Quandle homology, volume and the Chern-Simons invariant

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Introduction

M : an oriented closed 3-mfd

$\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$: 基本群の表現

$\mathrm{Vol}(M,\rho) \in \mathbb{R}, \qquad \mathrm{CS}(M,\rho) \in \mathbb{R}/\pi^2\mathbb{Z}$

Mが双曲多様体, ρがdiscrete faithful表現のときは、それぞれ双曲計量に関する体積とChern-Simons不変量.

Vol，CS は境界がトーラスである3-mfdの場合（e.g．knot complements）にも定義できる.

$i(\mathrm{Vol} + i\mathrm{CS}) \in \mathbb{C}/\pi^2\mathbb{Z}$ の公式は W．Neumann によって triangulation の言葉で与えられている.

Quandleを利用する事でknotのダイアグラムの言葉でVol，CSを記述する.

Quandle structure on $\mathbb{C}^2 \setminus \{0\}$

Define a binary operation $*$ on $\mathbb{C}^2 \setminus \{0\}$ by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} * \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} 1 + x_2 y_2 & -x_2^2 \\ y_2^2 & 1 - x_2 y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

This satisfies the quandle axioms:

1. $x * x = x \qquad (\forall x \in \mathbb{C}^2 \setminus \{0\})$,
2. $*y : x \mapsto x * y$ is bijective $\qquad (\forall y \in \mathbb{C}^2 \setminus \{0\})$,
3. $(x * y) * z = (x * z) * (y * z) \qquad (\forall x, y, z \in \mathbb{C}^2 \setminus \{0\})$.

$\mathcal{P}$: the set of parabolic elements of $\mathrm{PSL}(2,\mathbb{C})$

Define a map $\mathbb{C}^2 \setminus \{0\} \xrightarrow{2:1} \mathcal{P}$ by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + xy & -x^2 \\ y^2 & 1 - xy \end{pmatrix}$$

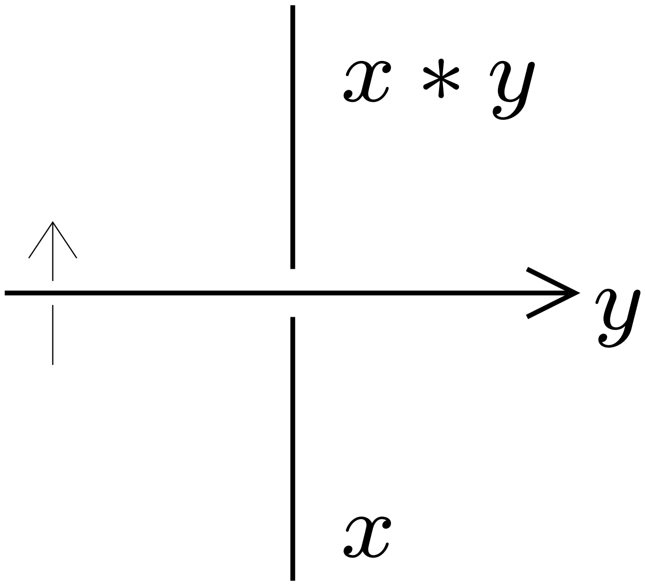
This map induces a quandle isomorphism

$$(\mathbb{C}^2 \setminus \{0\})/\pm \cong \mathcal{P}$$

Arc coloring by $\mathcal{P}$

K : knot in S^3, D : a diagram of K.

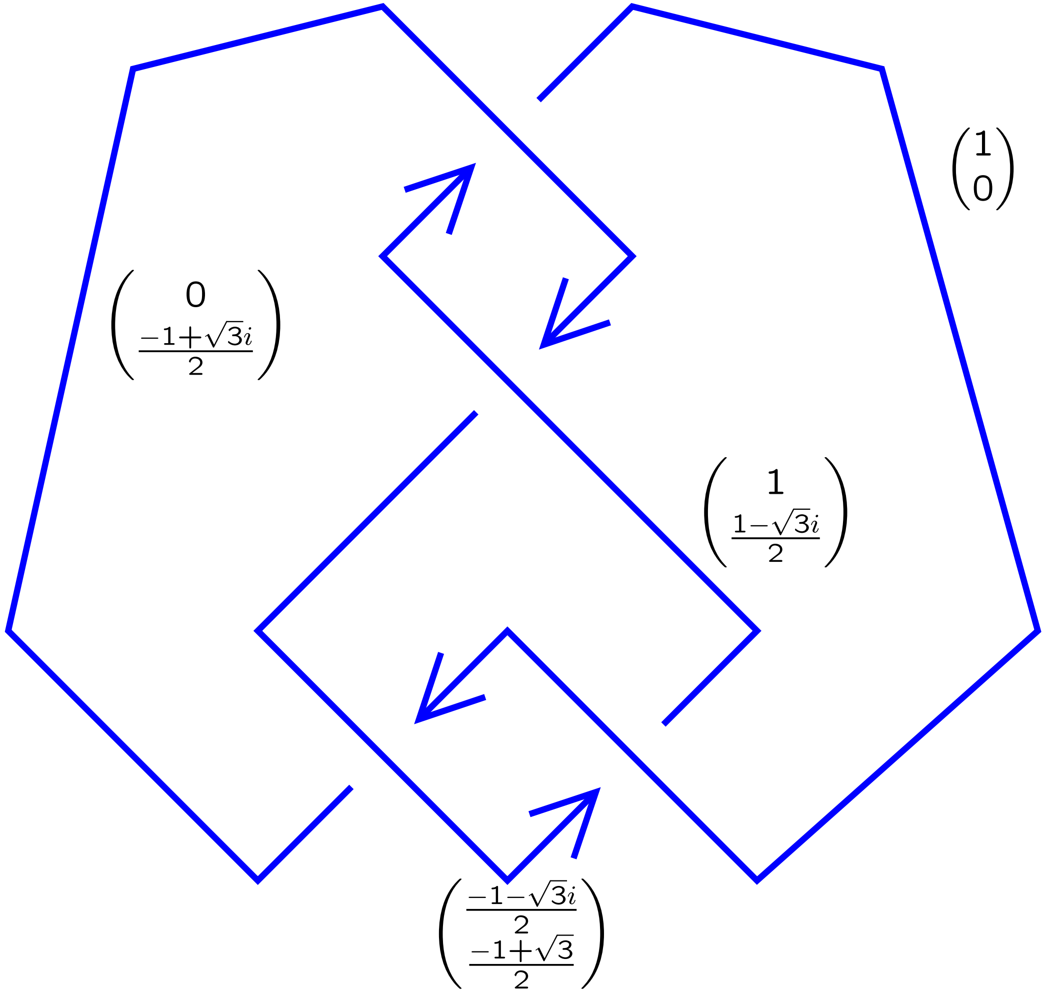
We call a map $\mathcal{A} : \{\text{arcs of } D\} \to \mathcal{P}$ an *arc coloring* if it satisfies



