Quandle cocycles from group cocycles

KABAYA, Yuichi (蒲谷 祐一)

(Osaka City University

Advanced Mathematical Institute (OCAMI))

Kyoto University, 15:40–16:30 June 2nd 2010

Introduction

X : a quandle (an algebraic object)

For a knot diagram D, we can color the arcs of D by X. This gives a cycle in some homology theory: *quandle homology theory*.

If we have a cocycle of X, we obtain an invariant of knots by evaluation of cycles by the cocycle. This is called *cocycle invariant*.

Introduction

Problem: How can we find quandle cocycles?

Can we construct quandle cocycle of X from a group cocycle of $\mathrm{Aut}(X)$ or other group related to X?

I will show a construction of a quandle cocycle from a group cocycle. Then the geometric meaning of the cocycle invariant for the cocycle obtained from our construction.

Quandle

The definition of quandles was introduced by Joyce in 1982.

A quandle X is a set with a binary operation $* : X \times X \to X$ satisfying

1. $x * x = x$ for any $x \in X$,
2. the map $*y : X \to X : x \mapsto x * y$ is bijective for any y,
3. $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

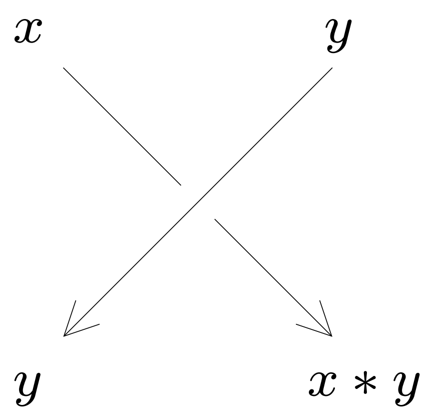
Example

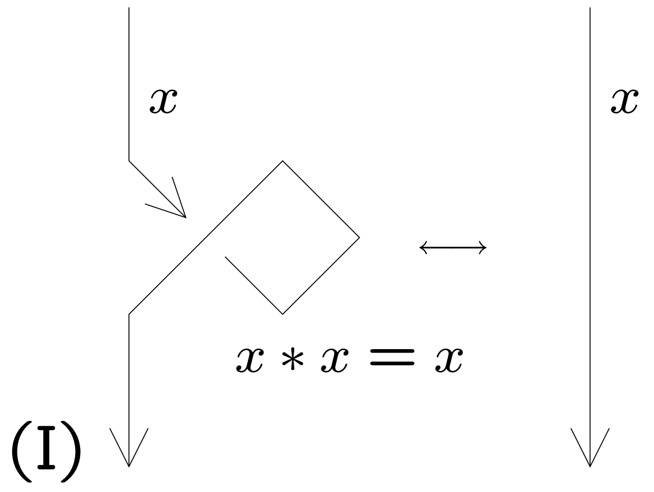
G : a group, $S \subset G$: a subset closed under conjugation.
S has a quandle structure by conjugation $x * y = y^{-1}xy$.

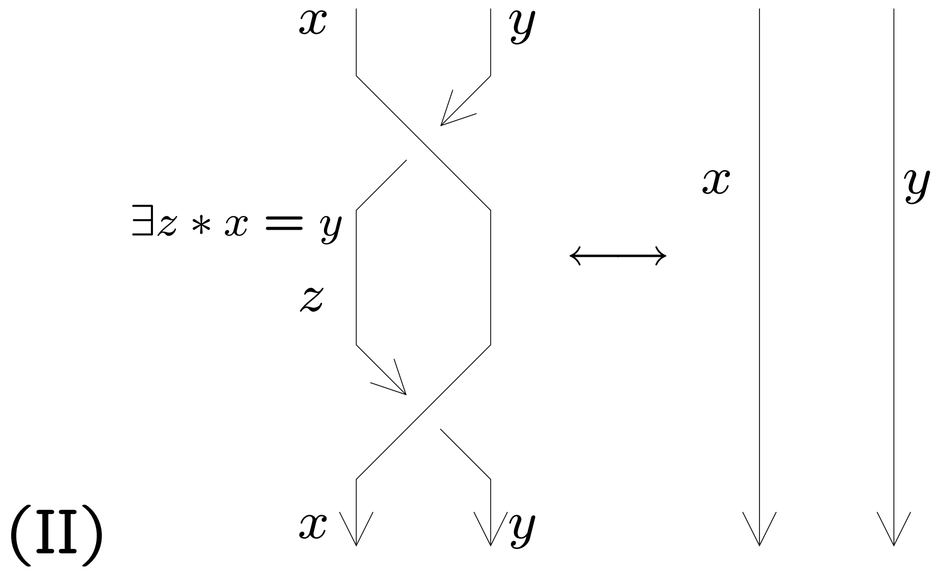
$$(x * y) * z = z^{-1}y^{-1}xyz = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz) = (x * z) * (y * z)$$

Relation with knot theory

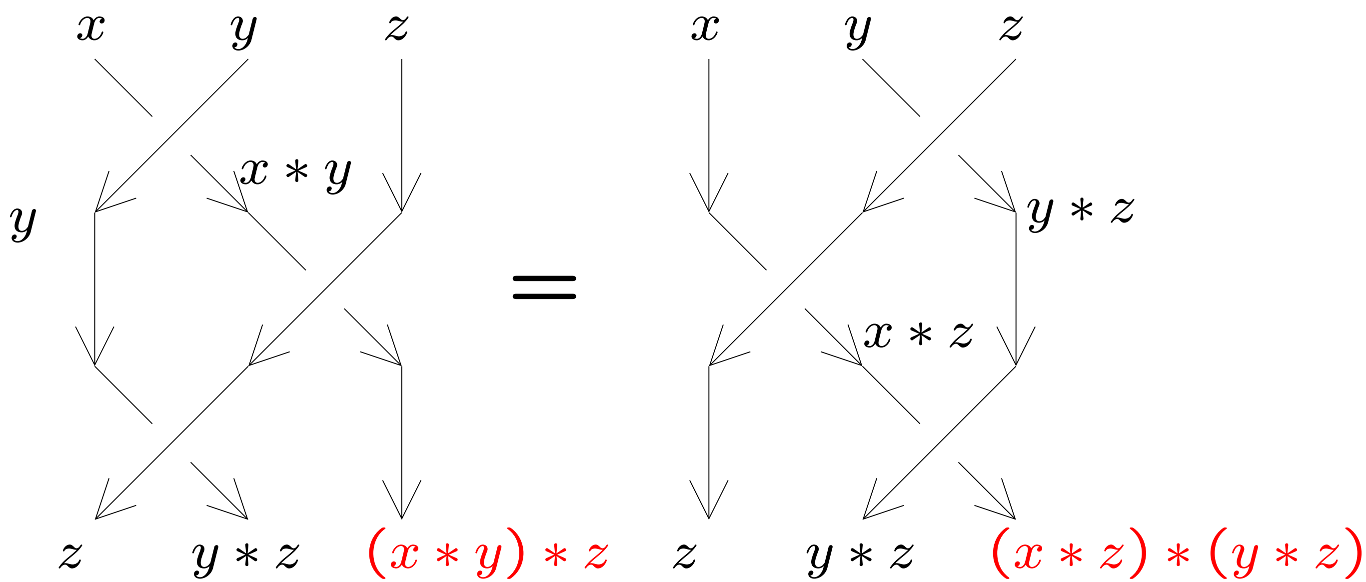
Assign an element of a quandle X for each arc of a knot diagram satisfying the following relation at each crossing. Then the axioms correspond to the Reidemeister moves:







Relation with knot theory



(III)

Quandle homology (Carter-Jelsovsky-Kamada-Langford-Saito, 2003)

For a quandle X, define the group G_X by $\langle x \in X | x * y = y^{-1}xy \rangle$. This is called the *associated group* of X.

Let $C_n^R(X) = \mathrm{span}_{\mathbb{Z}[G_X]}\{(x_1, \dots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\begin{aligned}\partial(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^i \{ & (x_1, \dots, \widehat{x_i}, \dots, x_n) \\ & - x_i(x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n)\}\end{aligned}$$

Let M be a right $\mathbb{Z}[G_X]$-module. The homology group of $M \otimes_{\mathbb{Z}[G_X]} C_n^R(X)$ is called the *rack homology* $H_n^R(X; M)$.

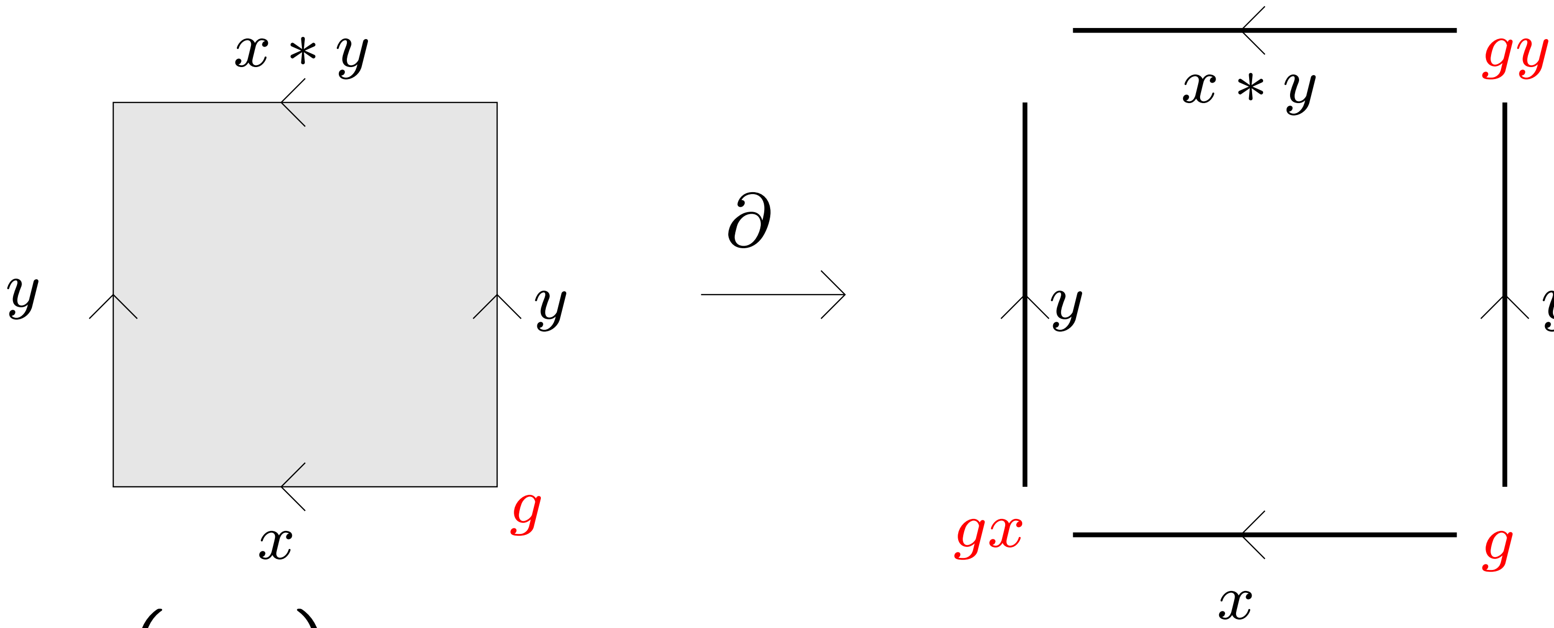
Factoring degenerate chains, we also define the quandle homology $H_n^Q(X; M)$.

