PSL(2,C)-representations via triangulations in dimension 2 and 3

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Introduction

$\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$ is

- $\mathrm{Isom}^+(\mathbb{H}^3)$ (orientation preserving isometries),
- the group of conformal transformations of $\mathbb{C}P^1$.

So $\mathrm{PSL}(2,\mathbb{C})$-representations are important in the study of

- 3-dimensional hyperbolic geometry (and topology),
- complex projective structure ($\mathbb{C}P^1$-structure) on surfaces
- Teichmüller spaces ($\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$)

Introduction

We study $\mathrm{PSL}(2, \mathbb{C})$-representations of 2- and 3-manifolds using *ideal triangulations*.

In 3-dim case, ideal triangulations are useful to study

- existence of hyperbolic structures,
- hyperbolic Dehn fillings,
- limits of representations (ideal points).

In 2-dim case, ideal triangulations are used to study Teichmüller spaces (cell decomposition, etc.)

Introduction

We will construct a parametrization of $\mathrm{PSL}(2,\mathbb{C})$-representations of surface groups using ideal triangulation, (which is different from Penner's work.)

Our parametrization is an analogue of the (complex) Fenchel-Nielsen coordinates using ideal triangulations. The construction is quite elementary. It is easy to give explicit matrix generators. It works for closed surfaces.

PSL$(2, \mathbb{C})$

$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$

$\mathbb{H}^3 = \{(x, y, t) | t > 0\}$ (metric : $\frac{dx^2 + dy^2 + dt^2}{t^2}$)

$\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \mathbb{C}P^1$ and $\partial \overline{\mathbb{H}^3} = \mathbb{C}P^1$

$\mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$

$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) / \{\pm I\}$

PSL$(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ by the linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot t = \frac{at + b}{ct + d} \quad (t \in \mathbb{C}P^1)$$

This extends to an isometry on $\mathbb{H}^3$ and $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$.

PSL$(2, \mathbb{C})$

Fact *There exists a unique element which sends any distinct three points of* $\mathbb{C}P^1$ *to other distinct three points.*

For example, the map which sends (x, y, z) to $(0, \infty, 1)$ is given by

$$\frac{1}{\sqrt{-(x-y)(y-z)(z-x)}} \begin{pmatrix} (z-y) & -x(z-y) \\ (z-x) & -y(z-x) \end{pmatrix}.$$

(The square root is well defined up to sign. Usually convenient to use PGL$(2, \mathbb{C})$ instead of PSL$(2, \mathbb{C})$.)

Actually, the matrix which sends (x_1, x_2, x_3) to (x'_1, x'_2, x'_3) is

$$\frac{1}{\sqrt{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)(x'_1 - x'_2)(x'_2 - x'_3)(x'_3 - x'_1)}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = x_1 x'_1 (x'_2 - x'_3) + x_2 x'_2 (x'_3 - x'_1) + x_3 x'_3 (x'_1 - x'_2),$$

$$a_{12} = x_1 x_2 x'_3 (x'_1 - x'_2) + x_2 x_3 x'_1 (x'_2 - x'_3) + x_3 x_1 x'_2 (x'_3 - x'_1),$$

$$a_{21} = x_1 (x'_2 - x'_3) + x_2 (x'_3 - x'_1) + x_3 (x'_1 - x'_2),$$

$$a_{22} = x_1 x'_1 (x_2 - x_3) + x_2 x'_2 (x_3 - x_1) + x_3 x'_3 (x_1 - x_2).$$

Set of $\mathrm{PSL}(2,\mathbb{C})$-representations

M : manifold (we are interested in $\dim M = 2$ or 3.)

$R(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$

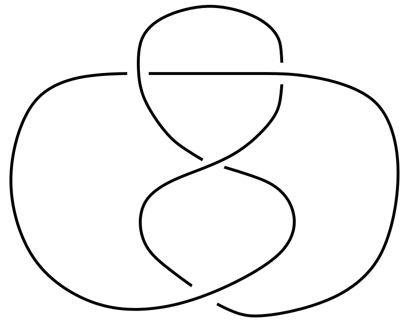
$\mathrm{PSL}(2,\mathbb{C})$ acts on $R(M)$ by $\rho \mapsto g^{-1}\rho g$ ($\rho \in R(M), g \in \mathrm{PSL}(2,\mathbb{C})$).

$X(M) = R(M)/\text{conj.}$ (quotient in algebraic sense)

A representation $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ is called *reducible* if $\rho(\pi_1(M))$ fixes a point of $\mathbb{C}P^1$.

Restricted to irreducible representations, $X(M)$ is nothing but the usual quotient of $R(M)$ by the action of $\mathrm{PSL}(2,\mathbb{C})$. $X(M)$ is called the *character variety*.

Usually the structure of $X(M)$ is complicated. A parametrization using an *ideal triangulation* is usually simpler.



E.g. let K be the figure eight knot, then $\pi_1(S^3 - K)$ has a complicated presentation:

$$\pi_1(S^3 - K) \cong \langle x_1, x_2 | x_2^{-1} x_1 x_2 x_1^{-1} x_2 x_1 x_2^{-1} x_1^{-1} x_2 x_1^{-1} = 1 \rangle.$$

But $X(M)$ is parametrized by the variety

$$\{(x, y) \in (\mathbb{C}^*)^2 | xy(1-x)(1-y) = 1\}.$$

using an ideal triangulation.

