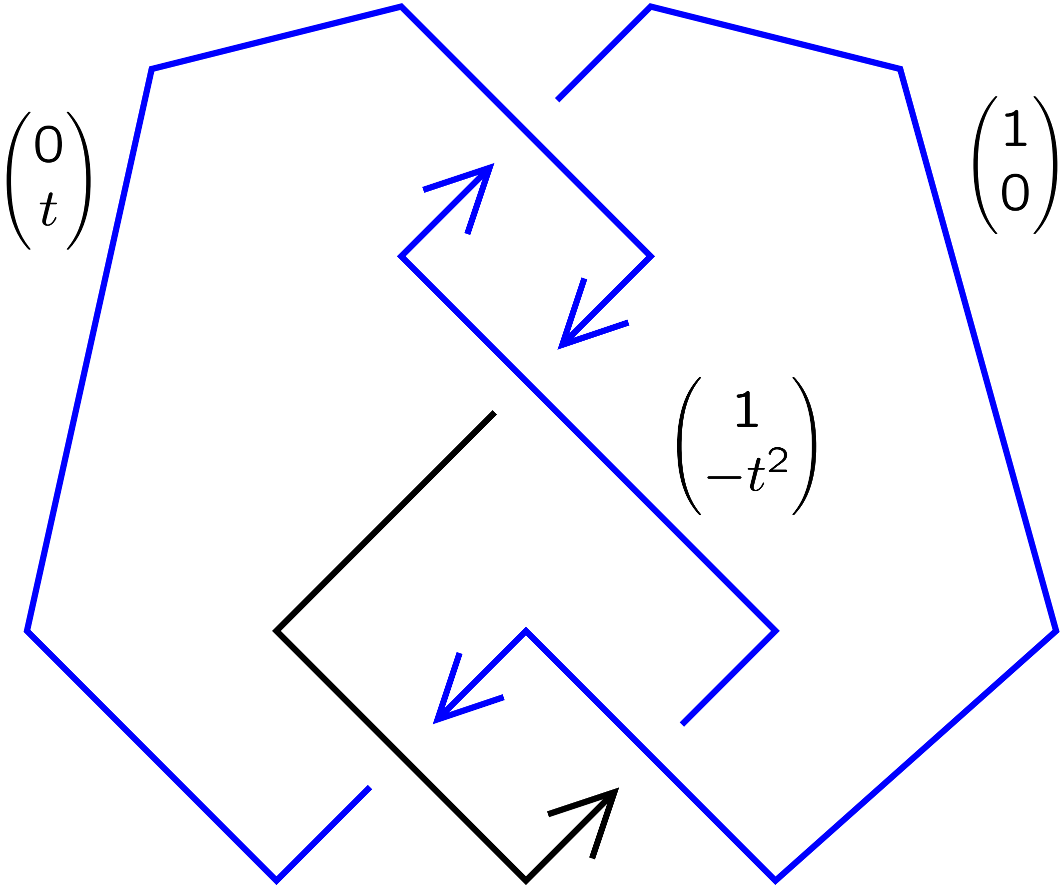
Color two arcs.

Example: Arc coloring of the figure eight knot



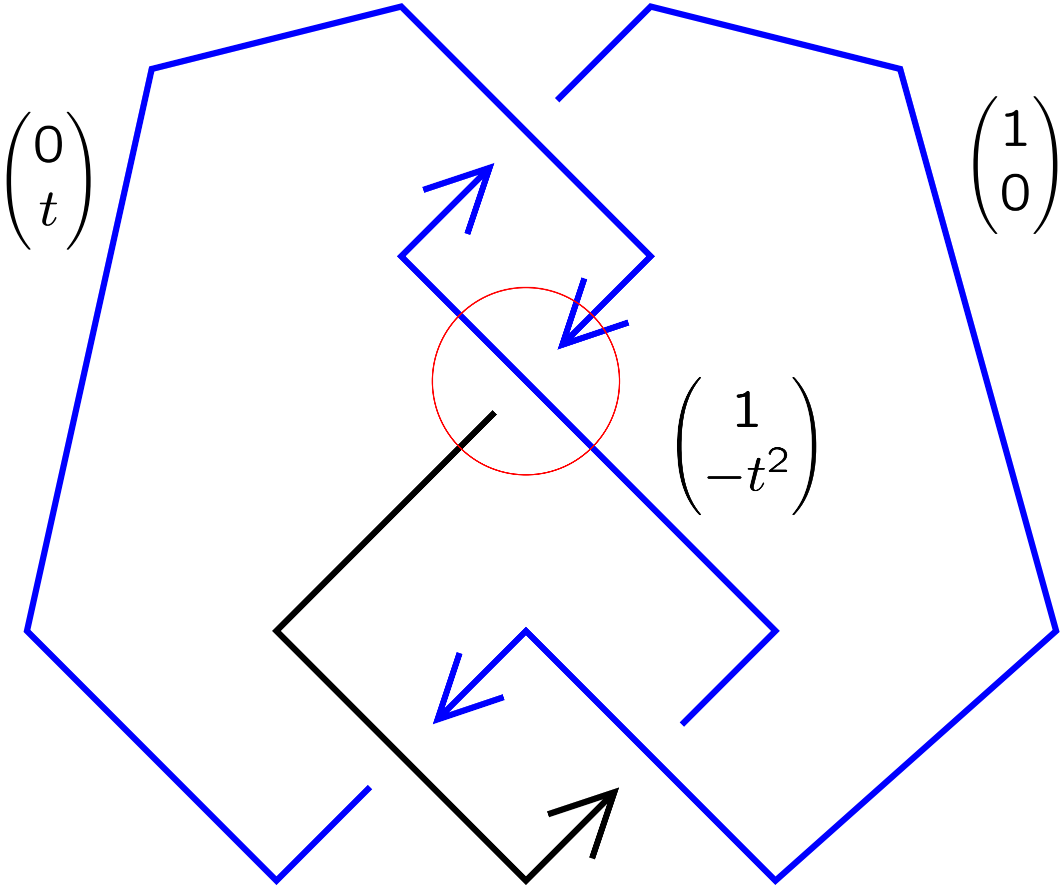
Consider the relation at a crossing.