Let

$$C_n^D(X) = \text{span}_{\mathbb{Z}[G_X]}\{(x_1, \ldots, x_n) | x_i \in X,$$
$$x_i = x_{i+1} (\text{for some } i)\}.$$

This is a subcomplex of $C_n^R(X)$. Let $C_n^Q(X)$ be the quotient $C_n^R(X)/C_n^D(X)$. The homology of $M \otimes_{\mathbb{Z}[G_X]} C_n^Q(X)$ is called the *quandle homology* $H_n^Q(X; M)$

Geometric interpretation $C_2^R(X) \to C_1^R(X)$

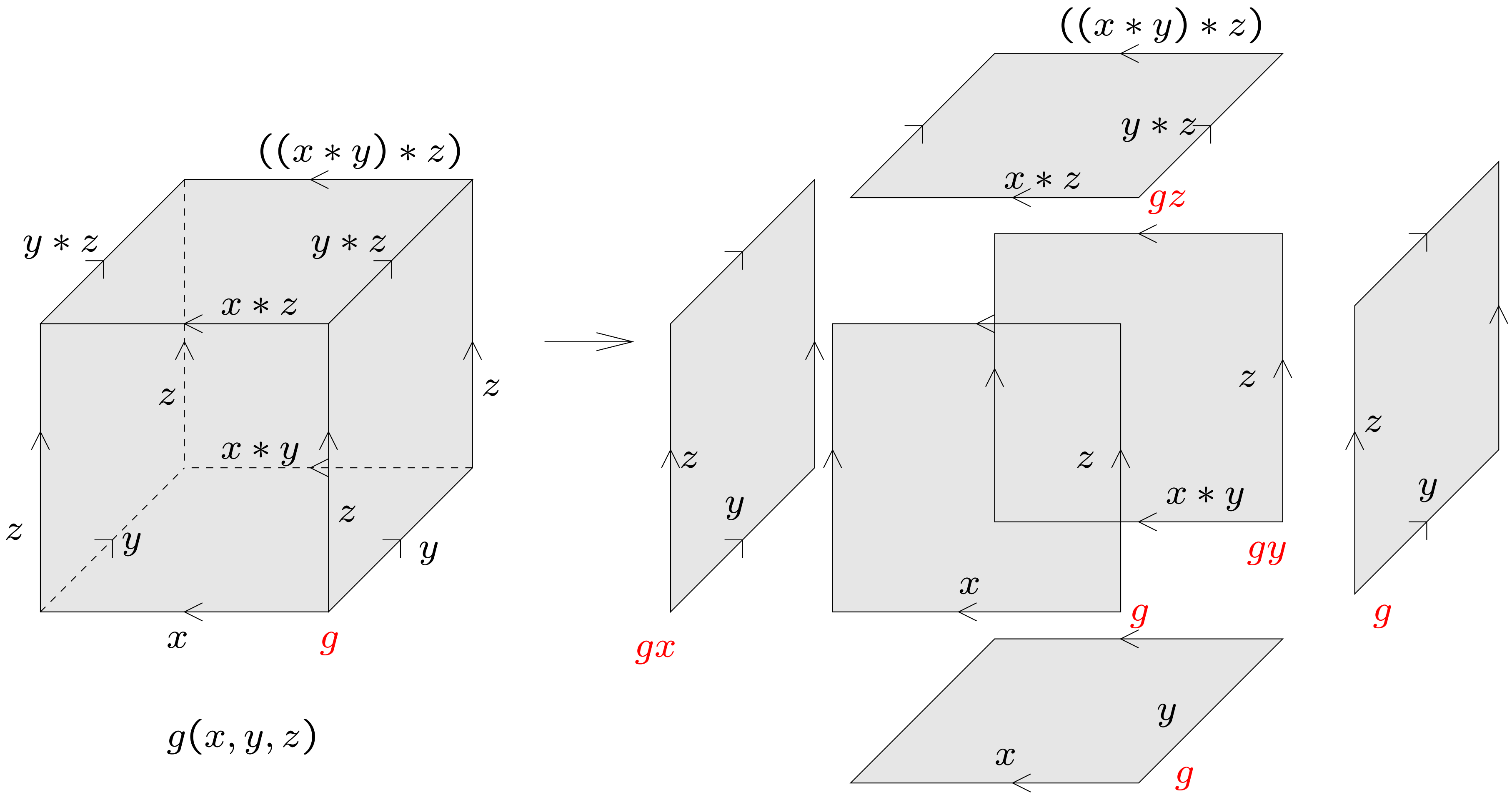


$g(x, y)$

$-g(y) + gx(y)$
$+g(x) - gy(x * y)$

$$\sum_{i=1}^{n} (-1)^i \{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\ - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

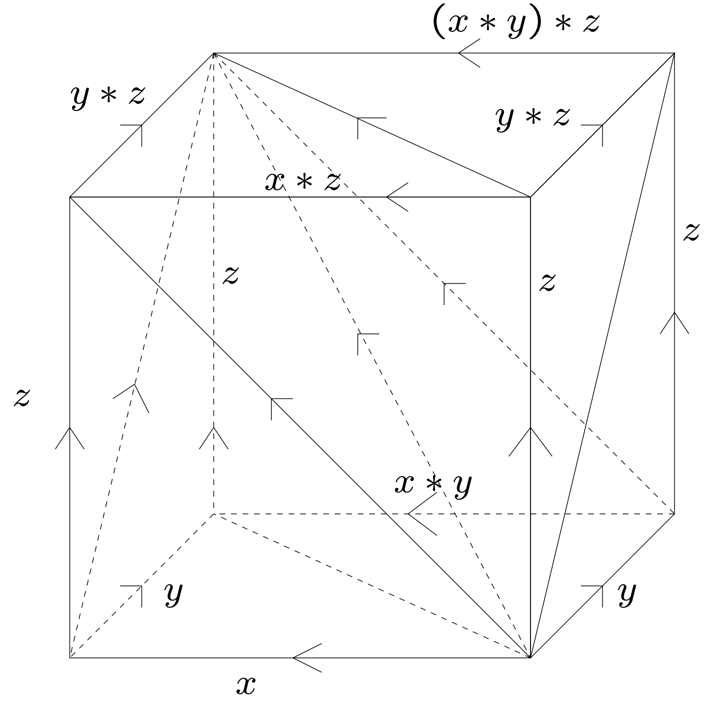
Geometric interpretation $C_3^R(X) \to C_2^R(X)$



$$g(x,y,z) \mapsto -g(y,z) + gx(y,z) + g(x,z) - gy(x*y,z)$$
$$-g(x,y) + gz(x*z, y*z)$$

A naive relationship with group homology

We can construct a map from the rack homology $H_n^R(X; A)$ to the group homology $H_n(G_X; A)$ by dividing an n-cube into $n!$ simplices.



For example, when $n = 3$,

$$(x, y, z) \mapsto [x|y|z] - [x|z|y * z] + [y|z|(x * y) * z]$$
$$-[y|x * y|z] + [z|x * z|y * z] - [z|y * z|(x * y) * z]$$

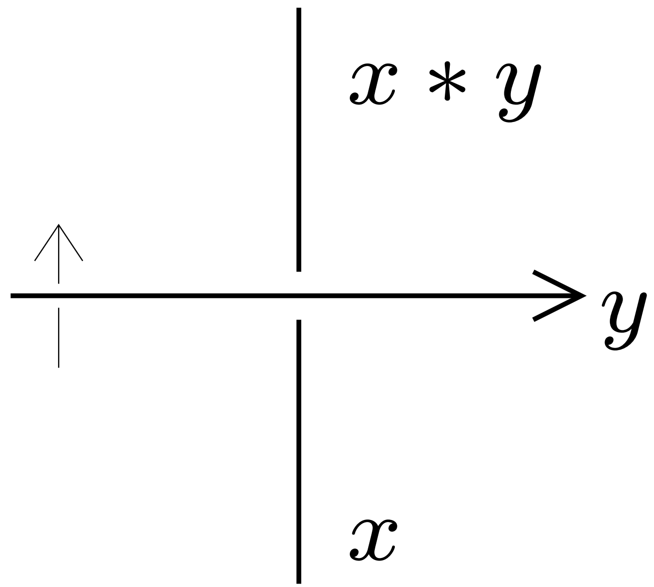
We will give another relationship between quandle homology and group homology.

Before mentioning the relation, we introduce the cycle associated to a knot diagram with *coloring*.