(x, y and $x * y \in \mathcal{P}$)

at each crossing.

Arc coloring of the figure eight knot by $\mathcal{P}$

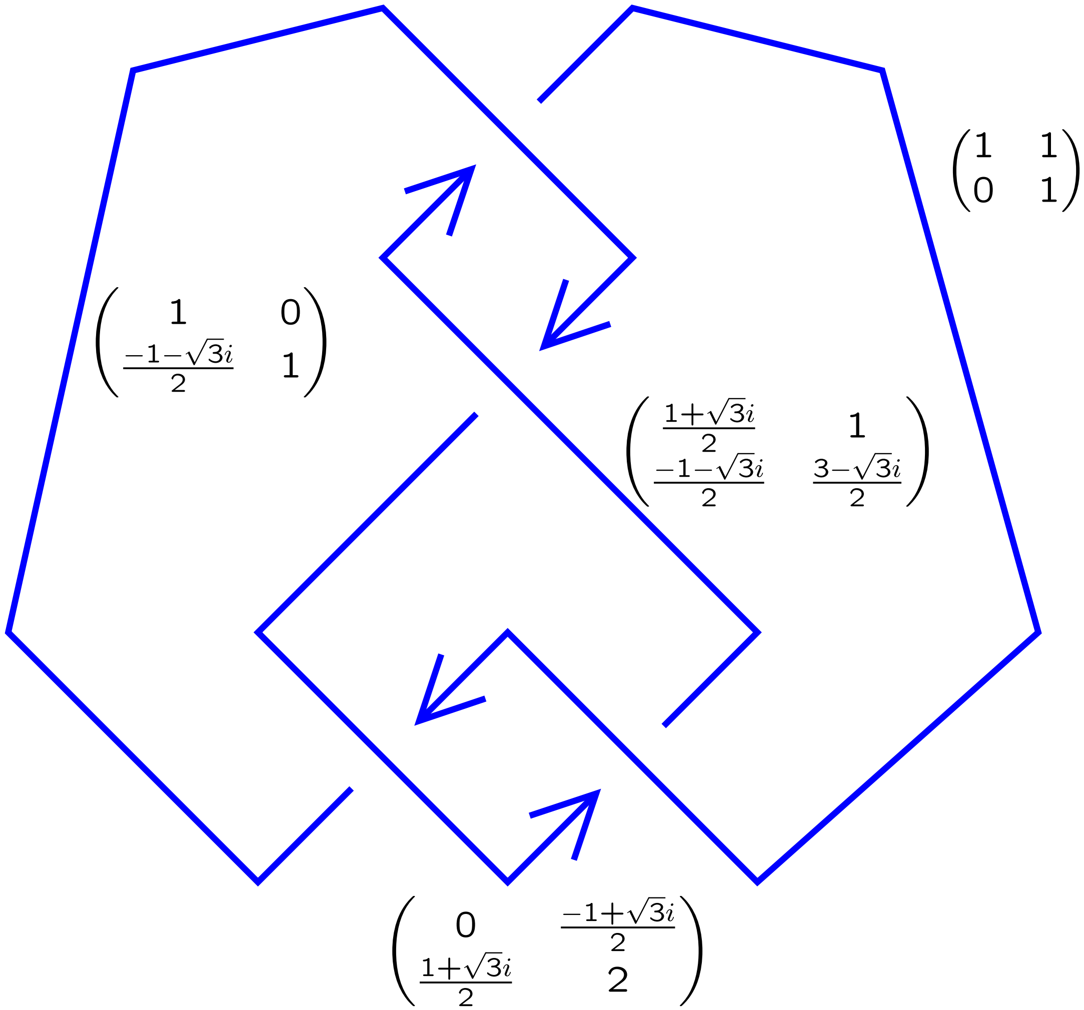


An arc coloring by $(\mathbb{C}^2 \setminus \{0\})/\pm$

A parabolic representation can be obtained by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - xy & x^2 \\ -y^2 & 1 + xy \end{pmatrix}$$

Arc coloring of the figure eight knot by $\mathcal{P}$



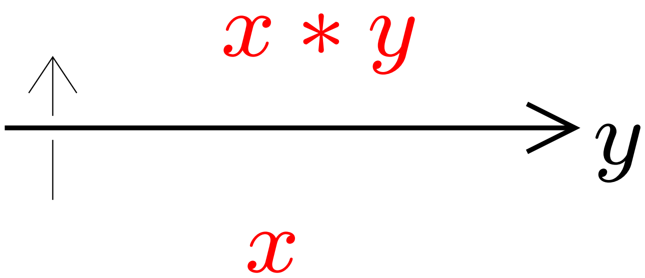
A parabolic representation can be obtained by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1-xy & x^2 \\ -y^2 & 1+xy \end{pmatrix}$$