Example: Arc coloring of the figure eight knot



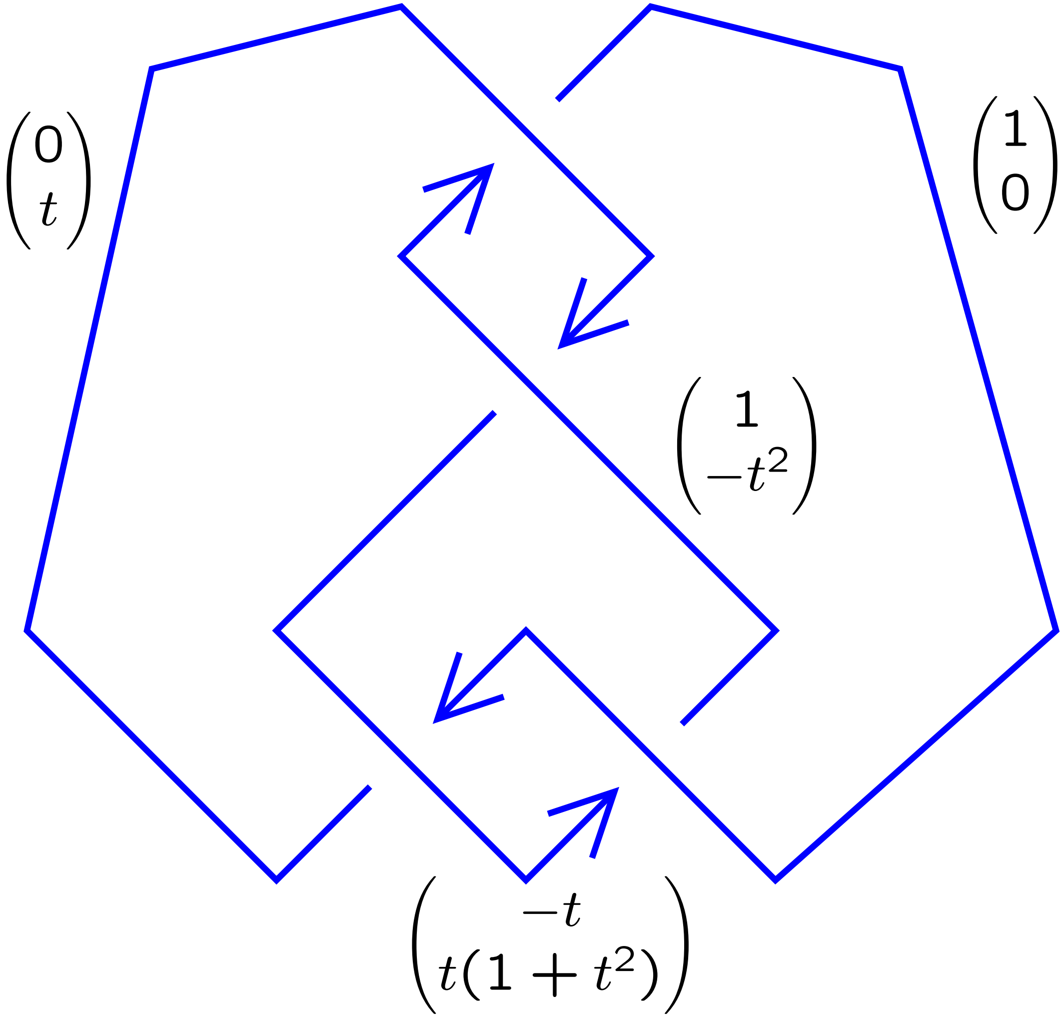
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} *^{-1} \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -t^2 \end{pmatrix}$$

Example: Arc coloring of the figure eight knot



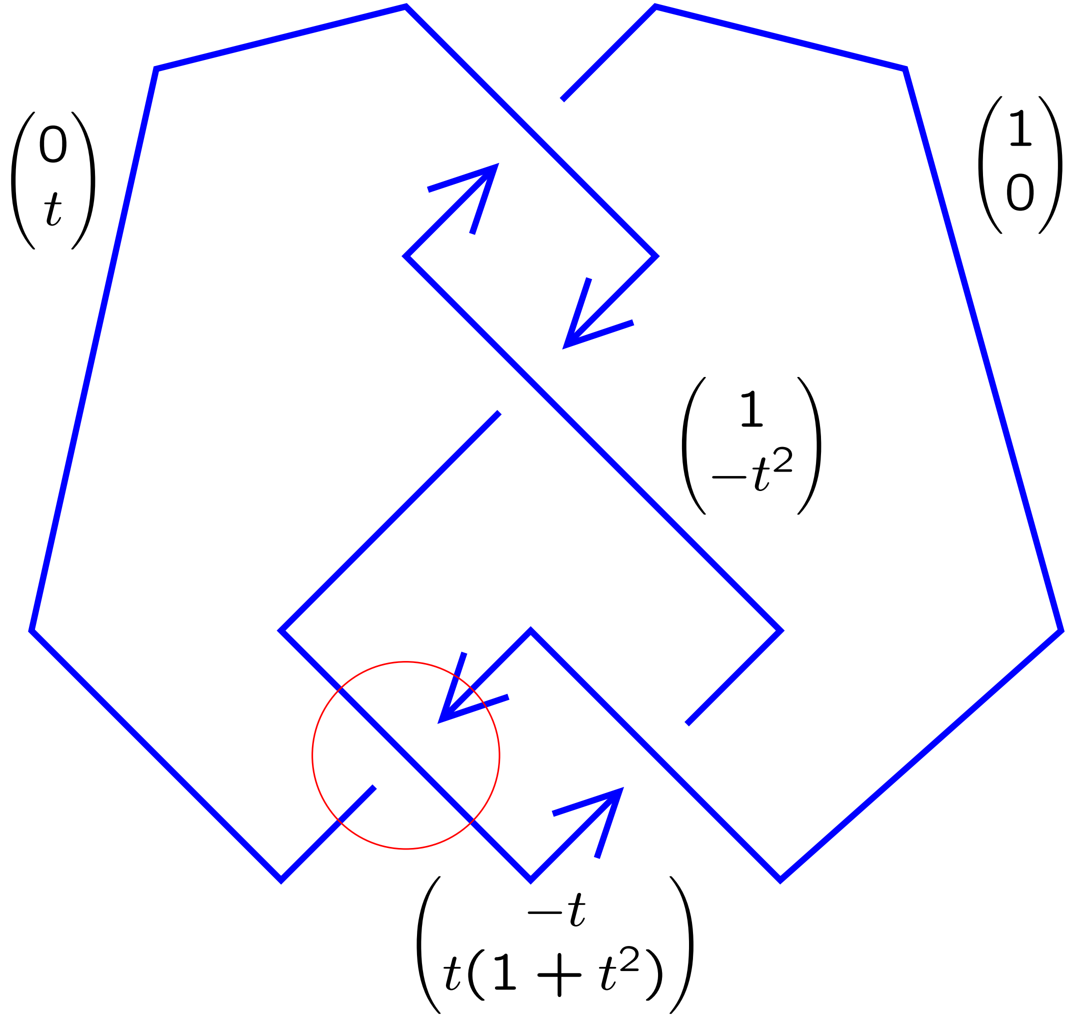
Consider the relation at another crossing.