Arc coloring

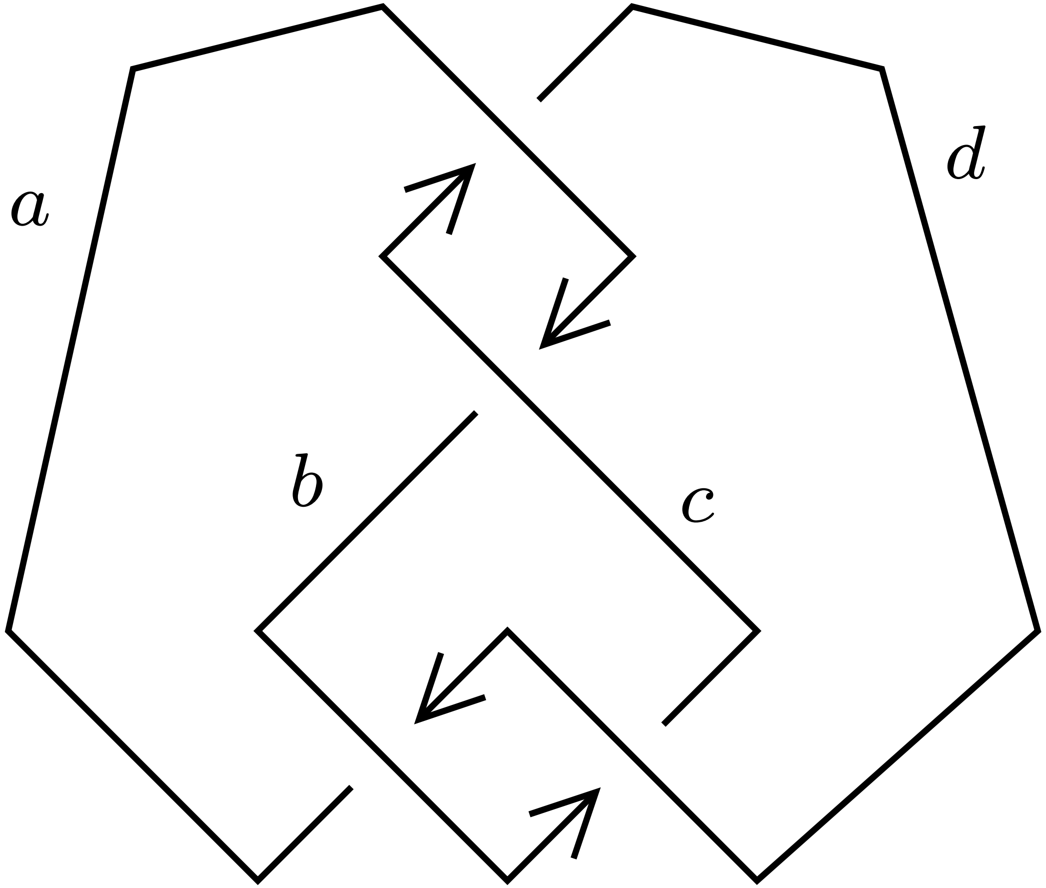
Let D be a diagram of a knot K.

We call a map $\mathcal{A} : \{\text{arcs of } D\} \to X$ *arc coloring* if it satisfies the following relation at each crossing.