Ideal tetrahedra

An *ideal tetrahedron* is the convex hull of distinct 4 points of $\mathbb{C}P^1$ in $\mathbb{H}^3$. Let z_0, z_1, z_2, z_3 be the vertices of an ideal tetrahedron. The ideal tetrahedron is parametrized by the cross ratio

$$[z_0 : z_1 : z_2 : z_3] = \frac{z_3 - z_0}{z_3 - z_1} \cdot \frac{z_2 - z_1}{z_2 - z_0} \in (\mathbb{C} - \{0, 1\}).$$

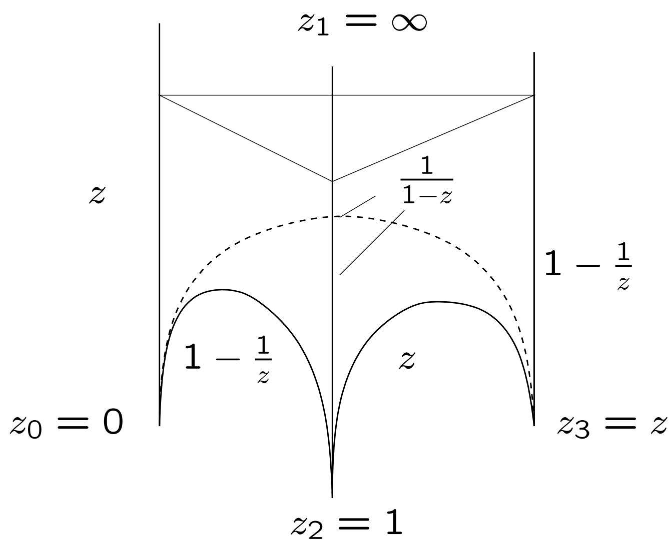
The cross ratio is invariant under the action of $\mathrm{PSL}(2, \mathbb{C})$. $([gz_0 : gz_1 : gz_2 : gz_3] = [z_0 : z_1 : z_2 : z_3]$ for any $g \in \mathrm{PSL}(2, \mathbb{C}))$

Meaning of Cross ratio

Denote the edge spanned by z_i and z_j by $[z_i z_j]$. Define the complex parameter of the edge $[z_i z_j]$ by the cross ratio

$$[z_i : z_j : z_k : z_l]$$

where (i, j, k, l) is an even permutation of $(0, 1, 2, 3)$.



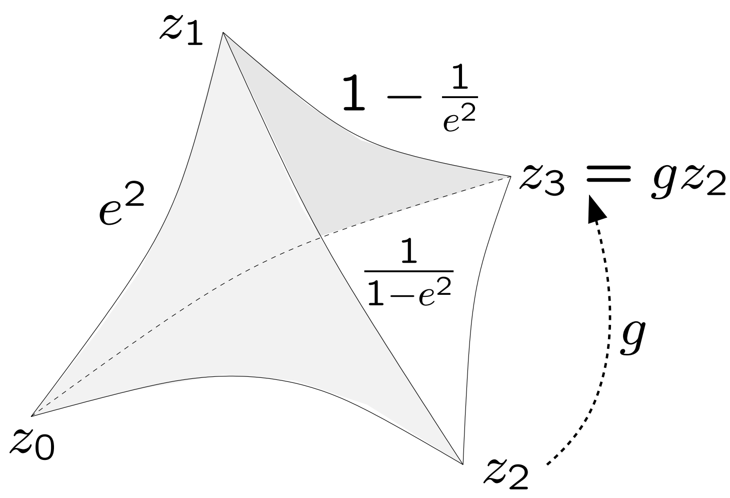
$$[0 : \infty : 1 : z] = [1 : z : 0 : \infty] = z$$

$$[1 : \infty : z : 0] = [z : 0 : 1 : \infty] = \frac{1}{1 - z}$$

$$[z : \infty : 0 : 1] = [0 : 1 : z : \infty] = 1 - \frac{1}{z}$$

The complex parameter of $[z_i z_j]$ is well-defined and the opposite edge has same parameter.

The cross ratio is interpreted as the square of the eigenvalue of the matrix which sends z_2 to z_3 fixing $[z_0 z_1]$. The matrix



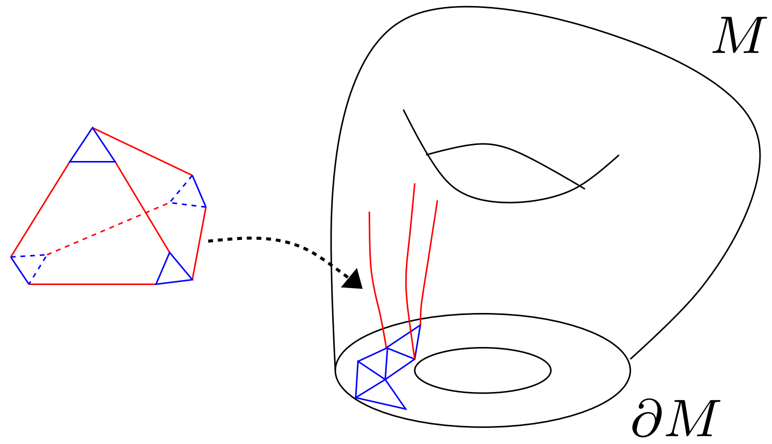
$$g = \begin{pmatrix} z_1 & z_0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} z_1 & z_0 \\ 1 & 1 \end{pmatrix}^{-1} \sim \begin{pmatrix} ez_1 - e^{-1}z_0 & -(e-e^{-1})z_0 z_1 \\ e - e^{-1} & -ez_0 + e^{-1}z_1 \end{pmatrix}.$$

sends z_2 to z_3, then $e^2 = [z_0 : z_1 : z_2 : z_3]$.