Example: Arc coloring of the figure eight knot



$$\begin{pmatrix} 0 \\ t \end{pmatrix} * \begin{pmatrix} 1 \\ -t^2 \end{pmatrix} = \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix}$$

Example: Arc coloring of the figure eight knot



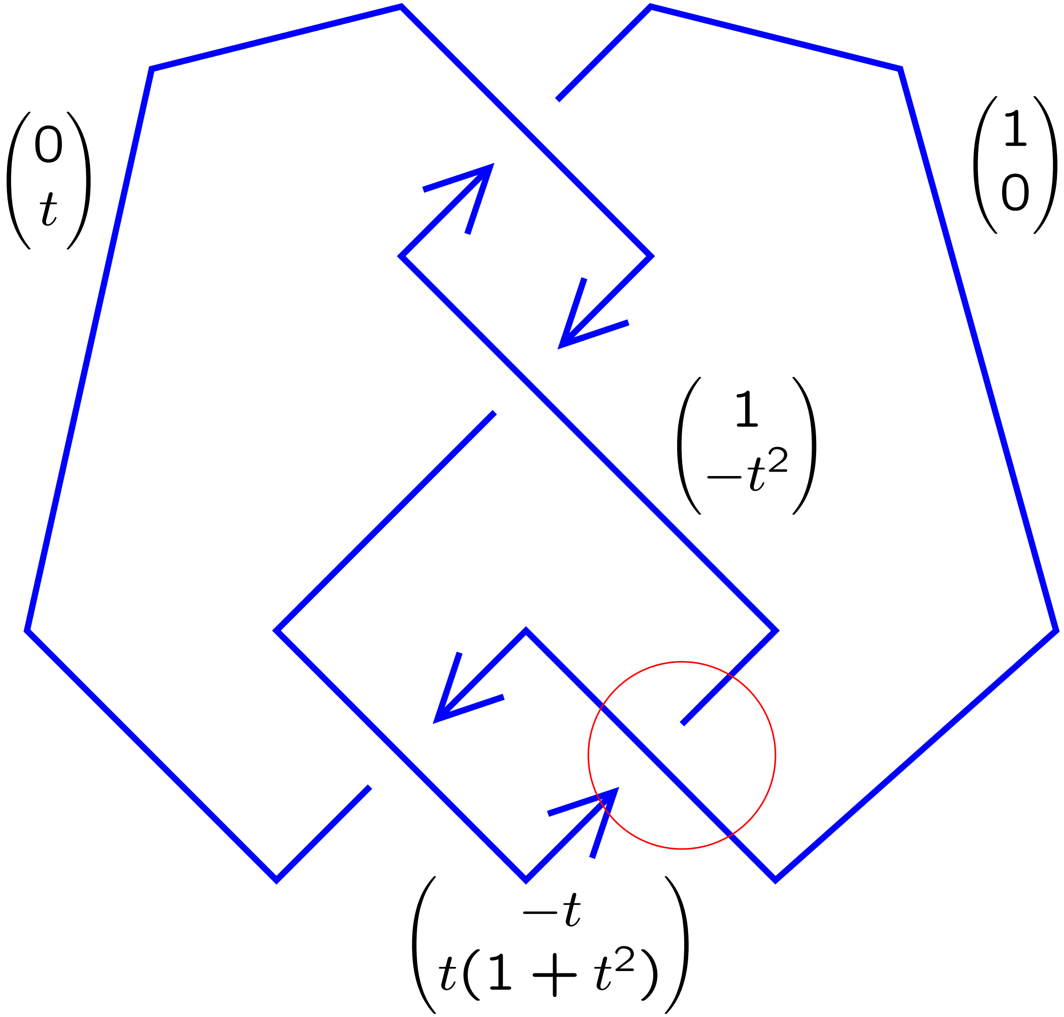
The relation at this crossing is

$$\left(\begin{pmatrix} 0 \\ t \end{pmatrix} * \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix} = \right)$$
$$\begin{pmatrix} -t^3 \\ t(1+t^2+t^4) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} (t+1)(t^2-t+1) = 0 \\ t(t^2+t+1)(t^2-t+1) = 0 \end{cases}$$

$$\therefore t^2 - t + 1 = 0$$

Example: Arc coloring of the figure eight knot



The relation at this crossing is

$$\left(\begin{pmatrix} 1 \\ -t^2 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \right)$$

$$\begin{pmatrix} 1+t^2 \\ -t^2 \end{pmatrix} = \begin{pmatrix} -t \\ t(1+t^2) \end{pmatrix}$$