x, y and $x * y \in X$

Example: Arc coloring



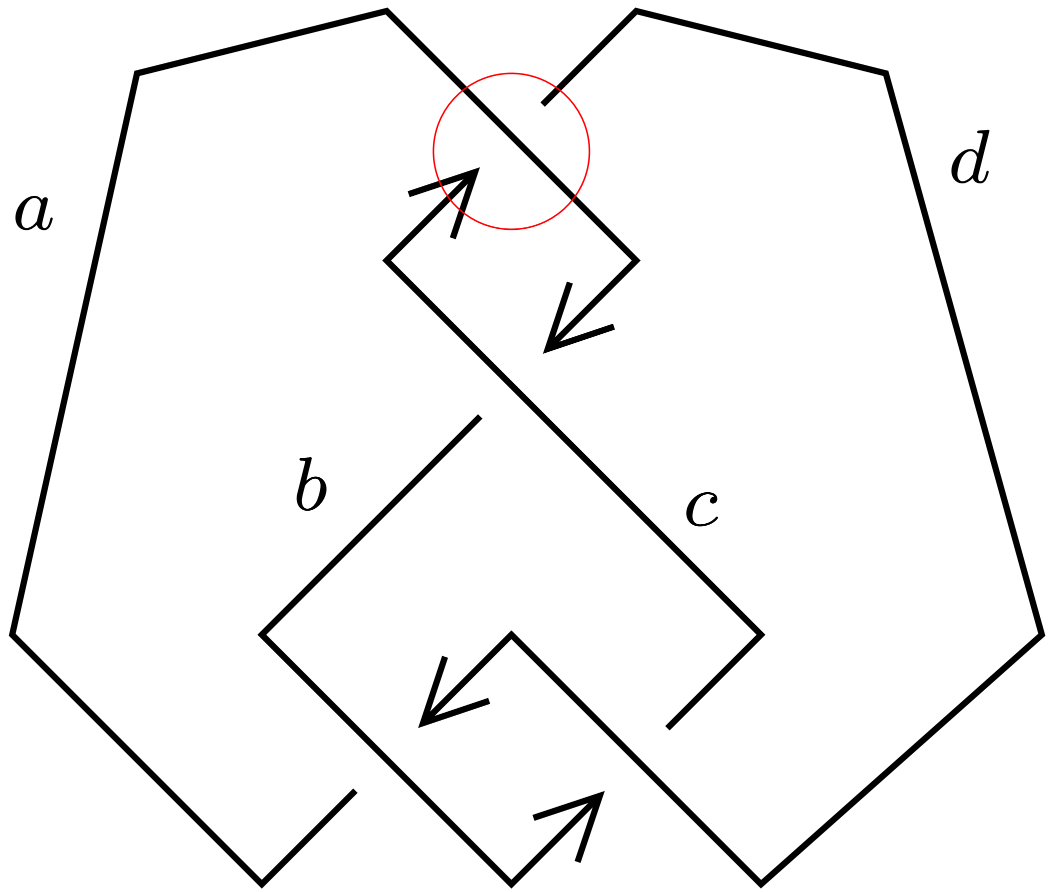
$$c * a = d,$$

$$a * c = b,$$

$$a * b = d,$$

$$c * d = b.$$

Example: Arc coloring



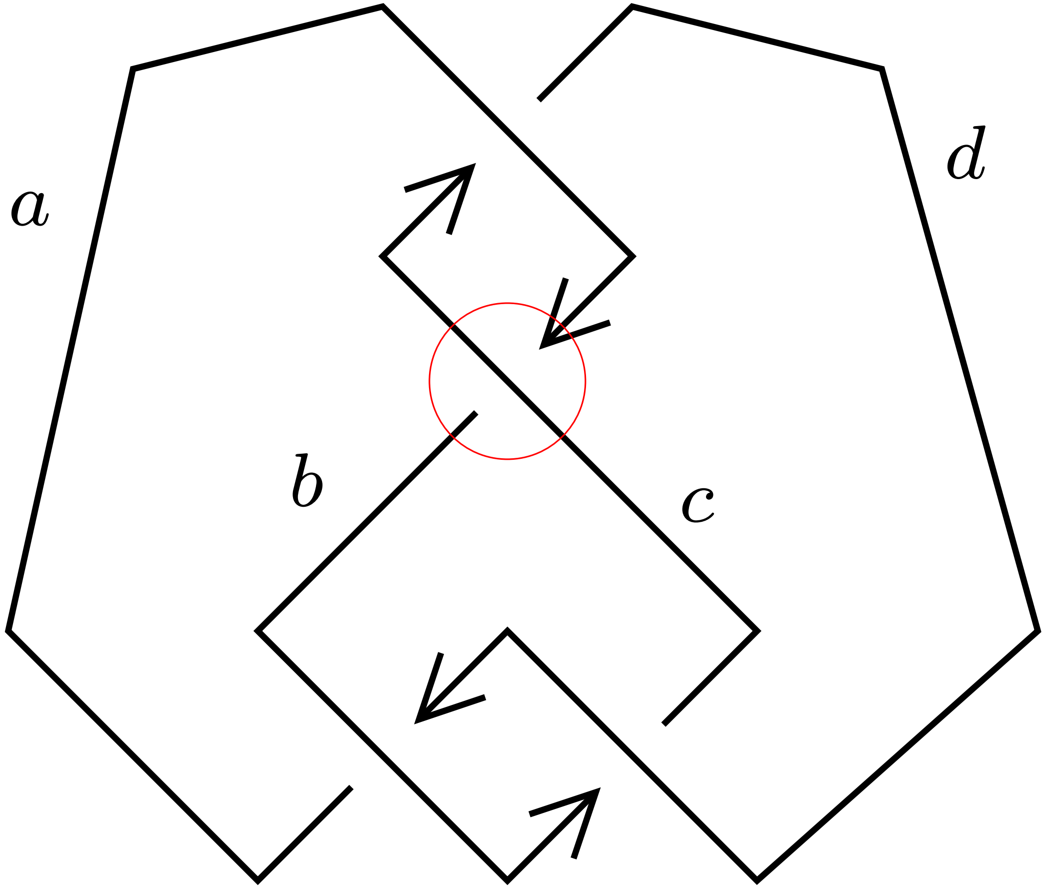
$c * a = d,$

$a * c = b,$

$a * b = d,$

$c * d = b.$

Example: Arc coloring



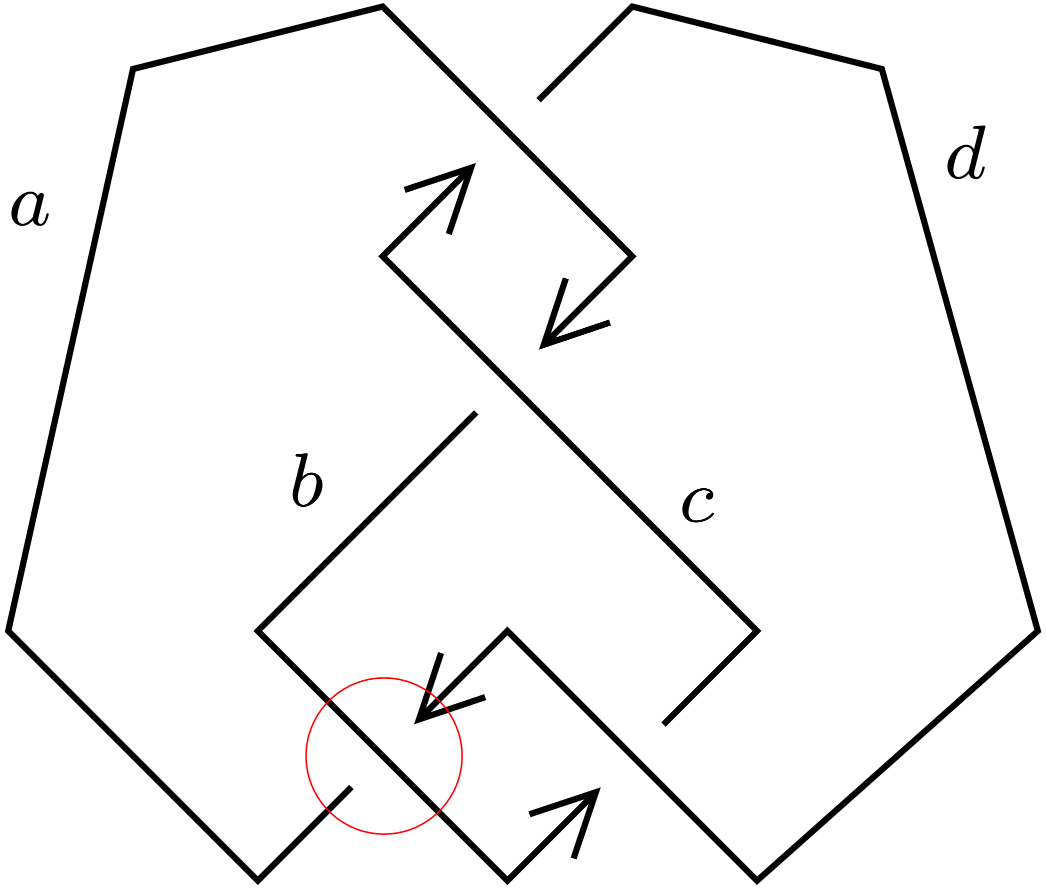
$c * a = d,$

$a * c = b,$

$a * b = d,$

$c * d = b.$

Example: Arc coloring



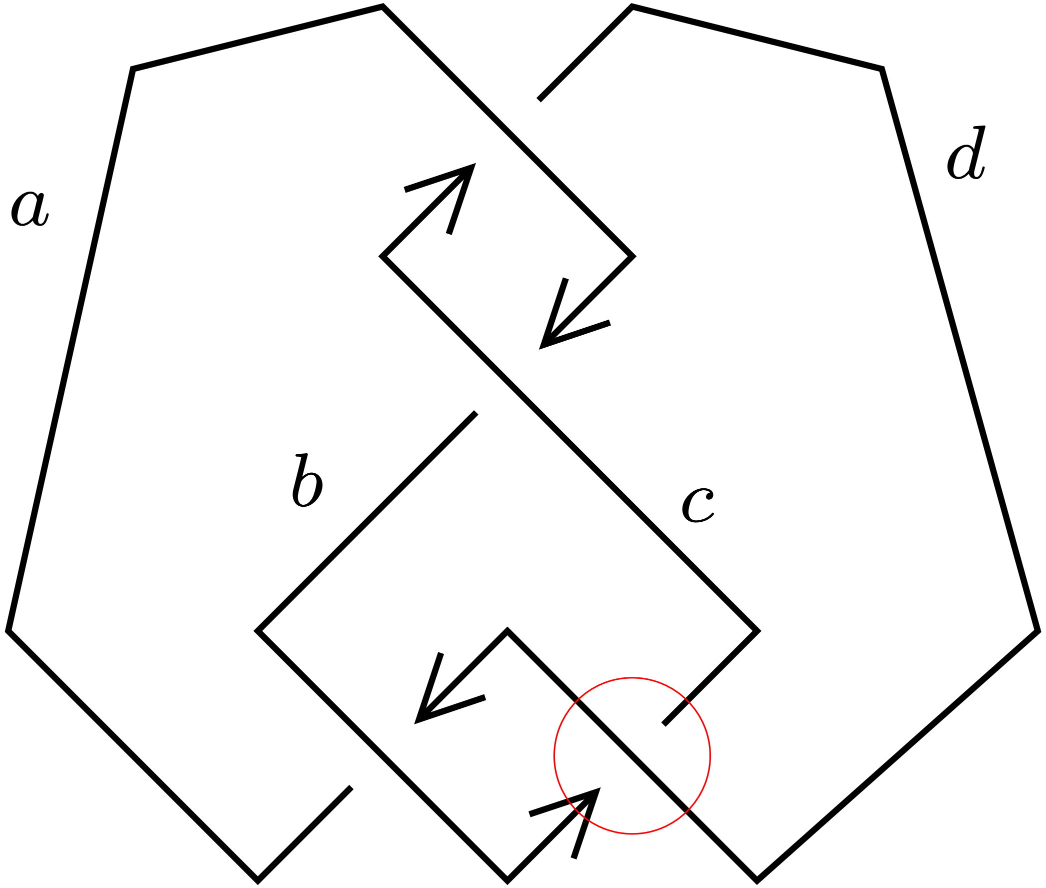
$c * a = d,$

$a * c = b,$

$a * b = d,$

$c * d = b.$

Example: Arc coloring



$c * a = d,$

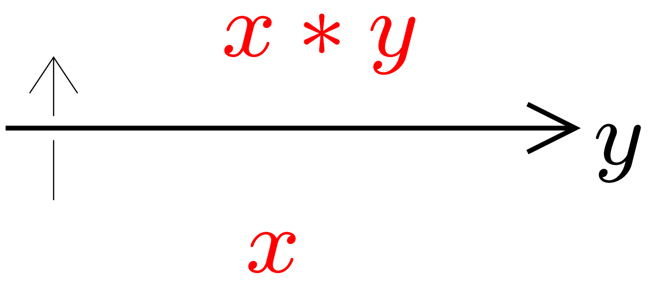
$a * c = b,$

$a * b = d,$

$c * d = b.$

Region coloring

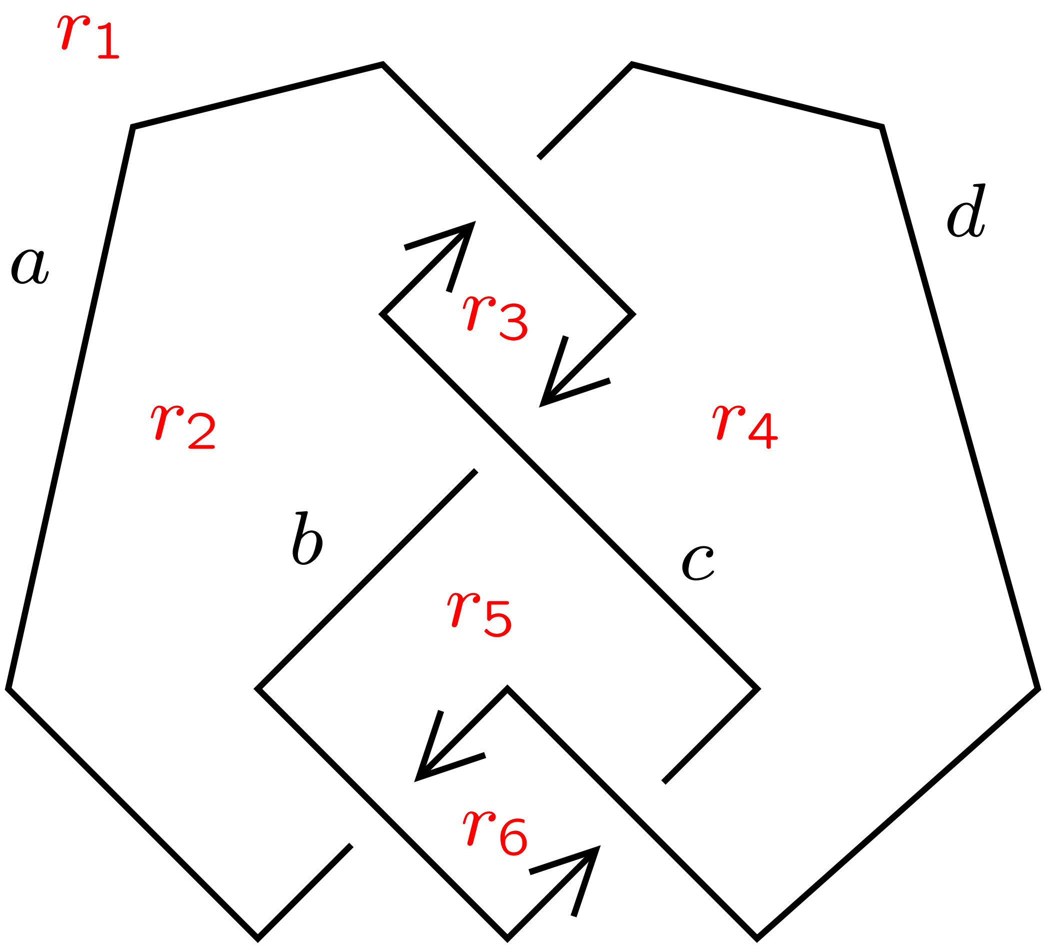
Let D be a diagram and $\mathcal{A}$ be an arc coloring by X. A map $\mathcal{D} : \{\text{regions of } D\} \to X$ is called a *region coloring* if it satisfies the following relation:



x, y and $x * y \in X$

We call a pair $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ ($\mathcal{A}$: arc coloring, $\mathcal{R}$: region coloring) a *shadow coloring*. (The notion of region coloring is defined for any set with right G_X-action.)

Example: Shadow coloring



$$r_2 * a = r_1, \quad r_3 * c = r_2,$$

$$r_3 * a = r_4, \quad r_2 * b = r_5,$$

$$r_5 * d = r_6,$$

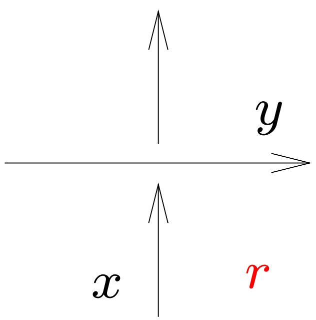
Cycles associated with quandle colorings

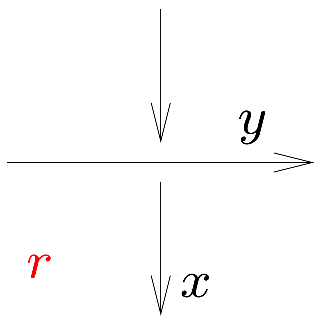
A quandle X itself has a right G_X-action defined by

$$x * (x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}) = (\dots((x *^{\varepsilon_1} x_1) *^{\varepsilon_2} x_2) \dots) *^{\varepsilon_n} x_n.$$

So the free abelian group $\mathbb{Z}[X]$ is a right $\mathbb{Z}[G_X]$-module.

Let $\mathcal{S} = (\mathcal{A}, \mathcal{R})$ be a shadow coloring by a quandle X. Assign

$+r \otimes (x, y)$ for 

and $-r \otimes (x, y)$ for 

.