(To fix a parametrization of e, we assume that z_0 is the repelling fixed point and z_1 is the attractive fixed point when $|e| > 1$.)

Ideal triangulation (3-dim)

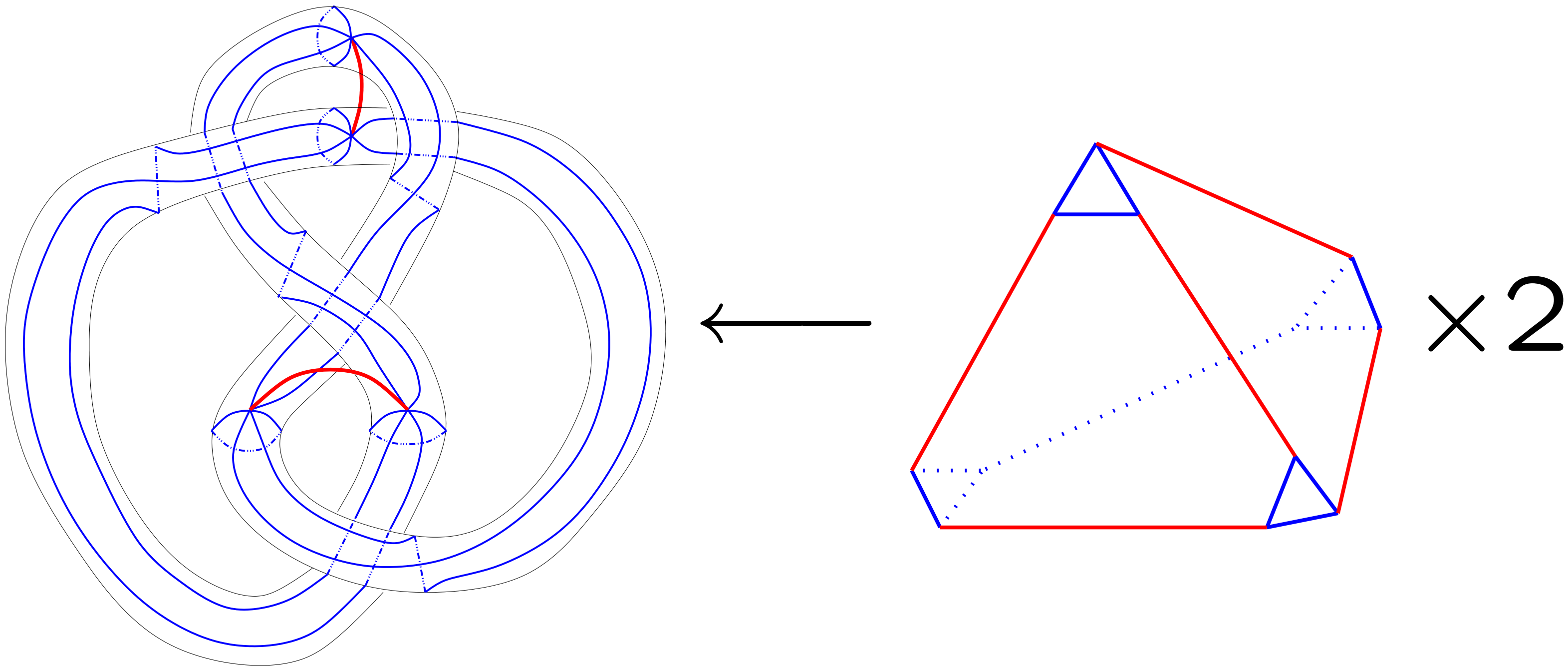
M : a compact 3-manifold ($\partial M \neq \emptyset$)



Def *A (topological) ideal triangulation of M is a cell complex T obtained from gluing tetrahedra along their faces so that $T - N(T^{(0)})$ is homeomorphic to M.*

We call a tetrahedron with 4 vertices deleted *ideal tetrahedron*.

Figure eight knot complement



$K \subset S^3$: the figure eight knot.

$S^3 - K$ is decomposed into two ideal tetrahedra.

Developing map

$\widetilde{M}$: the universal cover of M

We will construct an equivariant map $\widetilde{M} \to \mathbb{H}^3$.

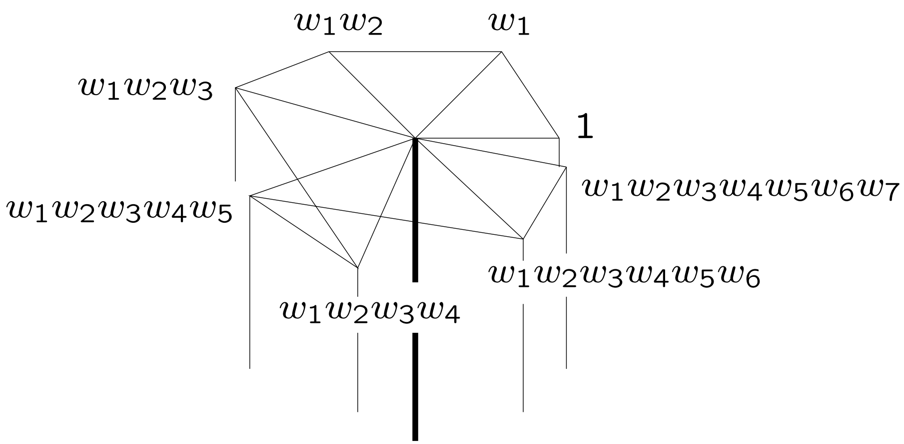
($\Rightarrow$ $\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$)

$T = \Delta(z_1) \cup \cdots \cup \Delta(z_n)$: ideal triangulation of M with complex parameters z_i.