$$\begin{cases} t^2+t+1=0 \\ t(t^2+t+1)=0 \end{cases}$$

$$\therefore t^2+t+1=0$$

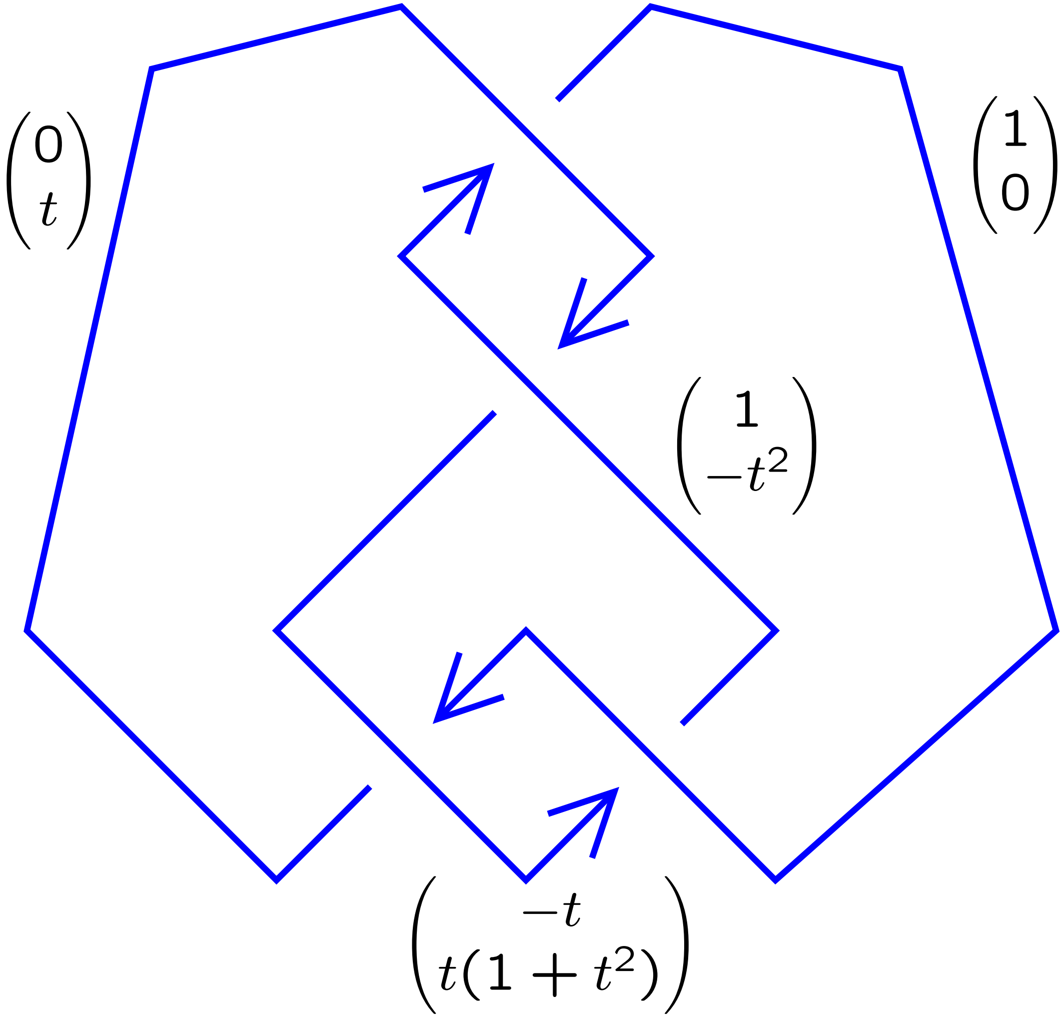
Example: Arc coloring of the figure eight knot

There are two relations

$$t^2 + t + 1 = 0, \quad t^2 - t + 1 = 0$$

which do not have any common solution. But we have a coloring by $(\mathbb{C}^2 \setminus \{0\})/\pm \cong \mathcal{P}$ ($t = \pm\frac{1+\sqrt{3}i}{2}$ or $\pm\frac{1-\sqrt{3}i}{2}$).

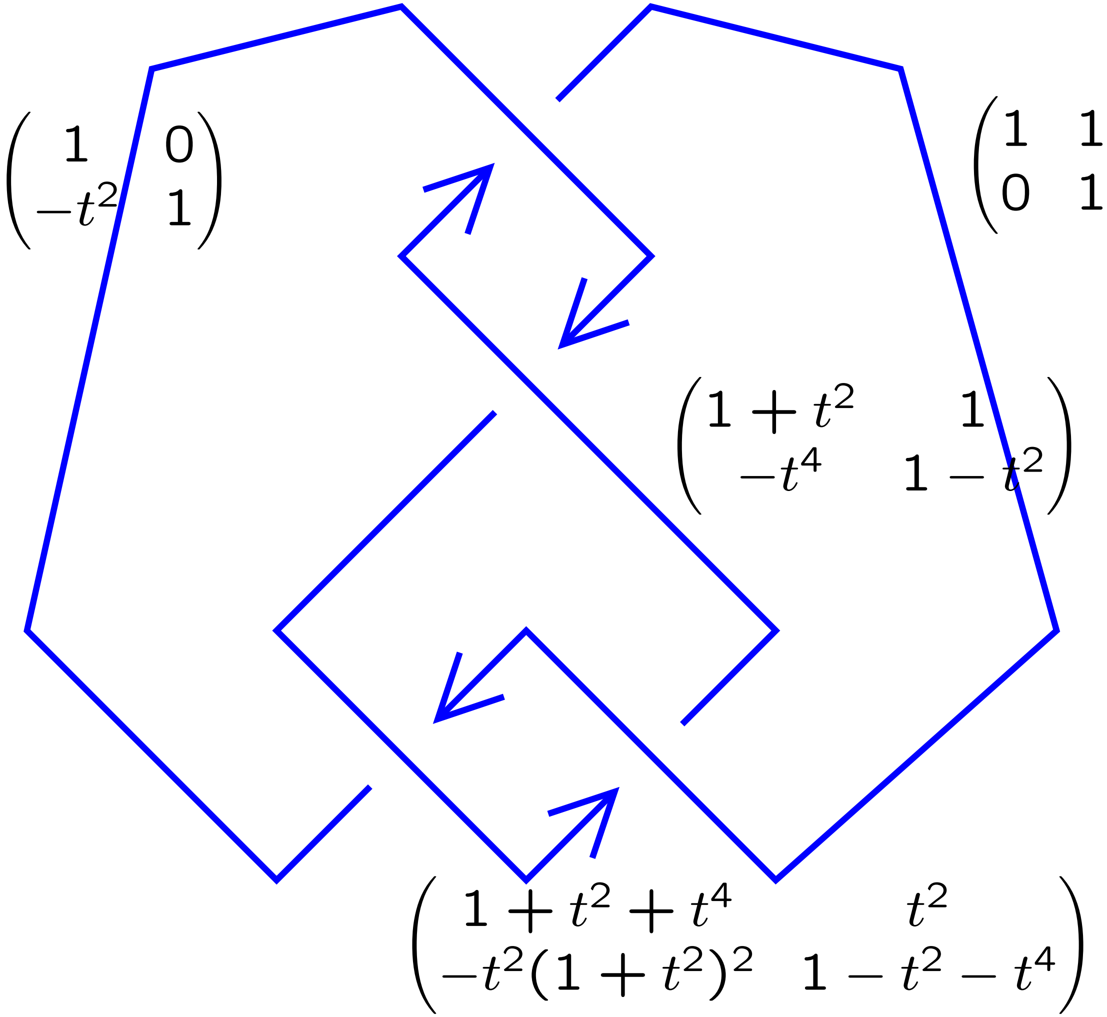
Example: Arc coloring of the figure eight knot