We define two chains associated with a shadow coloring

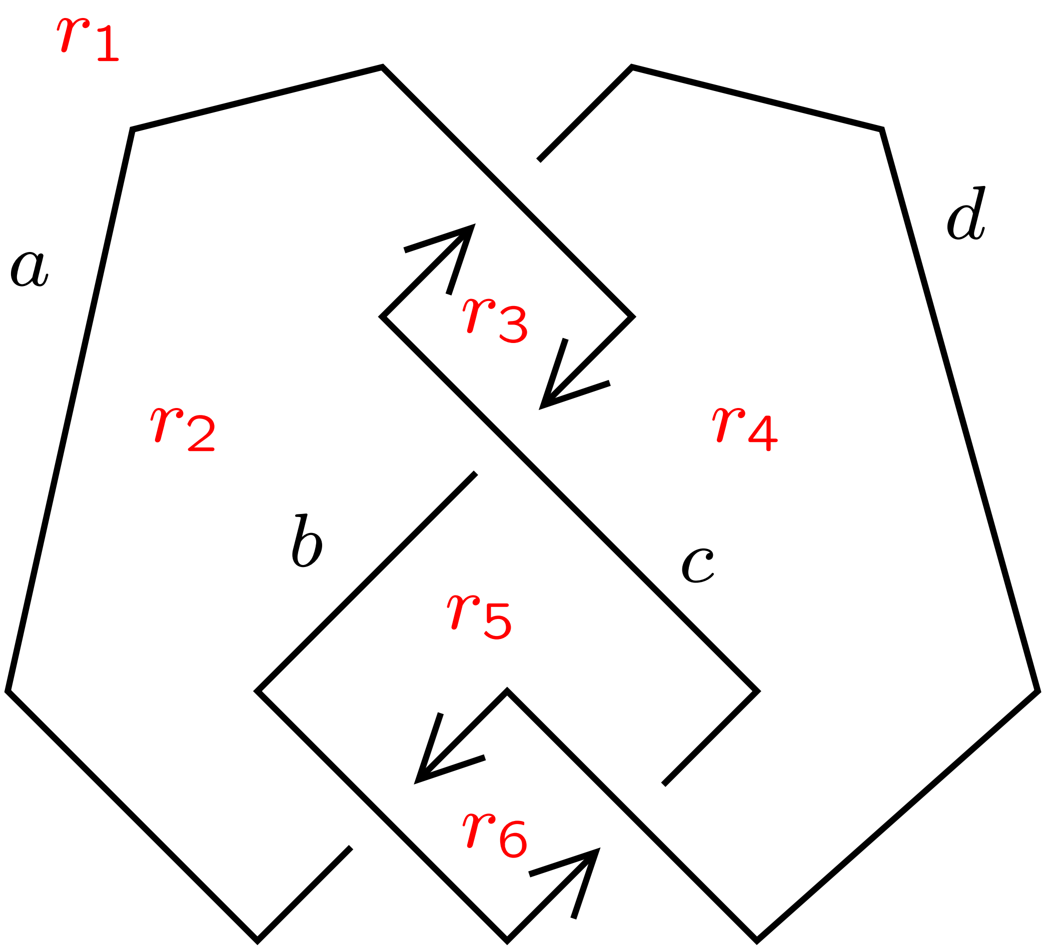
$$C_s(\mathcal{S}) = \sum_{c\text{:crossing}} \varepsilon_c r_c \otimes (x_c, y_c) \in C_2^Q(X; \mathbb{Z}[X])$$

$$C_a(\mathcal{A}) = \sum_{c\text{:crossing}} \varepsilon_c (x_c, y_c) \in C_2^Q(X; \mathbb{Z}).$$

We can show that $C_s(\mathcal{S})$ and $C_a(\mathcal{A})$ are cycles. Moreover the homology class $[C_s(\mathcal{S})]$ does not depend on the region coloring.

Eisermann showed that the cycle $[C_a]$ is essentially equivalent to the monodoromy along the longitude of some representation of the knot group. So we study the invariant $[C_s(\mathcal{S})]$.

Example: $C_s(\mathcal{S})$ **and** $C_a(\mathcal{A})$



$$C_s(\mathcal{S}) =$$

$$r_3 \otimes (c, a) + r_3 \otimes (a, c)$$

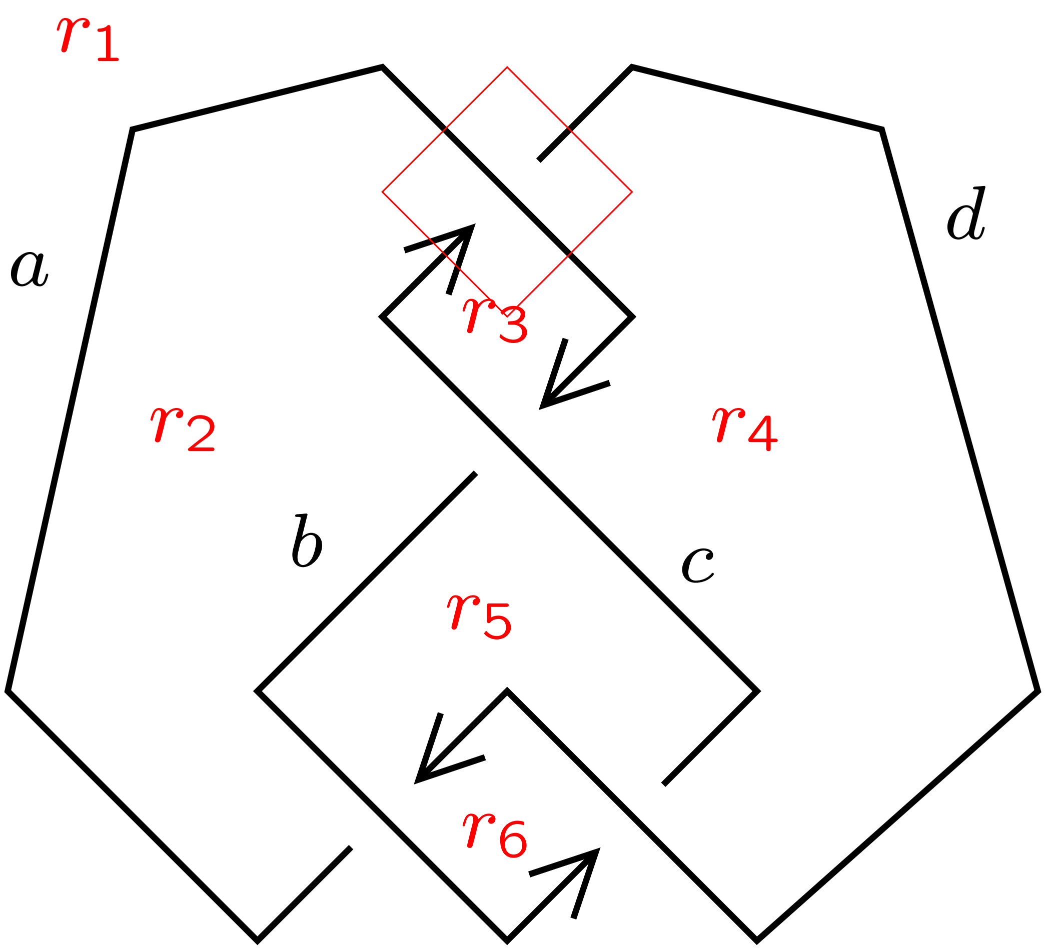
$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$$C_a(\mathcal{S}) =$$

$$(c, a) + (a, c)$$

$$- (a, b) - (c, d)$$

Example: $C_s(\mathcal{S})$ and $C_a(\mathcal{A})$



$C_s(\mathcal{S}) =$

$$r_3 \otimes (c, a) + r_3 \otimes (a, c)$$

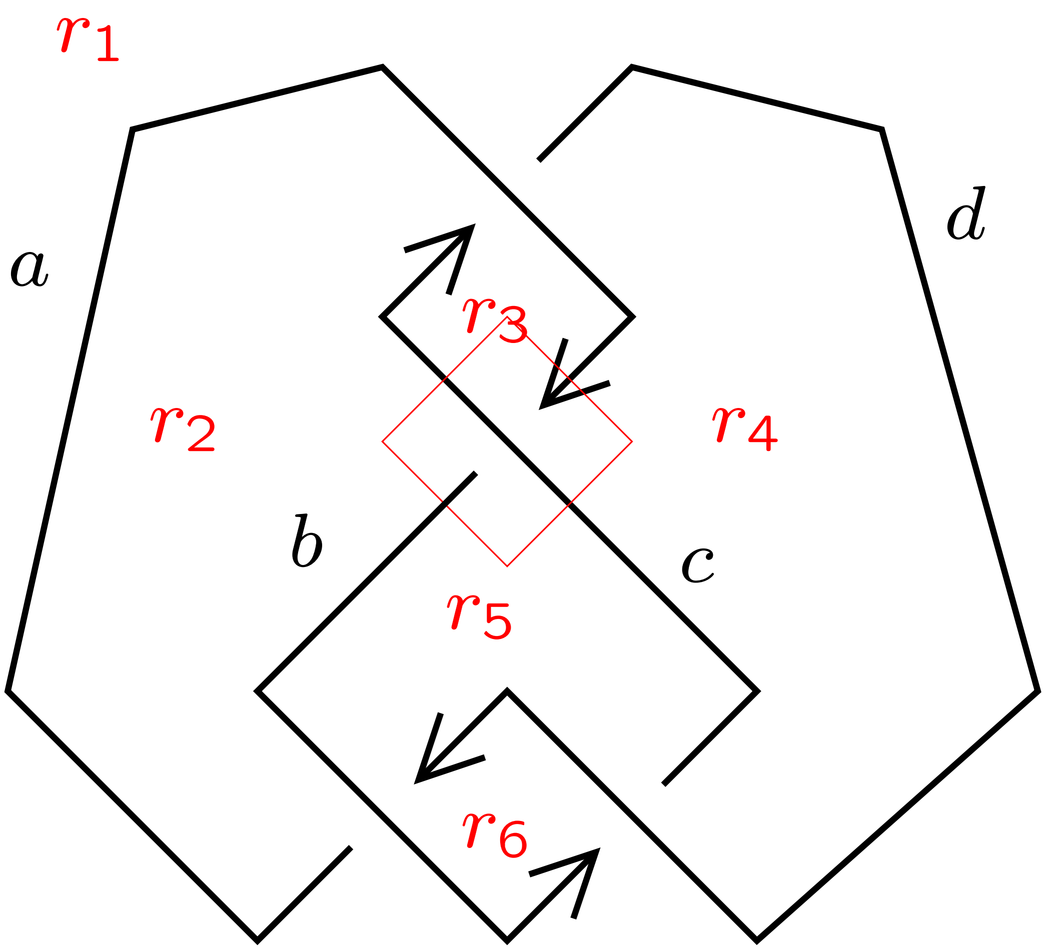
$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$C_a(\mathcal{S}) =$