Develop T in $\mathbb{H}^3$ according with their complex parameters. We obtain a map from the universal cover of $T - T^{(1)}$ to $\mathbb{H}^3$.

To obtain a map from the universal cover $\widetilde{M}$, which is homeomorphic to the universal cover of $T - N(T^{(0)})$, we have to impose the *gluing equation* around each 1-simplex of T.

Around a 1-simplex of T, there are some ideal tetrahedra parametrized by w_i. To extends the map, we have to make sure that the product of w_i's is equal to 1.



Since w_i is one of z_k, $\frac{1}{1-z_k}$ or $1-\frac{1}{z_k}$,

$$\prod_{i=1}^{n} z_i^{p_{ji}} \left(\frac{1}{1-z_i}\right)^{p'_{ji}} \left(1-\frac{1}{z_i}\right)^{p''_{ji}} = \pm \prod_{i=1} z_i^{r'_{ji}} (1-z_i)^{r''_{ji}} = 1.$$

(for each 1-simplex indexed by j.)

We call these equations *gluing equations*.

Let

$$\mathcal{D}(M,T) = \{(z_1,\ldots,z_n) \in (\mathbb{C} - \{0,1\})^n \\ \mid \pm \prod_{i=1} z_i^{r'_{ji}} (1-z_i)^{r''_{ji}} = 1 \quad (\forall j)\}.$$

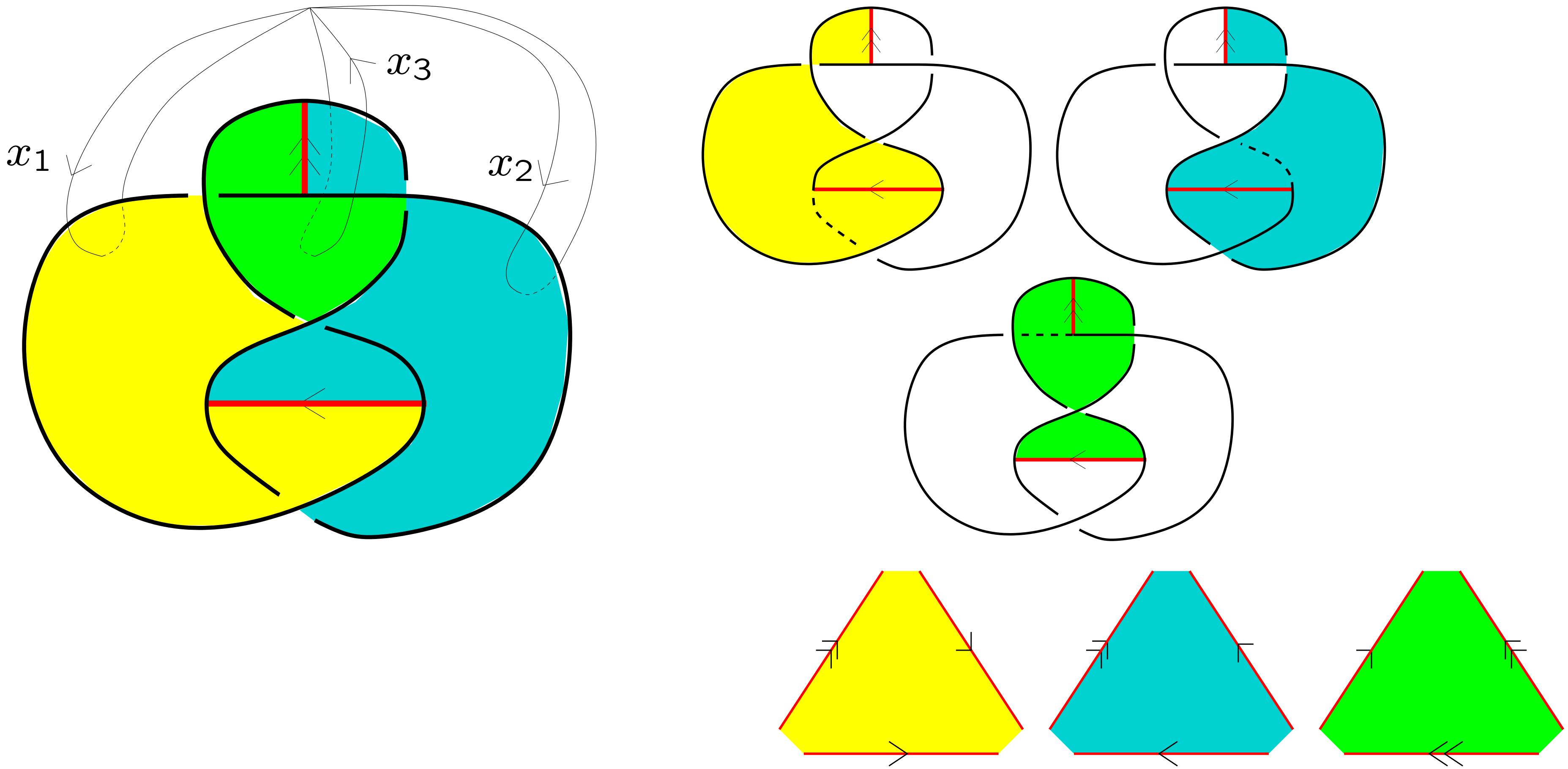
We have a *developing map* $D : \widetilde{M} \to \mathbb{H}^3$ for an element of $\mathcal{D}(M)$. For $\gamma \in \pi_1(M)$, there exists a unique element $\rho(\gamma) \in \mathrm{PSL}(2,\mathbb{C})$ such that $D(\gamma p) = \rho(\gamma) D(p)$ for any $p \in \widetilde{M}$. Then ρ is a homomorphism $\pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, which is called the *holonomy representation* of D.