Region coloring by $\mathcal{P}$

D : a diagram of K. $\mathcal{A}$: an arc coloring by $\mathcal{P}$

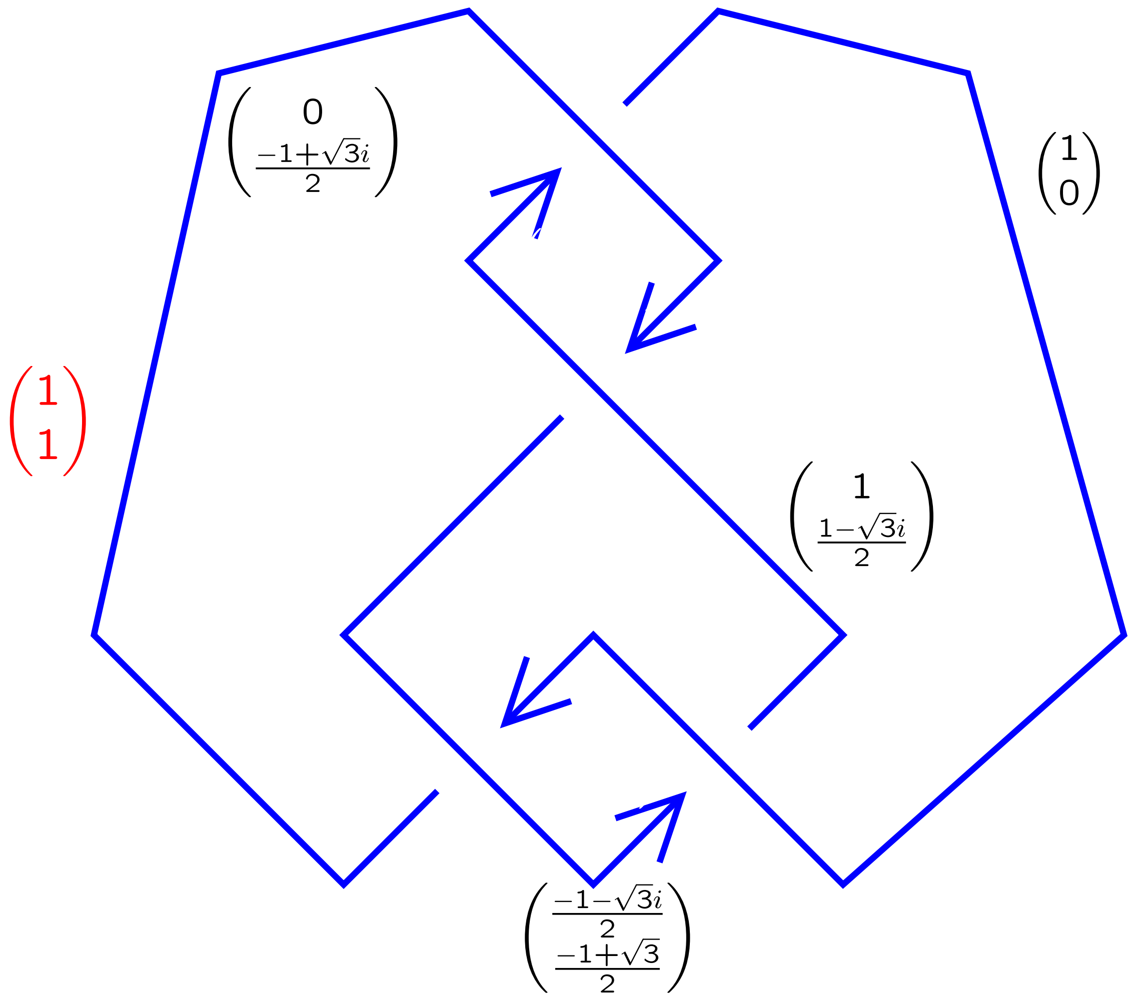
We call a map $\mathcal{R}$: {regions of D} $\to \mathcal{P}$ a *region coloring* if it satisfies