$$(c, a) + (a, c)$$

$$- (a, b) - (c, d)$$

Example: $C_s(\mathcal{S})$ and $C_a(\mathcal{A})$



$C_s(\mathcal{S}) =$

$$r_3 \otimes (c, a) + r_3 \otimes (a, c)$$

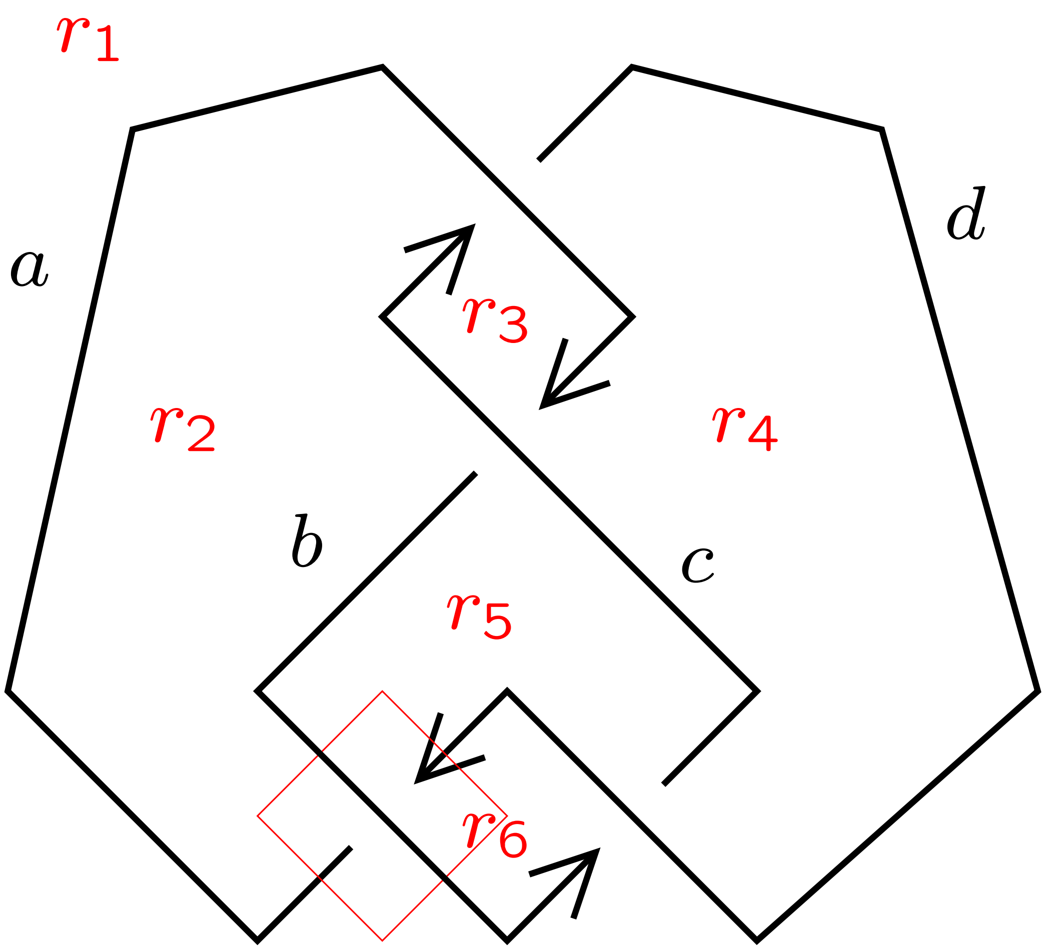
$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$C_a(\mathcal{S}) =$

$$(c, a) + (a, c)$$

$$- (a, b) - (c, d)$$

Example: $C_s(\mathcal{S})$ and $C_a(\mathcal{A})$



$C_s(\mathcal{S}) =$

$$r_3 \otimes (c, a) + r_3 \otimes (a, c)$$

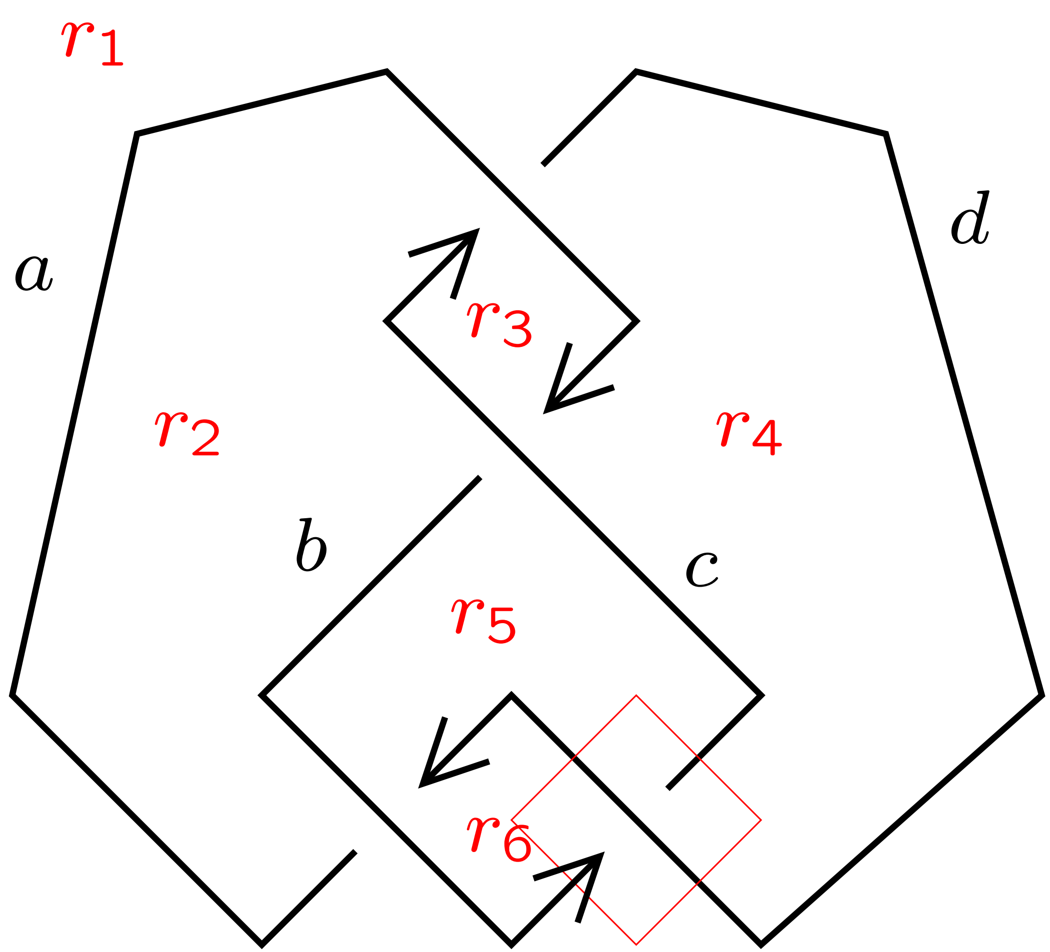
$$-r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$C_a(\mathcal{S}) =$

$$(c, a) + (a, c)$$

$$-(a, b) - (c, d)$$

Example: $C_s(\mathcal{S})$ and $C_a(\mathcal{A})$



$C_s(\mathcal{S}) =$

$$r_3 \otimes (c, a) + r_3 \otimes (a, c)$$

$$- r_2 \otimes (a, b) - r_4 \otimes (c, d)$$

$C_a(\mathcal{S}) =$

$$(c, a) + (a, c)$$

$$- (a, b) - (c, d)$$

Quandle cocycle invariants

Assume $|X| < \infty$. Let A be an abelian group. For any quandle cocycle $f \in H^2_Q(X; \mathrm{Func}(X, A))$ (or $f \in H^3_Q(X; A)$),

$$\sum_{\mathcal{S}:\text{colorings}} \langle f, C_s(\mathcal{S}) \rangle \in \mathbb{Z}[A]$$

is an invariant of knots. This is called *quandle cocycle invariant*.

We can also define an invariant for C_a by using a cocycle of $H^2_Q(X; A)$.

Simplicial quandle homology $H_n^{\Delta}(X)$

Let $C_n^{\Delta}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) | x_i \in X\}$. Define the boundary operator $\partial : C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ by

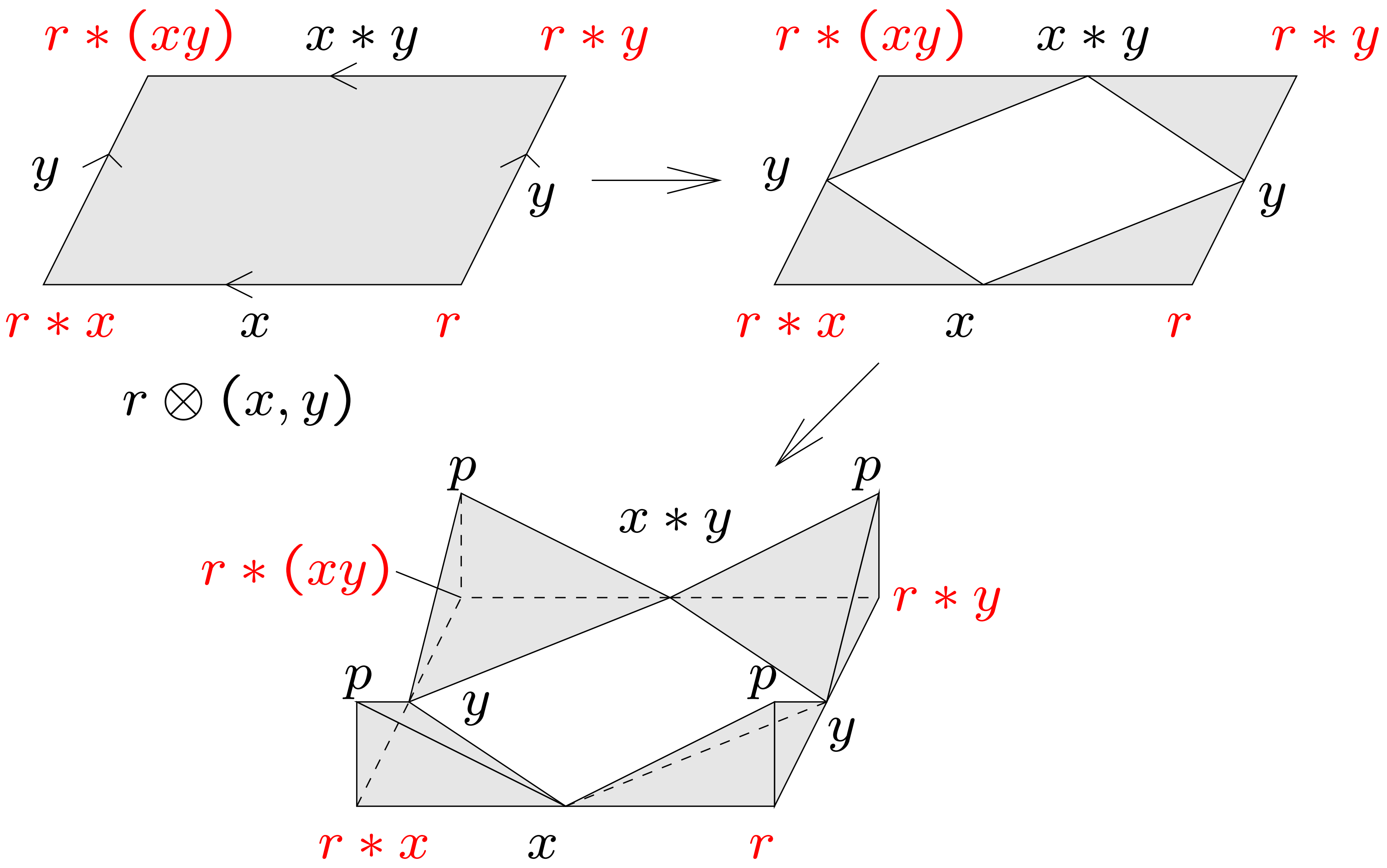
$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