A parabolic representation can be obtained by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1-xy & x^2 \\ -y^2 & 1+xy \end{pmatrix}$$

Example: Arc coloring of the figure eight knot



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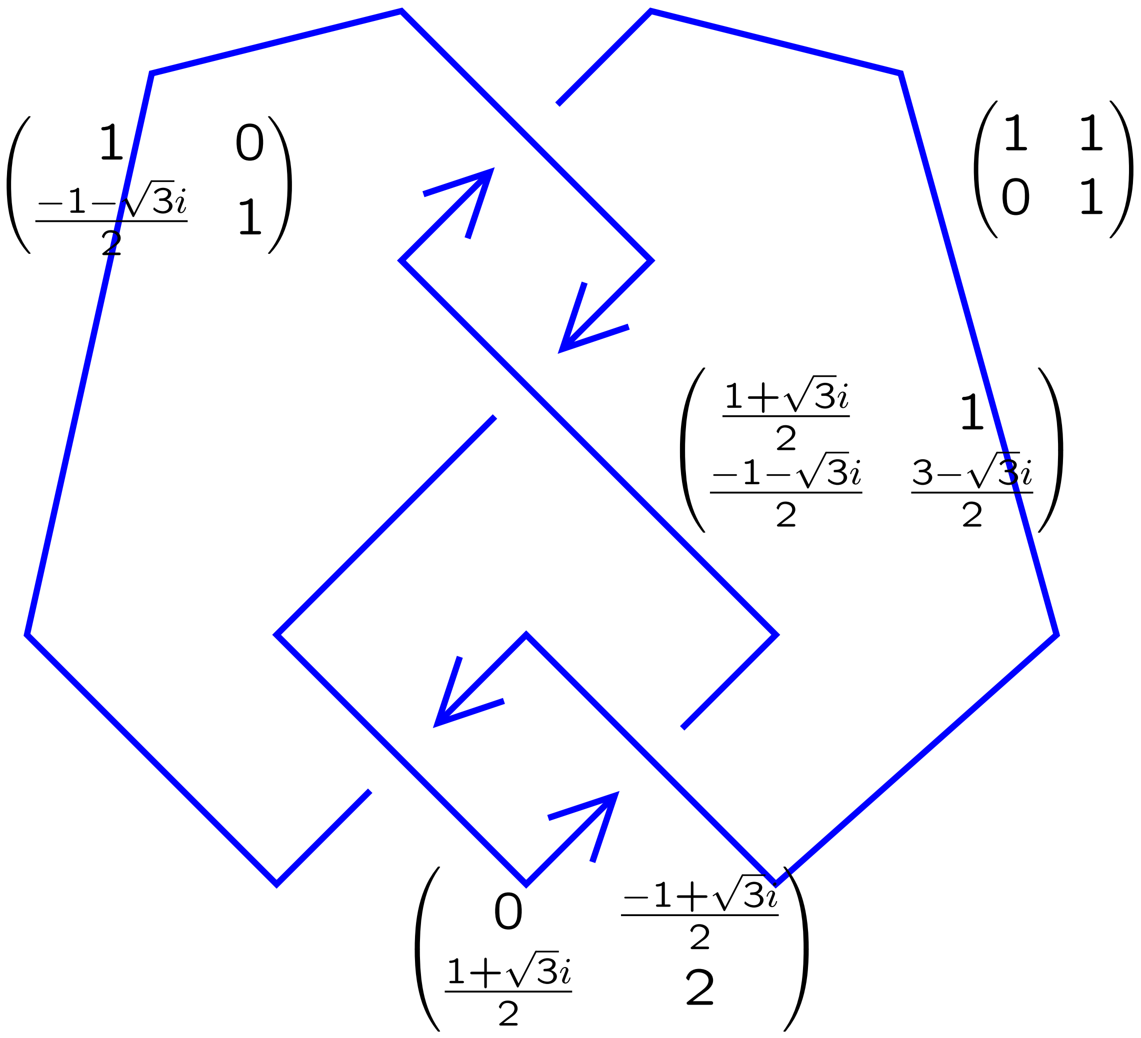
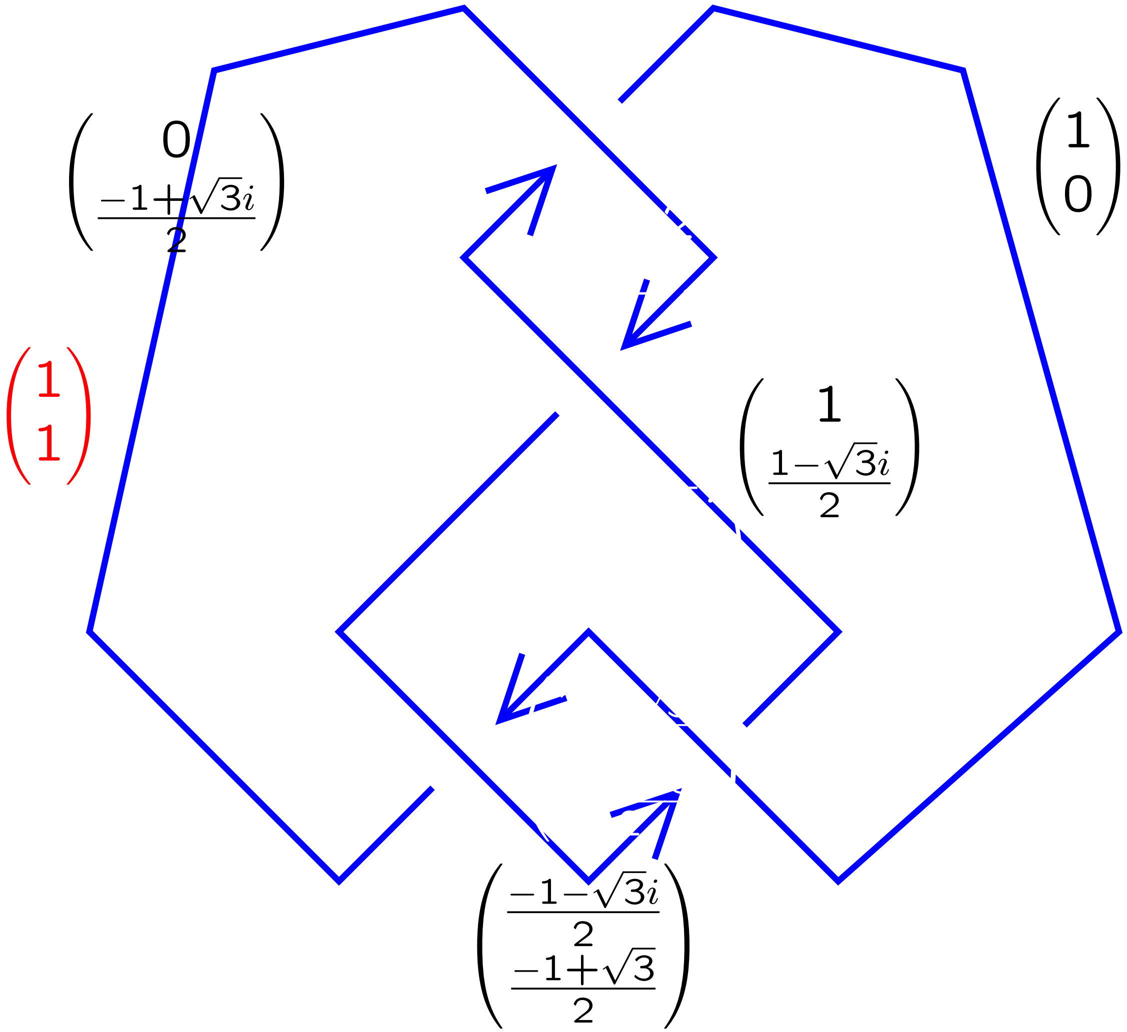
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1-xy & x^2 \\ -y^2 & 1+xy \end{pmatrix}$$