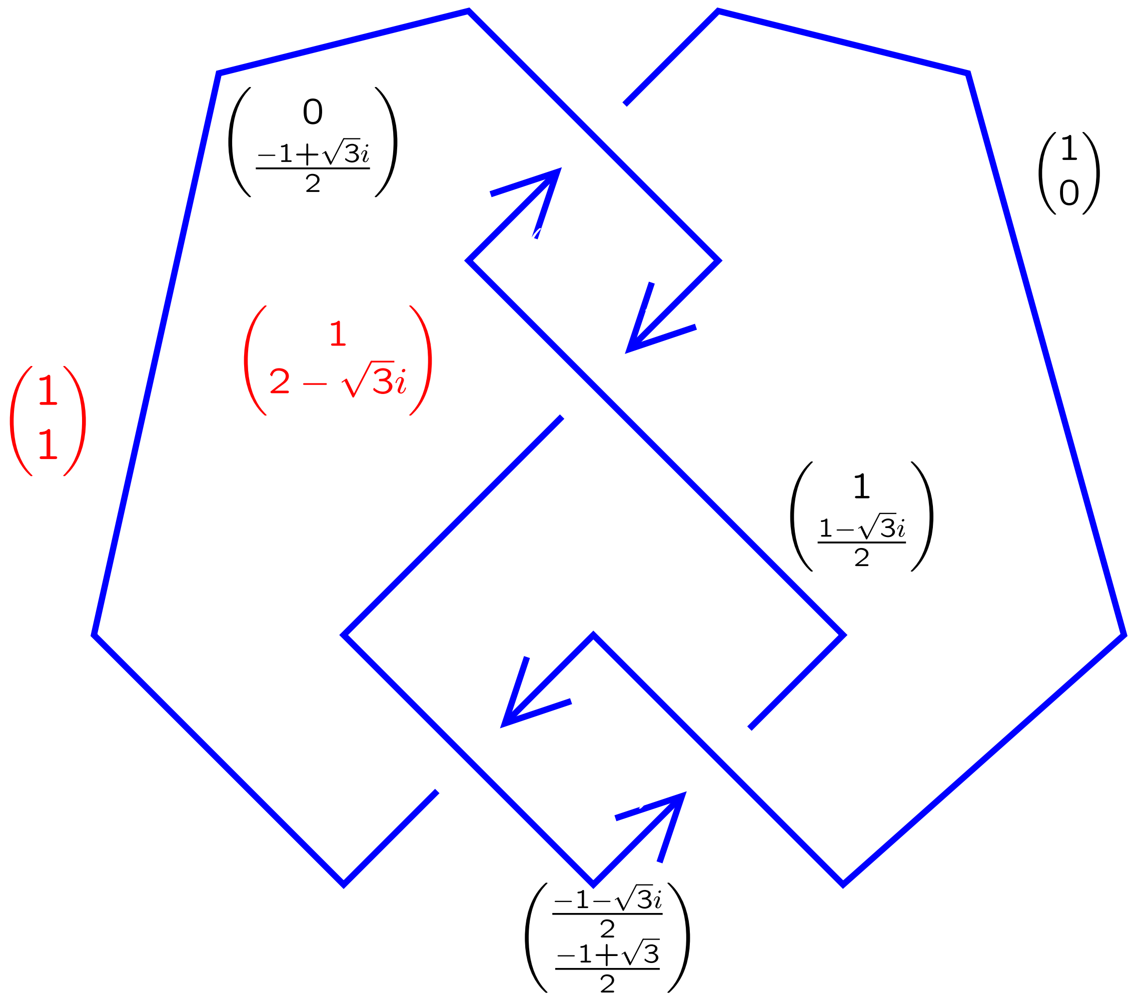
$(x, y$ and $x * y \in \mathcal{P})$

We call a pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ ($\mathcal{A}$: arc coloring, $\mathcal{R}$: region coloring) a *shadow coloring*.