$C_n^{\Delta}(X)$ has a natural right action by $\mathbb{Z}[G_X]$. Denote the homology of $C_n^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by $H_n^{\Delta}(X)$. We can construct a map

$$\varphi_* : H_n^R(X; \mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$$

in the following way:

$n = 2$ $\varphi : C_2^R(X;\mathbb{Z}[X]) \to C_3^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$



$$(p, r, x, y) - (p, r * x, x, y)$$
$$-(p, r * y, x * y, y) + (p, r * (xy), x * y, y)$$

For general case, let I_n be the set of maps $\iota : \{1, 2, \cdots, n\} \to \{0, 1\}$. Let $|\iota|$ denote the cardinality of the set $\{k \mid \iota(k) = 1,\ 1 \leq k \leq n\}$. For $r \otimes (x_1, x_2, \cdots, x_n) \in C_n^R(X; \mathbb{Z}[X])$ and $\iota \in I_n$, define

$$r(\iota) = r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)})$$
$$x(\iota, i) = x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)}).$$

Fix $p \in X$. Define $\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ by

$$\varphi(r \otimes (x_1, x_2, \cdots, x_n))$$
$$= \sum_{\iota \in I_n} (-1)^{|\iota|} (p, r(\iota), x(\iota, 1), x(\iota, 2), \cdots, x(\iota, n)).$$

Thm (Inoue-K.) $\varphi : C_n^R(X;\mathbb{Z}[X]) \longrightarrow C_{n+1}^{\triangle}(X) \otimes_{\mathbb{Z}[G_X]} \mathbb{Z}$ *is a chain map.*

The map φ induces a homomorphism

$$H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\triangle}(X).$$

So we can construct a quandle cocycle from a cocycle of $H_{n+1}^{\triangle}(X)$.

If we have a function f from X^{k+1} to some abelian group A satisfying

1. $\sum_i (-1)^i f(x_0, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = 0$,
2. $f(x_0 * y, \ldots, x_k * y) = f(x_0, \ldots, x_k)$ for any y, and
3. $f(x_0, \ldots, x_k) = 0$ if $x_i = x_{i+1}$ for some i,

then f gives a cocycle of $H_k^{\Delta}(X)$ and a cocycle of $H_{k-1}^{Q}(X; \mathbb{Z}[X])$. Moreover f can be regarded as a cocycle in $H_Q^k(X; A)$

We will construct functions satisfying these conditions from *group cocycles*.

Group cocycle

Let G be a group and A be an abelian group.

A map $f : G^n \to A$ is called a group n-cocycle if it satisfies

$$f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n) = 0.$$

Define $f' : G^{n+1} \to A$ by

$$f'(g_0, g_1, g_2, \ldots, g_n) := f(g_0 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{n-1} g_n^{-1})$$

The map f' satisfies following properties:

(a) $\sum_{i=0}^{n+1}(-1)^i f'(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1}) = 0$

(b) $f'(g_0 g, \ldots, g_n g) = f'(g_0, \ldots, g_n)$ (right invariance)

Conversely, any map satisfying these two properties gives a group n-cocycle. We call this presentation of a group cocycle *homogeneous presentation*.

Example: Dihedral quandle

$R_p = \{0, 1, \ldots, p-1\}$ ($p > 2$: odd) has a quandle structure by

$$x * y = 2y - x \mod p$$

This is called the *dihedral quandle*.

We will construct quandle cocycles of R_p from group cocycles of $\mathbb{Z}/p$. Regard $\mathbb{Z}/p$ as R_p. Then a (normalized) group cocycle f in homogeneous notation satisfies

1. $\sum_i (-1)^i f(x_0, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = 0$,
3. $f(x_0, \ldots, x_k) = 0$ if $x_i = x_{i+1}$ for some i.

So we only have to check the property:

2. $f(x_0 * y, \ldots, x_k * y) = f(x_0, \ldots, x_k)$ for any y

But f does not satisfy this property in general. Let

$$\tilde{f}(x_0, \ldots, x_n) := f(x_0, \ldots, x_n) + f(-x_0, \ldots, -x_n)$$

Then we have

$$\begin{aligned}
&\tilde{f}(x_0 * y, \ldots, x_n * y) \\
&\quad = f(2y - x_0, \ldots, 2y - x_n) + f(2y + x_0, \ldots, 2y + x_n) \\
&\quad = f(-x_0, \ldots, -x_n) + f(x_0, \ldots, x_n) \quad \text{(right invariance)} \\
&\quad = \tilde{f}(x_0, \ldots, x_n)
\end{aligned}$$

Therefore $\tilde{f}$ satisfies the properties 1, 2 and 3. So we obtain a quandle n-cocycle.

Cohomology of cyclic groups

Let $G = \mathbb{Z}/p$ be a cyclic group (p is a positive integer). The first cohomology $H^1(G;\mathbb{Z}/p)$ is generated by

$$b_1(x) = x$$

and the second cohomology $H^2(G;\mathbb{Z}/p)$ is generated by

$$b_2(x,y) = \begin{cases} 1 & \text{if } \bar{x} + \bar{y} \geq p \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{x}$ is an integer $0 \leq \bar{x} < p$ with $\bar{x} \equiv 0 \mod p$. Moreover any element of $H^*(G;\mathbb{Z}/p)$ is generated by a cup product of b_1's and b_2's.

Let

$$d(x,y)=\begin{cases}1 & \text{if } \bar{x}+\bar{y}>p\\ -1 & \text{if } \bar{x}+\bar{y}<p \text{ and } xy\neq 0\\ 0 & \text{otherwise}\end{cases}$$

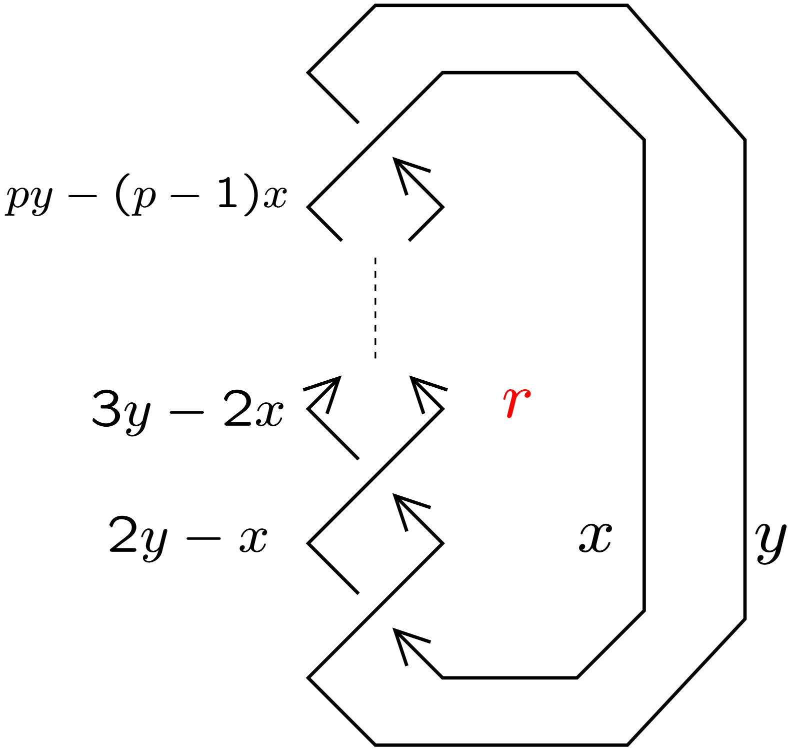
Prop *The quandle 3-cocycle obtained from $b_1 b_2$ is given by*

$$(x,y,z)\mapsto 2z(d(y-x,z-y)+d(y-x,y-z))$$

By computer calculation, I checked that this is 4 times the Mochizuki's 3-cocycle up to coboundary.

Next we will compute the quandle cocycle invariant of $(2,p)$-torus knot for this quandle 3-cocycle.

Quandle cycle invariant of the $(2, p)$-torus knot



For any x, y and r, the left figure is a shadow coloring of the $(2, p)$-torus knot.

Then

$$C_s(\mathcal{S}) = \sum_{i=0}^{p-1} r \otimes (x + i(y-x), \quad y + i(y-x))$$

Prop *The quandle cocycle invariant of the* $(2,p)$*-torus knot constructed from* $b_1 b_2 \in H^3(G;\mathbb{Z}/p)$ *is equal to*

$$p^2 \sum_{i=0}^{p-1} t^{-i^2} \in \mathbb{Z}[t]/(t^p - 1).$$

($\mathbb{Z}[\mathbb{Z}/p] \cong \mathbb{Z}[t]/(t^p - 1)$)

Remark

Let $L(p,q)$ be the lens space. The Dijkgraaf-Witten invariant of $L(p,q)$ for $G = \mathbb{Z}/p$ is equal to

$$\sum_{i=0}^{p-1} t^{-q\cdot i^2} \in \mathbb{Z}[t]/(t^p - 1)$$

(Usually Dijkgraaf-Witten invariant is defined with values in $\mathbb{C}$ and normalized by multiplying $\frac{1}{|G|}$. I also used different orientation convention)

Since the double branched covering of the $(2,p)$-torus knot is $L(p,1)$, it is natural to ask a relation with quandle cocycle invariant.