Example: Arc coloring of the figure eight knot



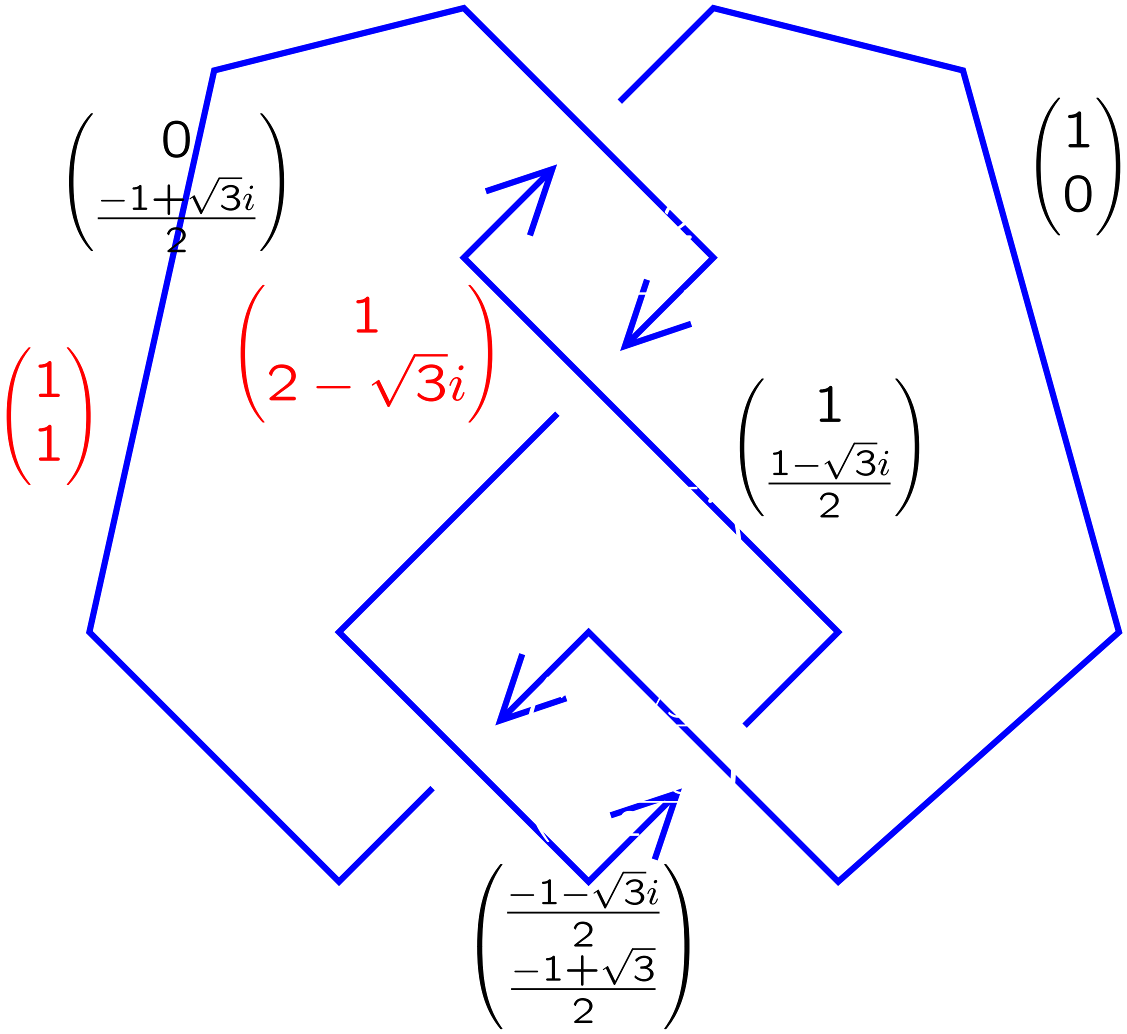
When $t^2 = \frac{-1+\sqrt{3}i}{2}$:

Example: Region coloring of the figure eight knot



Put a region color at a region e.g. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