If we change the position of an ideal tetrahedron in $\mathbb{H}^3$, we obtain a conjugate representation. So we obtain an algebraic map

$$\mathcal{D}(M) \to X(M)$$

Example: Figure eight knot complement

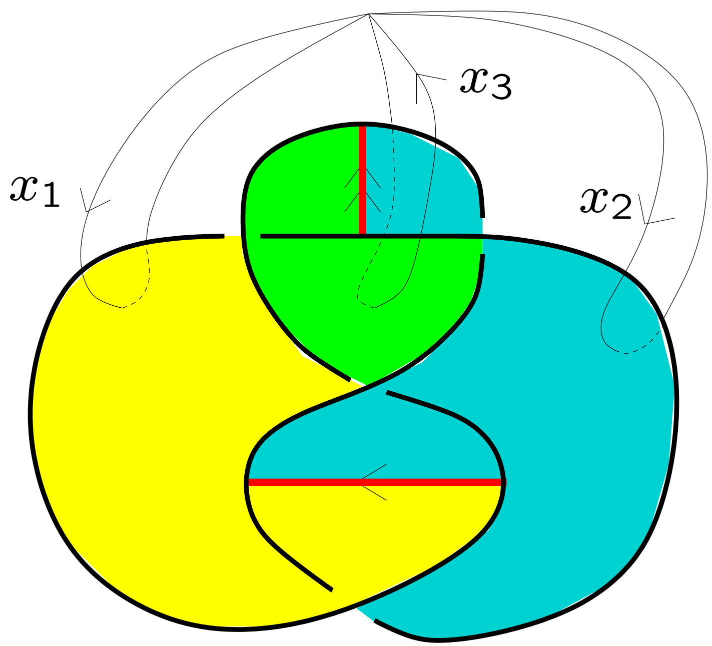
Span three truncated triangles in $S^3 - N(K)$.

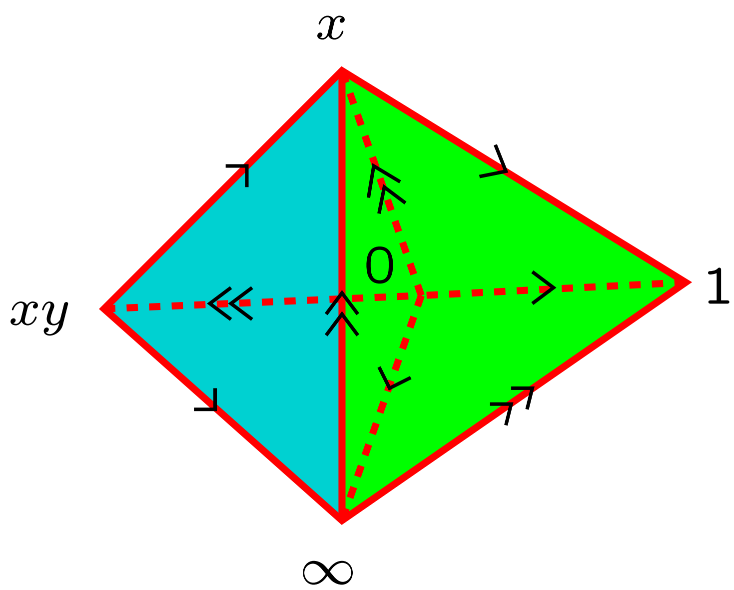


Cut $S^3 - N(K)$ along these triangles. We obtain a polyhedron.

Example: Figure eight knot complement

The polyhedron decomposed into to ideal tetrahedra. Assign complex parameters x and y.