Region coloring of the figure eight knot

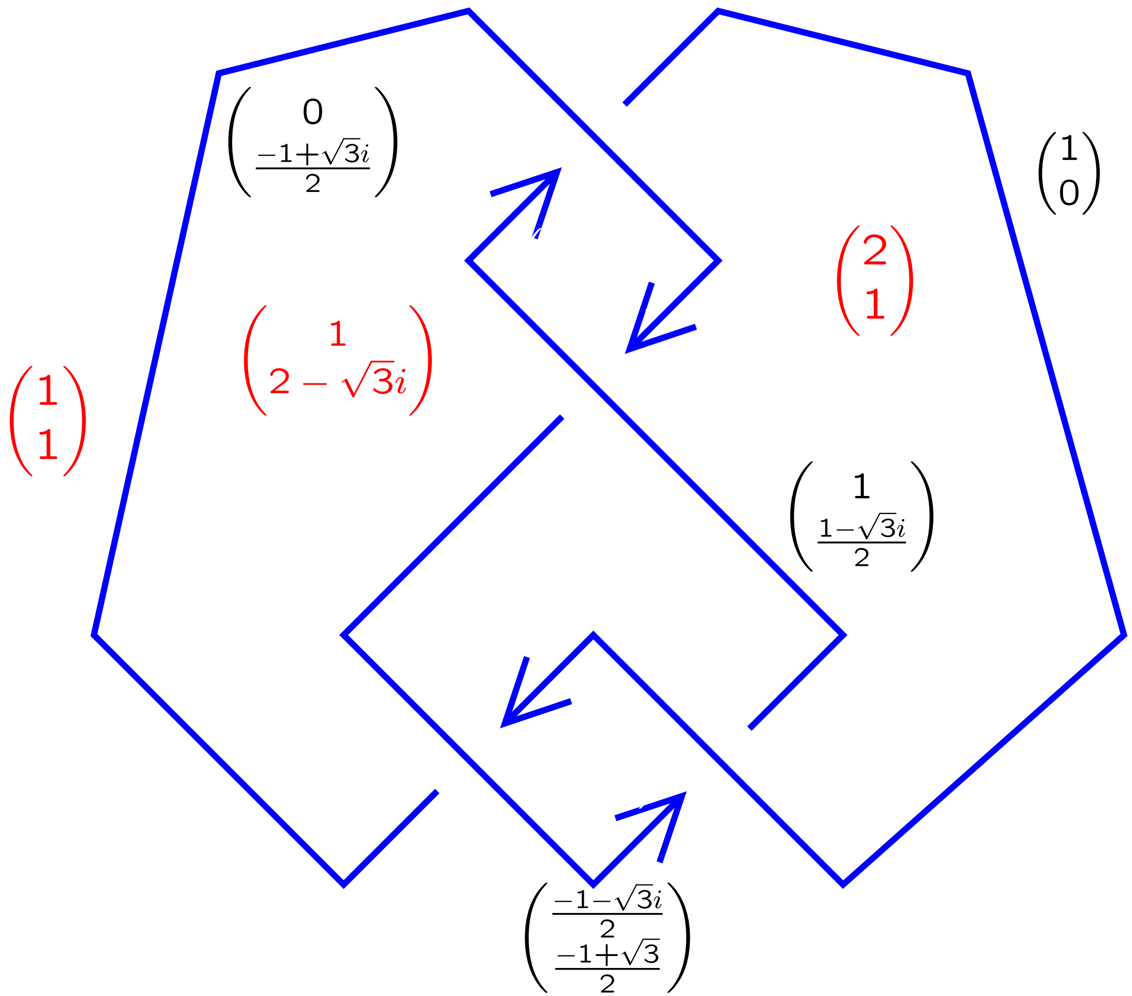


Region coloring of the figure eight knot



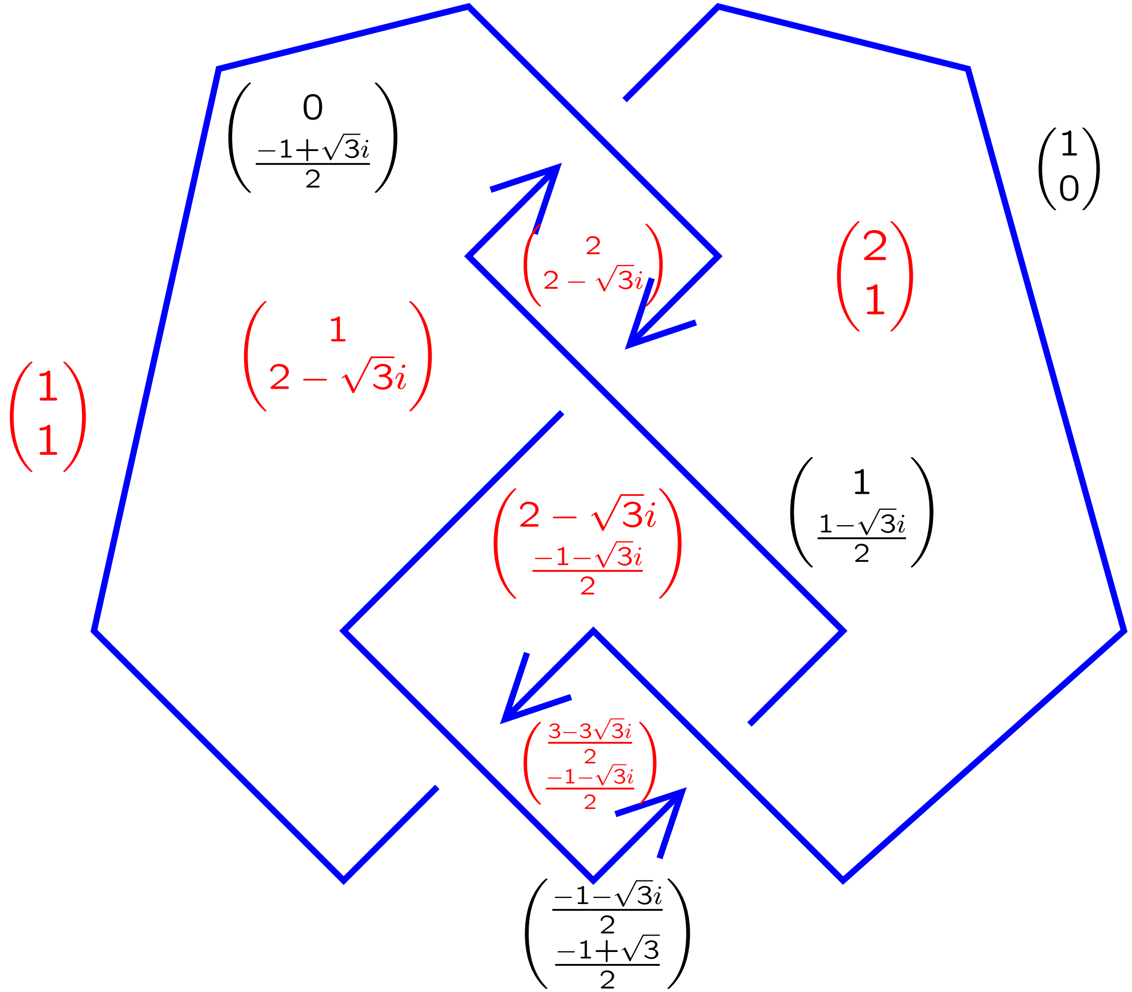
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *^{-1} \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}i}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\sqrt{3}i \end{pmatrix}$$

Region coloring of the figure eight knot

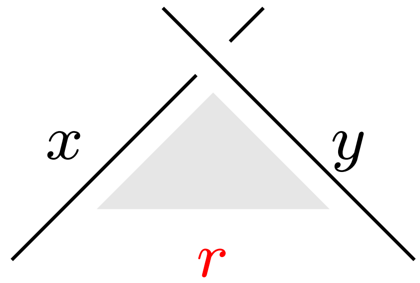


$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} *^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Region coloring of the figure eight knot



Fix an element p_0 of $\mathbb{C}^2 \setminus \{0\}$.