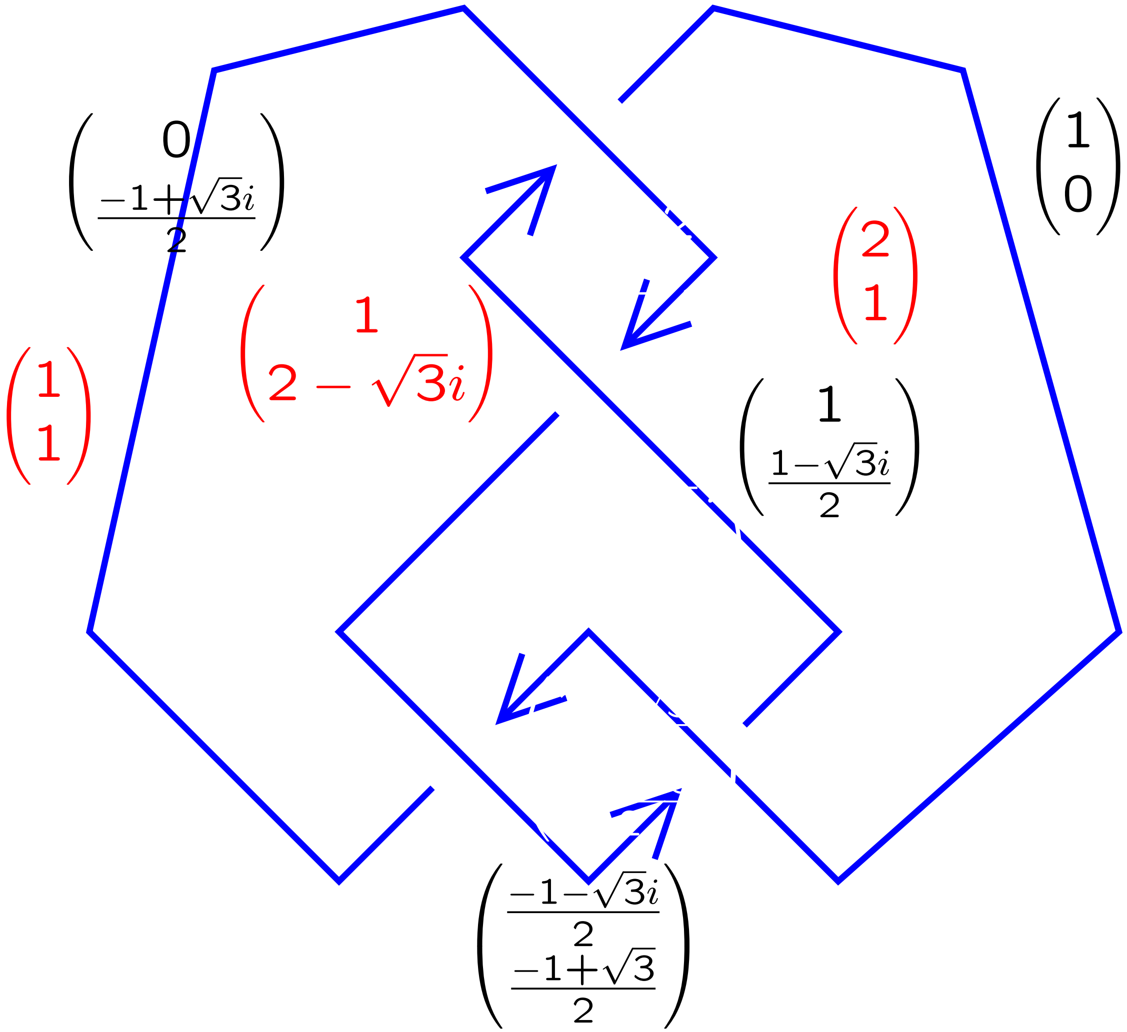
Example: Region coloring of the figure eight knot



The color of an adjacent region is determined by the relation.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *^{-1} \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}i}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\sqrt{3}i \end{pmatrix}$$

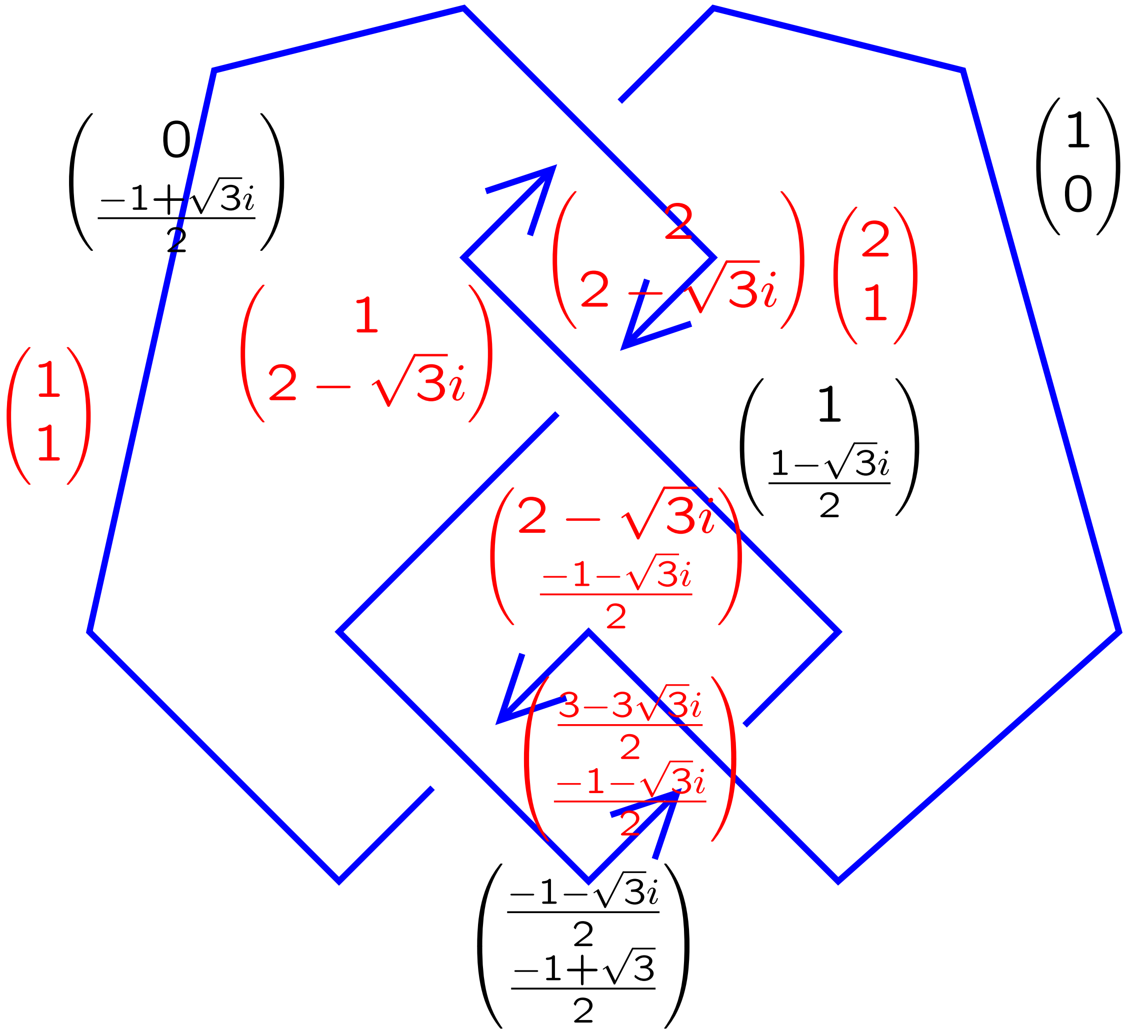
Example: Region coloring of the figure eight knot