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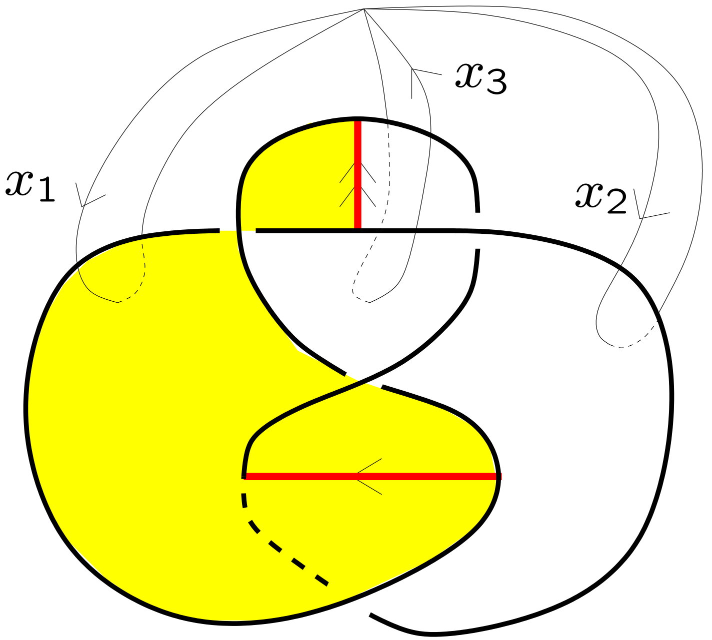
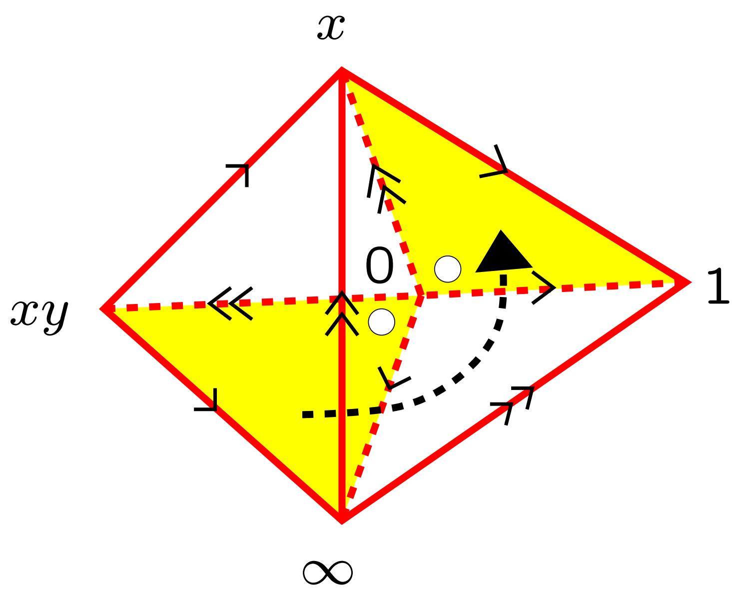


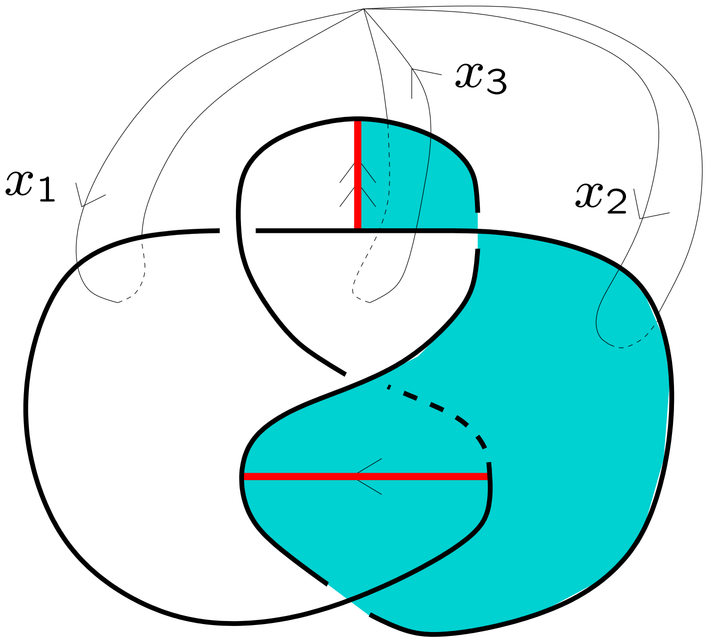


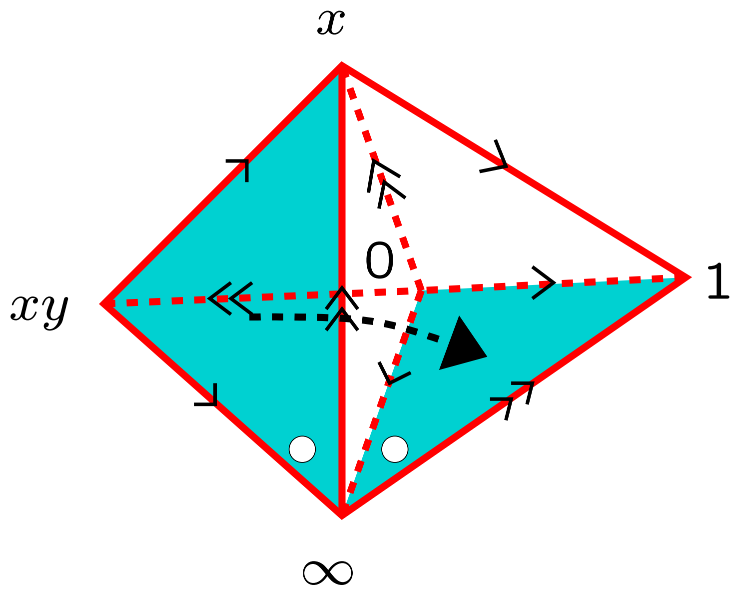
$\rho(x_1)$

$$\rho(x_1) = \frac{\pm 1}{\sqrt{y(1-x)}} \begin{pmatrix} 1 & 0 \\ 1 & y(1-x) \end{pmatrix},$$

Example: Figure eight knot complement

The polyhedron decomposed into to ideal tetrahedra. Assign complex parameters x and y.