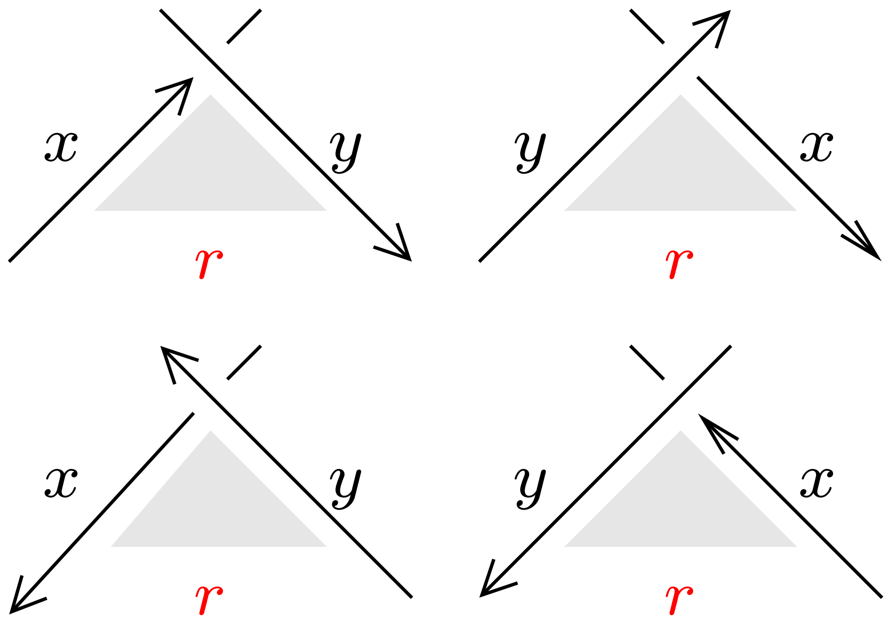
At a corner colored by 

($x \leftrightarrow$ under arc, $y \leftrightarrow$ over arc), we let

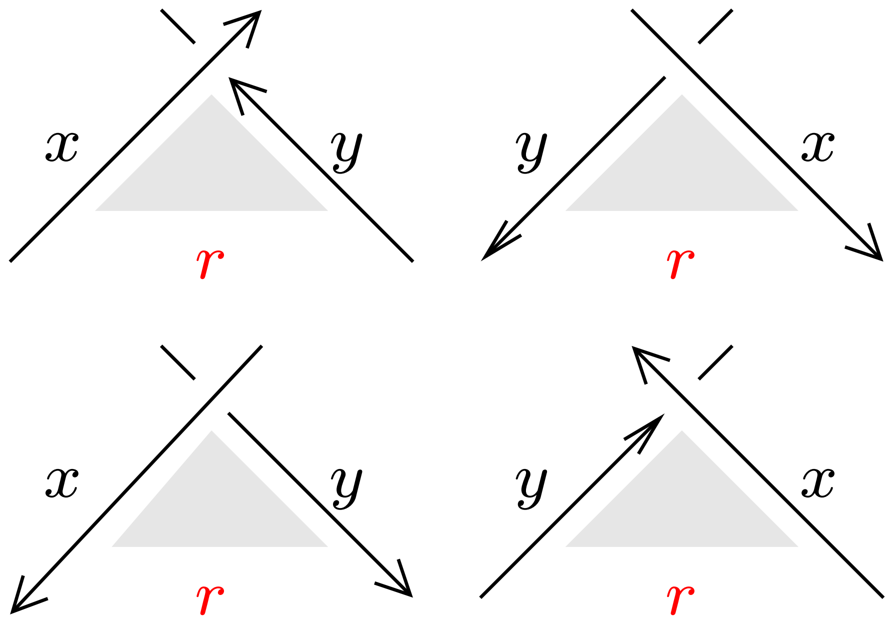
$$z = \frac{\det(p_0, y)\det(r, x)}{\det(r, y)\det(p_0, x)}, \quad (0 \leq \arg(\det) < \pi)$$
$$p\pi i = \mathrm{Log}(\det(p_0, y)) + \mathrm{Log}(\det(r, x)) - \mathrm{Log}(\det(r, y)) - \mathrm{Log}(\det(p_0, x)) - \mathrm{Log}(z),$$
$$q\pi i = \mathrm{Log}(\det(p_0, x)) + \mathrm{Log}(\det(r, y)) - \mathrm{Log}(\det(p_0, r)) - \mathrm{Log}(\det(x, y)) - \mathrm{Log}(\frac{1}{1-z})$$

where $\mathrm{Log}(z) = \log|z| + i\arg(z)$ $(-\pi < \arg(z) \leq \pi)$.

Then define the sign in the following rule:



and



$+[z; p, q]$

(in-out or out-in)

$-[z; p, q]$

(in-in or out-out)

Thm (Inoue-K.)

$$\sum_{c:\text{corners}} \varepsilon_c[z_c; p_c, q_c] \in \widehat{\mathcal{B}}(\mathbb{C})$$

is the extended Bloch invariant.

$R : \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}$: the Rogers dilogarithmic function

$$R(z; p, q) = \mathcal{R}(z) + \frac{\pi i}{2}\left(q\mathrm{Log}(z) - p\mathrm{Log}\left(\frac{1}{1-z}\right)\right) - \frac{\pi^2}{6},$$
$$\mathcal{R}(z) = -\int_0^z \frac{\mathrm{Log}(1-t)}{t}dt + \frac{1}{2}\mathrm{Log}(z)\mathrm{Log}(1-z)$$

When the arc coloring corresponding to the discrete faithful representation of a hyperbolic knot K, then we have

$$\sum_{c:\text{corners}} \varepsilon_c R(z_c; p_c, q_c) = i(\mathrm{Vol}(S^3 \setminus K) + i\mathrm{CS}(S^3 \setminus K)).$$

Outline

1.

$\rho : \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2,\mathbb{C})$ parabolic representations	1:1 $\longleftrightarrow$	*Arc colorings* $\mathcal{A}$ by the quandle $\mathcal{P}$

2. Define a shadow coloring $\mathcal{S}$ and construct an invariant $[C(\mathcal{S})]$ with values in the *quandle homology* $H_2^Q(\mathcal{P};\mathbb{Z}[\mathcal{P}])$.

3.

Quandle homology	our work $\downarrow$	Simplicial quandle homology	Dupont -Zickert $\downarrow$	Extended Bloch group
$H_2^Q(\mathcal{P};\mathbb{Z}[\mathcal{P}])$	$\xrightarrow{\varphi_*}$	$H_3^\Delta(\mathcal{P})$	$\longrightarrow$	$\widehat{\mathcal{B}}(\mathbb{C})$
$\Cup$				$R \downarrow$ Neumann
$[C(\mathcal{S})]$				$\mathbb{C}/\pi^2\mathbb{Z}$
				$i(\mathrm{Vol} + i\mathrm{CS})$

Quandle homology

X: quandle

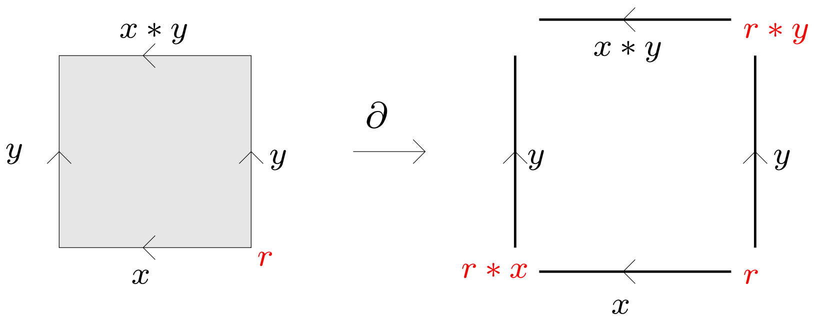
$C_n^Q(X) = \text{span}_{\mathbb{Z}[G_X]}\{(x_1,\dots,x_n) | x_i \in X\}/\sim$

Define $\partial : C_n^Q(X) \to C_{n-1}^Q(X)$ by

$$\partial(x_1,\dots,x_n) = \sum_{i=1}^{n} (-1)^i \{(x_1,\dots,\widehat{x_i},\dots,x_n) \\ - x_i(x_1 * x_i,\dots,x_{i-1} * x_i, x_{i+1},\dots,x_n)\}$$