The color of an adjacent region is determined by the relation.

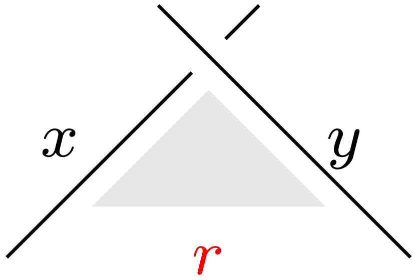
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Example: Region coloring of the figure eight knot



The color of an adjacent region is determined by the relation.

Fix an element p_0 of $\mathbb{C}^2 \setminus \{0\}$ e.g. $p_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

At a corner colored by 

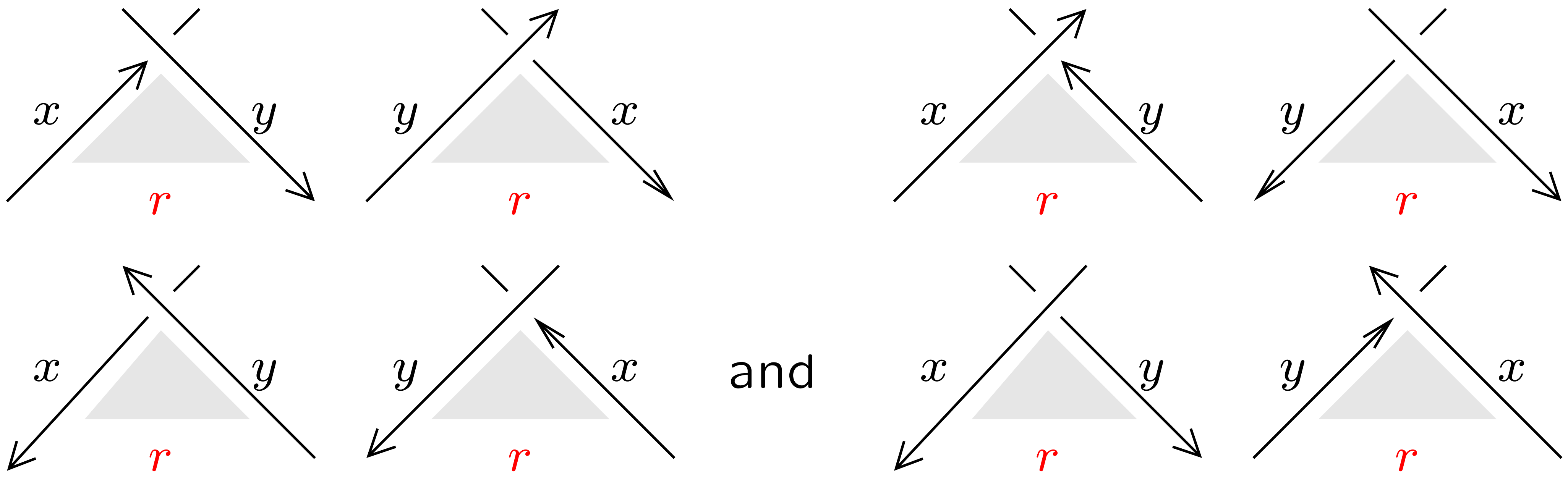
($x \leftrightarrow$ under arc, $y \leftrightarrow$ over arc), we let

$$\begin{aligned} z =& \frac{\det(p_0, y)\det(r, x)}{\det(r, y)\det(p_0, x)} \\ p\pi i =& \mathrm{Log}(\det(p_0, y)) + \mathrm{Log}(\det(r, x)) \\ & - \mathrm{Log}(\det(r, y)) - \mathrm{Log}(\det(p_0, x)) - \mathrm{Log}(z) \\ q\pi i =& \mathrm{Log}(\det(p_0, x)) + \mathrm{Log}(\det(r, y)) \\ & - \mathrm{Log}(\det(p_0, r)) - \mathrm{Log}(\det(x, y)) - \mathrm{Log}(\frac{1}{1-z}) \end{aligned}$$

where $\mathrm{Log}(z) = \log|z| + i\arg(z)$ $(-\pi < \arg(z) \leq \pi)$

We remark that $p, q \in \mathbb{Z}$.

Then define the sign in the following rule:



$+[z; p, q]$ (in-out or out-in)

$-[z; p, q]$ (in-in or out-out)

Theorem

$$\sum_{c:\text{corners}} \varepsilon_c [z_c; p_c, q_c]$$

is the extended Bloch invariant.

Let $\widehat{L} : \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}$ be the Rogers dilogarithmic function defined by Neumann. When the arc coloring corresponding to the faithful discrete representation of a hyperbolic knot K, then we have

$$\sum_{c:\text{corners}} \varepsilon_c \widehat{L}(z_c; p_c, q_c) = i(\mathrm{Vol}(S^3 \setminus K) + i\mathrm{CS}(S^3 \setminus K)).$$