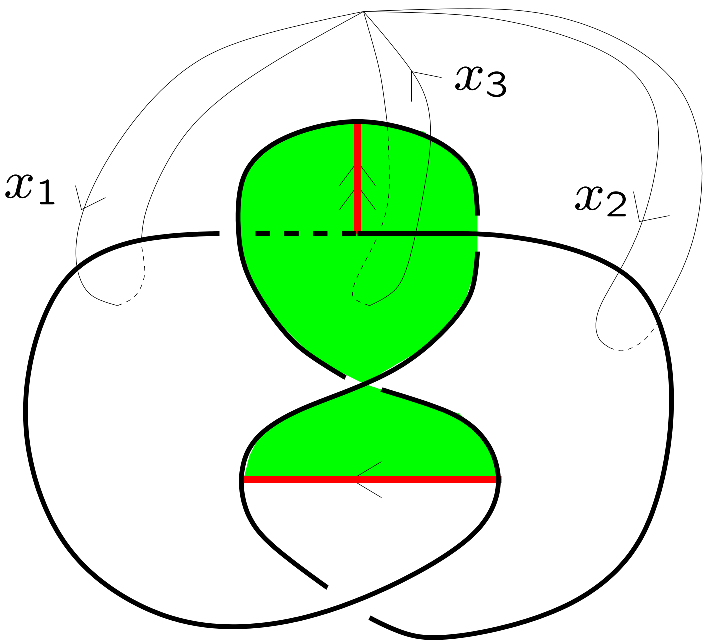


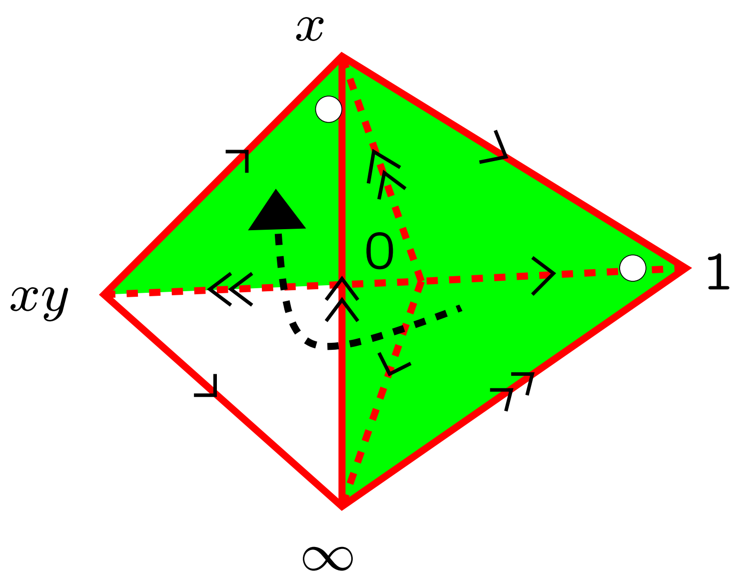
$\rho(x_2)$

$$\rho(x_2) = \frac{\pm 1}{\sqrt{x(1-y)}} \begin{pmatrix} 1 & -xy \\ 0 & x(1-y) \end{pmatrix}.$$

Example: Figure eight knot complement

The polyhedron decomposed into to ideal tetrahedra. Assign complex parameters x and y.

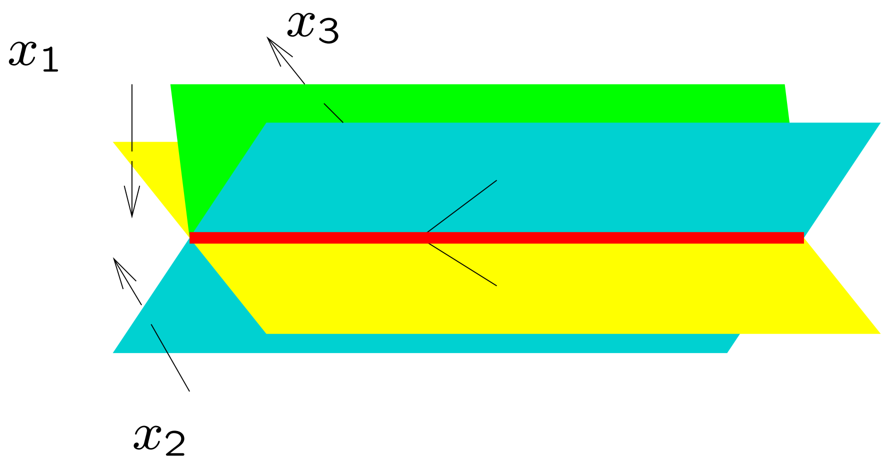


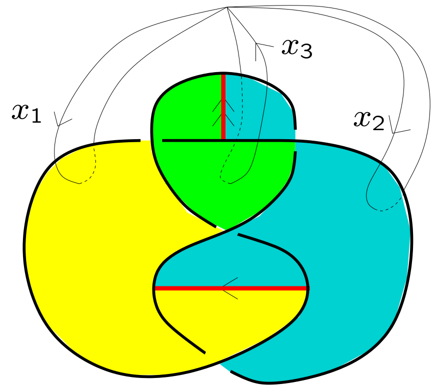


$\rho(x_3)$

$$\rho(x_3) = \frac{\pm 1}{\sqrt{xy(1-x)(1-y)}} \begin{pmatrix} 0 & (x-1)xy \\ 1-y & xy-1 \end{pmatrix}.$$

Example: Figure eight knot complement





$x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_1^{-1} \cdot x_3^{-1} = 1$: the relation at the single arrow

$y \cdot (1 - \frac{1}{x}) \cdot (xy) \cdot (1 - \frac{1}{y}) \cdot x = 1$: the gluing equation