General case

G : a group. Fix an element $h \in G$.

$\text{Conj}(h) = \{g^{-1}hg | g \in G\}$

$\text{Conj}(h)$ has a quandle operation by $x * y = y^{-1}xy$.

Let $Z(h) = \{g \in G | gh = hg\}$ be the centralizer of h in G.

Lemma *As a set* $\text{Conj}(h) \cong Z(h) \backslash G$ *by*

$$g^{-1}hg \leftrightarrow Z(h)g \text{ (right coset)}$$

Idea

$$\mathsf{Conj}(h) \qquad\qquad \leftrightarrow \qquad\qquad Z(h)\backslash G$$

Idea

$$\text{Conj}(h) \quad \leftrightarrow \quad Z(h)\backslash G$$

Use for representation of π_1 | Space which G acts on

Idea

Conj(h)	$\leftrightarrow$	$Z(h)\backslash G$
Use for representation of π_1		Space which G acts on
		Construct a group cycle

Idea

Conj(h) $\leftrightarrow$ $Z(h)\backslash G$

Use for representation of π_1 — Space which G acts on

Construct a group cycle

From now on we study the quandle structure on $Z(h)\backslash G$ and construct a lift of $\pi : G \to Z(h)\backslash G$.

$(Z(h)g_0, \ldots, Z(h)g_n) \rightsquigarrow (g_0, \ldots, g_n)$ lift to a group cycle

The quandle structure on Conj(h) induces a quandle operation on $Z(h)\backslash G$.

$$\begin{aligned}(g_1^{-1}hg_1) * (g_2^{-1}hg_2) &= (g_2^{-1}hg_2)^{-1}(g_1^{-1}hg_1)(g_2^{-1}hg_2) \\ &= (g_1g_2^{-1}hg_2)^{-1}h(g_1g_2^{-1}hg_2) \\ &\leftrightarrow Z(h)g_1(g_2^{-1}hg_2)\end{aligned}$$

Let $\pi : G \to Z(h)\backslash G$ be the projection map. The quandle operation on $Z(h)\backslash G$ lifts to the quandle operation on G by:

$$g_1 \bullet g_2 := h^{-1}g_1(g_2^{-1}hg_2) \quad (g_1, g_2 \in G)$$

This $\bullet$ satisfies the quandle axioms.

The projection map $\pi : G \to Z(h)\backslash G$ is a quandle homomorphism. Let $s : Z(h)\backslash G \to G$ be a section of π ($\pi \circ s = \mathrm{Id}$). Since $s(x * y)$ and $s(x) \bullet s(y)$ are in the same coset in $Z(h)\backslash G$, there exists an element $c(x, y) \in Z(h)$ satisfying

$$s(x * y) = c(x, y)s(x) \bullet s(y)$$

Fact *If $Z(h)$ is an abelian group, $c : X \times X \to Z(h)$ is a quandle 2-cocycle. If the cycle c is cohomologous to zero, we can change the section s so that $s(x * y) = s(x) \bullet s(y)$.*

Example (dihedral group D_{2p}, p: odd)

$G = D_{2p} = \langle h, x | h^2 = x^p = hxhx = 1 \rangle$: dihedral group

$Z(h) = \{1, h\}$

$\mathrm{Conj}(h) = \{x^{-i}hx^i | i = 0, 1, \ldots, p-1\} = \{hx^{2i} | i = 0, \ldots, p-1\}$

$$\begin{array}{ccccc} \mathrm{Conj}(h) & \leftrightarrow & Z(h)\backslash G & \xrightarrow{s} & G \\ \Cup & & \Cup & & \Cup \\ x^{-i}hx^i & & Z(h)x^i & \mapsto & hx^i \end{array}$$

$$\begin{aligned} s(Z(h)x^i * Z(h)x^j) &= s(Z(h)x^{2j-i}) = hx^{2j-i} \\ &= h^{-1}(hx^i)(x^{-j}hx^j) = s(Z(h)x^i) \bullet s(Z(h)x^j) \end{aligned}$$

Therefore $c(x, y) = 0$ for any $x, y \in R_p$.

Construction of quandle cocycles

G : a group. Fix $h \in G$ with $h^l = 1$.

We assume that $Z(h)$ is abelian and the 2-cocycle corresponding to the quandle extension $G \to Z(h)\backslash G$ is cohomologous to zero.

Let $f : G^{n+1} \to A$ be a group n-cocycle in homogeneous notation. Define $\tilde{f} : G^{n+1} \to A$ by

$$\tilde{f}(x_0. \dots, x_n) = \sum_{i=0}^{l-1} f(h^i s(x_0), \dots, h^i s(x_n))$$

for $x_0, \dots, x_n \in \mathrm{Conj}(h)$.

Construction of quandle cocycles

G : a group. Fix $h \in G$ with $h^l = 1$.

We assume that $Z(h)$ is abelian and the 2-cocycle corresponding to the quandle extension $G \to Z(h)\backslash G$ is cohomologous to zero. (Too strong assumption?)

Let $f : G^{n+1} \to A$ be a group n-cocycle in homogeneous notation. Define $\tilde{f} : G^{n+1} \to A$ by

$$\tilde{f}(x_0. \dots, x_n) = \sum_{i=0}^{l-1} f(h^i s(x_0), \dots, h^i s(x_n))$$

for $x_0, \dots, x_n \in \mathrm{Conj}(h)$.

Prop *This satisfies the 3 conditions of quandle n-cocycle of $H_n^{\Delta}(\mathrm{Conj}(h))$.*

We only have to check the second property.

$$
\begin{aligned}
&\tilde{f}(x_0 * y, \ldots, x_n * y) \\
&\quad = \sum_{i=0}^{l-1} f(h^i s(x_0 * y), \ldots, h^i s(x_n * y)) \\
&\quad = \sum_{i=0}^{l-1} f(h^i s(x_0) \bullet s(y), \ldots, h^i s(x_n) \bullet s(y)) \\
&\quad = \sum_{i=0}^{l-1} f(h^{i-1} s(x_0)(s(y)^{-1} h s(y)), \ldots, h^{i-1} s(x_n)(s(y)^{-1} h s(y))) \\
&\quad = \sum_{i=0}^{l-1} f(h^{i-1} s(x_0), \ldots, h^{i-1} s(x_n)) \quad \text{(right invariance)} \\
&\quad = \tilde{f}(x_0, \ldots, x_n)
\end{aligned}
$$

Dual objects

Considering the dual of this construction, we obtain a group cocycle of cyclic branched covering along K

Presentation of cyclic branched covering space

Let m_i $(i = 1, 2, \ldots, n)$ be the Wirtinger generators of a knot diagram. We denote the relations in the following form:

$$m_i = m_{\kappa i}^{-\varepsilon i} m_{i-1} m_{\kappa i}^{\varepsilon i}$$

where $\kappa : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and $\varepsilon : \{1, \ldots, n\} \to \{\pm 1\}$. Let C_l be the manifold corresponding to the kernel of

$$\pi_1(S^3 \setminus K) \to H_1(\pi_1(S^3 \setminus K)) \cong \mathbb{Z} \to \mathbb{Z}/l$$

$\pi_1(C_l)$ has the following presentation.

Generators: $m_{i,s} \quad (i = 1, 2, \ldots, n, \quad s = 0, 1, \ldots, l-1)$

Relations: $m_{i,s} = m_{\kappa(i),s-1}^{-\varepsilon i} m_{i-1,s-1} m_{\kappa(i),s}^{\varepsilon i},$

$$m_{0,1} = m_{0,2} = \ldots m_{0,l-1} = 1$$

If we add a relation $m_{0,0} = 1$, we obtain a presentation of the cyclic branched covering $\widehat{C}_l$.

For a representation $\rho : \pi_1(S^3 \setminus K) \to G$, we have

$$\rho|_{\pi_1(C_l)}(m_{i,s}) = \rho(m_0)^s \rho(m_i) \rho(m_0)^{-(s+1)}$$

If $\rho(m_i)^l = 1$, it reduces to a representation $\hat{\rho} : \pi_1(\widehat{C}_l) \to G$.

Group cycles represented by the cyclic branched covering

$X = \mathsf{Conj}(h) \quad (\cong Z(h)\backslash G)$.

$C_n(G) = \mathsf{span}_{\mathbb{Z}}\{(g_0, \ldots, g_n) | g_i \in G\}$

$\iota : C_n^{\Delta}(X) \to C_n(G) : (x_0, \ldots, x_n) \mapsto (s(x_0), \ldots, s(x_n))$

We can define a map $\varphi : C_n^Q(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X)$ and

$$C_2^Q(X; \mathbb{Z}[X]) \xrightarrow{\varphi} C_3^{\Delta}(X) \xrightarrow{\iota} C_3(G)$$

$\iota\varphi(C_s(\mathcal{S}))$ is not a group cycle in general.

Let a be the color of m_0. Then

$$(\mathcal{A} * a)(m_i) = \mathcal{A}(m_i) * a, \quad (\mathcal{R} * a)(m_i) = \mathcal{R}(m_i) * a,$$

is also an arc coloring and a region coloring. We denote $\mathcal{S} * a = (\mathcal{A}(m_i) * a, \mathcal{R} * a)$

Thm $\iota\varphi(C_s(\mathcal{S})) + \iota\varphi(C_s(\mathcal{S} * a)) + \iota\varphi(C_s(\mathcal{S} * a^2)) + \cdots + \iota\varphi(C_s(\mathcal{S} * a^{l-1}))$ *is a group cycle represented by the cyclic branched covering along the knot.*

Conclusion

By Eisermann's work, the quandle cocycle invariant associated to a cocycle of $H^2_Q(X;A)$ essentially comes from the monodoromies along the longitude.

On the other hand, the quandle cocycle invariant associated to a cocycle of $H^2_Q(X;\mathsf{Func}(X,A))$ $(= H^2_Q(X;A)_X)$ is closely related to representations of the cyclic branched covering.