$H_n^Q(X;\mathbb{Z}[X]) := H_n(\mathbb{Z}[X] \otimes_{G_X} C_*^Q(X))$: *quandle homology*

with $\mathbb{Z}[X]$ coefficient

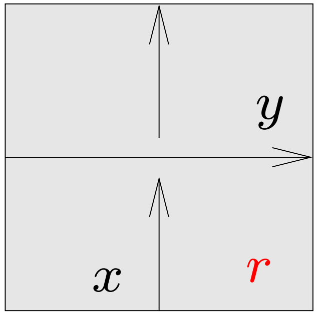


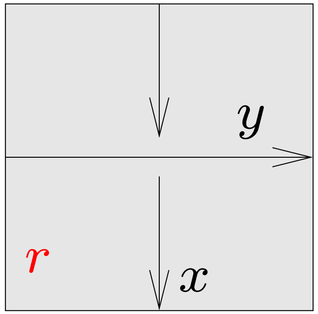
$\mathbb{Z}[X] \otimes_{G_X} C_2^Q(X) \to \mathbb{Z}[X] \otimes_{G_X} C_1^Q(X)$

Cycle $[C(\mathcal{S})]$ associated with a shadow coloring

$\mathcal{S} = (\mathcal{A}, \mathcal{R})$: a shadow coloring by a quandle X

Assign

$+r \otimes (x, y)$ for 

and $-r \otimes (x, y)$ for 

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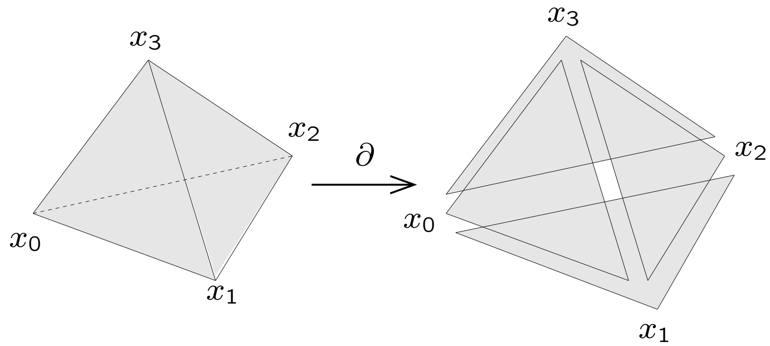
Let

$$C(\mathcal{S}) = \sum_{c:\text{crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[X]).$$

Prop *The homology class* $[C(\mathcal{S})]$ *in* $H_2^Q(\mathcal{P}, \mathbb{Z}[\mathcal{P}])$ *only depends on the conjugacy class of* $\pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$.

Simplicial quandle homology $H_n^{\Delta}(X)$



$C_n^{\Delta}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) | x_i \in X\}$

Define $\partial : C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ by

$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

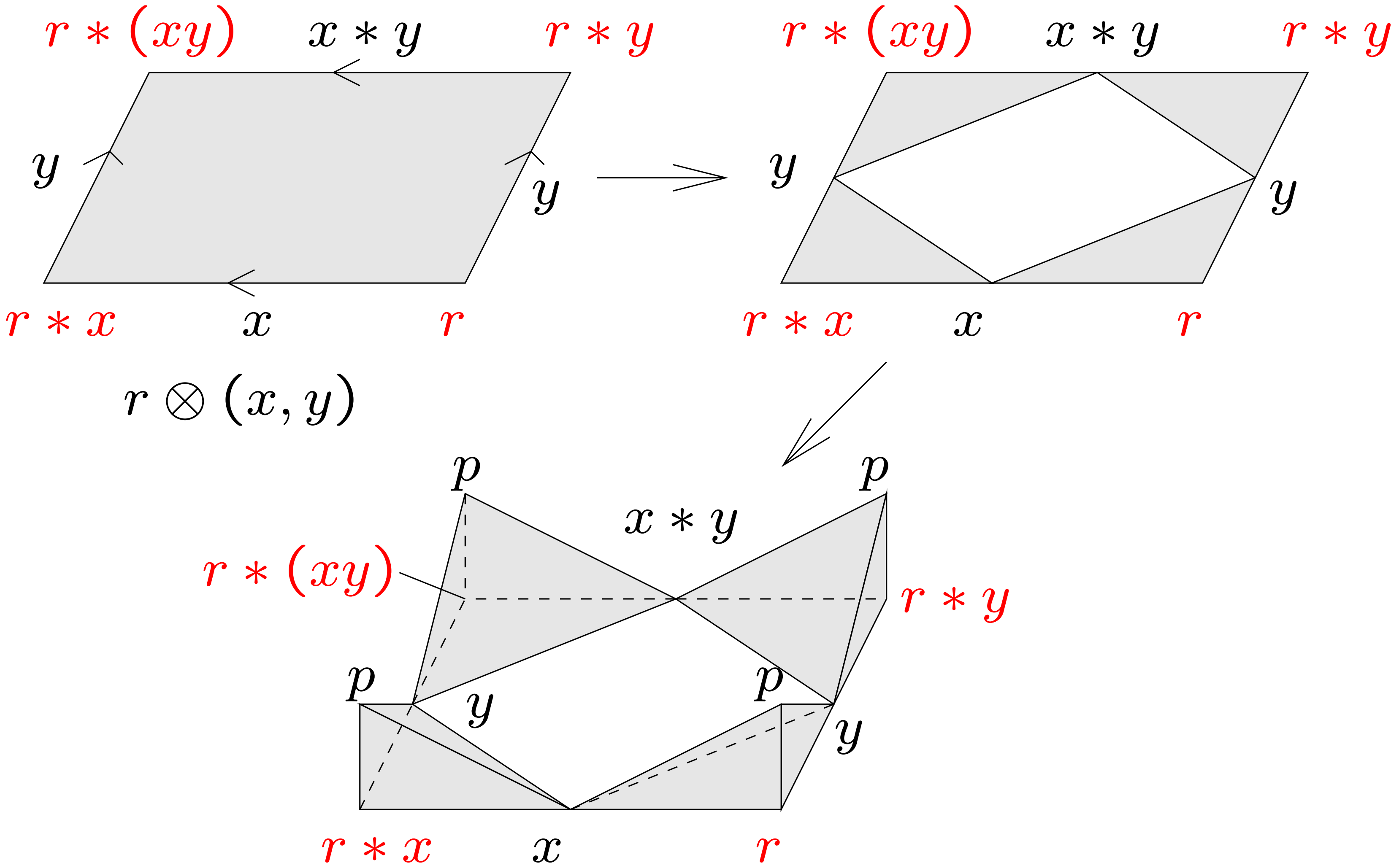
$H_n^{\Delta}(X) := H_n(C_*^{\Delta}(X)_{G_X})$

We can construct a homomorphism

$$\varphi_* : H_n^R(X; \mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$$

in the following way:

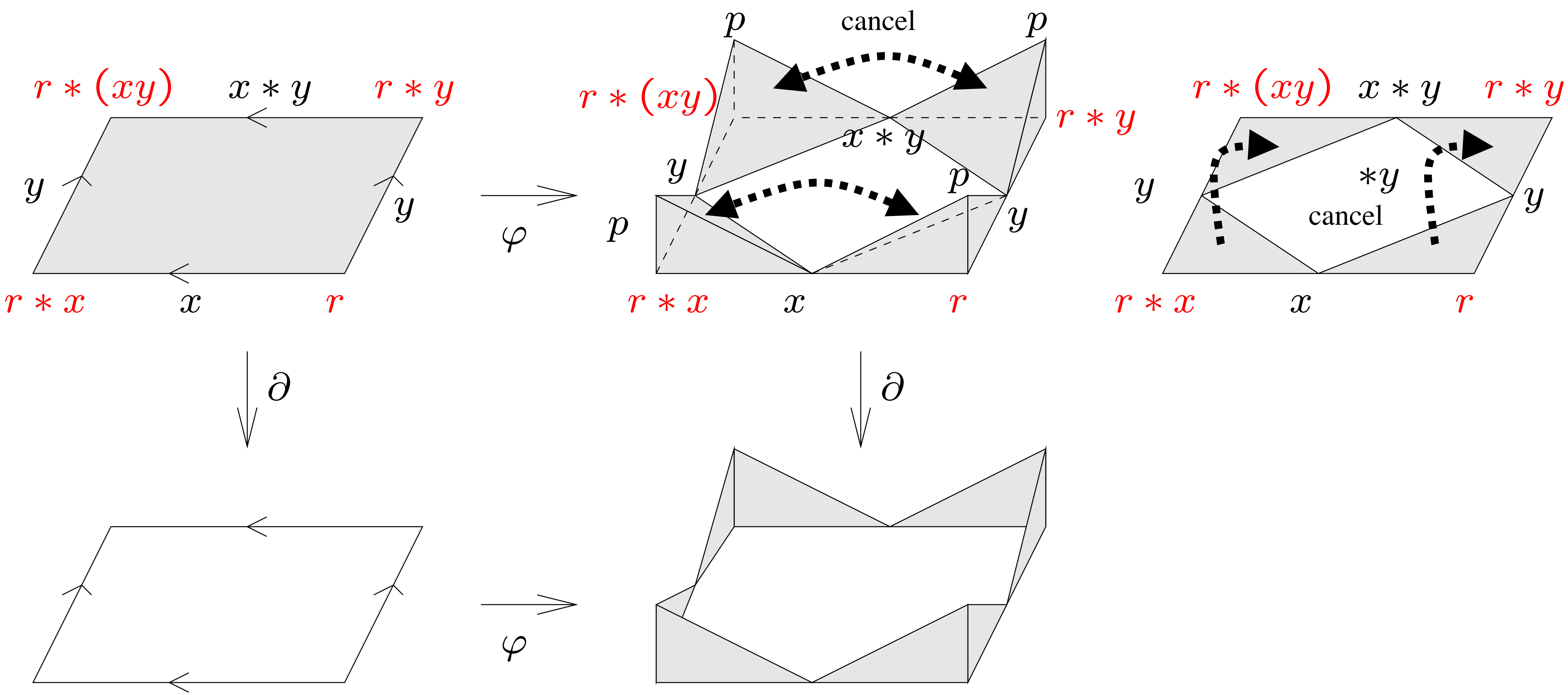
$n = 2$ $\quad \varphi : C_2^R(X; \mathbb{Z}[X]) \to C_3^{\Delta}(X)_{G_X}$



$$(p, r, x, y) - (p, r * x, x, y)$$
$$-(p, r * y, x * y, y) + (p, r * (xy), x * y, y)$$

Thm $\varphi : C_n^R(X;\mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X)_{G_X}$ *is a chain map.*

Proof.



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The map φ induces a homomorphism

$$\varphi_* : H^R_n(X;\mathbb{Z}[X]) \to H^{\Delta}_{n+1}(X).$$

If we have a function f from X^{k+1} to some abelian group A satisfying

1. $\sum_i (-1)^i f(x_0, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = 0$,
2. $f(x_0 * y, \ldots, x_k * y) = f(x_0, \ldots, x_k)$,
3. $f(x_0, \ldots, x_k) = 0$ if $x_i = x_{i+1}$ for some i,

then f gives a cocycle of $H^{\Delta}_k(X)$ and also a cocycle of $H^Q_{k-1}(X;\mathbb{Z}[X])$.

If X has a 'geometric structure', we can construct a cocycle for $H_k^{\Delta}(X)$.

$\mathcal{P}_n$: the quandle formed by parabolic elements of $\mathrm{Isom}^+(\mathbb{H}^n)$.
For $x \in \mathcal{P}_n$, let $(x)_\infty$ be the fixed point at $\partial\overline{\mathbb{H}^n}$ of x.
The function $(\mathcal{P}_n)^{n+1} \longrightarrow \mathbb{R}$ defined by

$$(x_0, x_1, \ldots, x_n) \mapsto \mathrm{Vol}(\mathrm{ConvHull}((x_0)_\infty, (x_1)_\infty, \ldots, (x_n)_\infty))$$

satisfies the previous three conditions.

Thm (Inoue-K.) *The n-dimensional hyperbolic volume is a quandle cocycle of $\mathcal{P}_n$.*