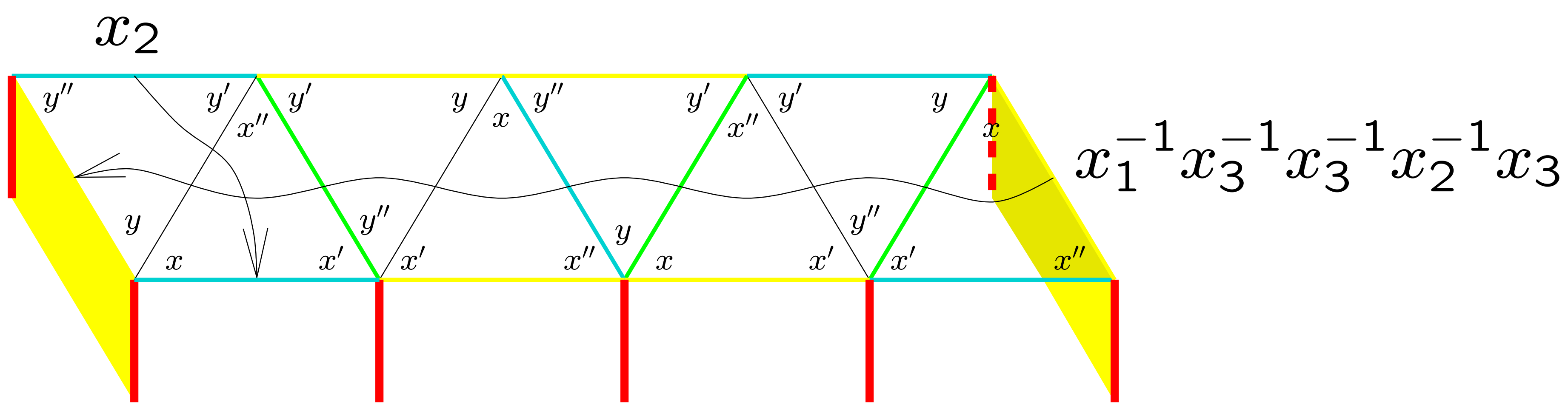
(Simplified to $xy(1-x)(1-y) = 1$.)

Similarly the gluing equation at the double arrow is

$$xy(1-x)(1-y) = 1.$$

Example: Figure eight knot complement

$\partial(S^3 - N(K))$ inherits a triangulation from the truncated vertices of T.



($x' = \frac{1}{1-x}$, $x'' = 1 - \frac{1}{x}$, $y' = \frac{1}{1-y}$ and $y'' = 1 - \frac{1}{y}$.)

Let

$$M = \frac{1}{x}\frac{1}{1-y}, \qquad LM^{-2} = \frac{1}{x}\frac{y-1}{y}\frac{x}{x-1}y\frac{1}{x}\frac{y-1}{y}\frac{x}{x-1}y.$$

$$(M = \frac{1}{x(1-y)}, \qquad L = \frac{1}{x^2(1-x)^2}.)$$

Example: Figure eight knot complement

Then the restriction of ρ to ∂M is given by

$$\rho(x_2) = \begin{pmatrix} \sqrt{M} & * \\ 0 & 1/\sqrt{M} \end{pmatrix}, \quad \rho(x_1^{-1}x_3^{-2}x_2^{-1}x_3) = \begin{pmatrix} \sqrt{LM^{-2}} & * \\ 0 & 1/\sqrt{LM^{-2}} \end{pmatrix}$$

Here $m = x_2$ and $lm^{-2} = x_1^{-1}x_3^{-2}x_2^{-1}x_3$ generate a boundary subgroup. Since the boundary subgroup fixes a point of $\mathbb{C}P^1$, the restriction is reducible. Therefore the restriction to ∂M is quite tractable:

$$\rho(m^p l^q) = \begin{pmatrix} \sqrt{M^p L^q} & * \\ 0 & 1/\sqrt{M^{-p}L^{-q}} \end{pmatrix}.$$

(Useful for hyperbolic Dehn fillings, etc.)

In summary,

Thm *Let M be a compact 3-manifold and T be an ideal triangulation of M. There exists an algebraic map $\mathcal{D}(M,T) \to X(M)$. The restriction of the representation to any boundary subgroup is reducible.*

Remark • *When ∂M consists of tori, usually we obtain all "generic" representations since $\pi_1(\partial M)$ is abelian.*

• *If $\mathrm{Im}(z_i) > 0$, there exists an incomplete hyperbolic metric on $\mathrm{Int}(M)$. If furthermore $L = 1$ and $M = 1$, there exists a complete hyperbolic metric.*

Surface group representations

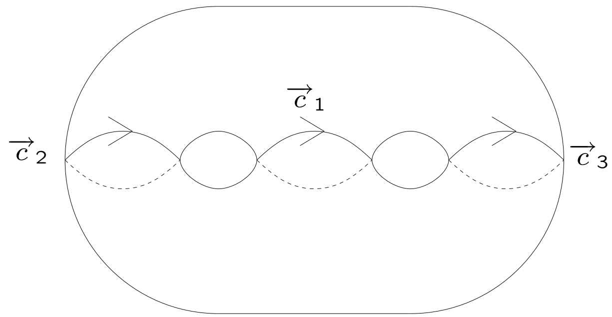
We apply these techniques to surface group representations.

Strategy: Decompose the surface into three holed spheres (pairs of pants). Then parametrize the representations of pairs of pants and glue them along common curves using developing maps.

Pants decomposition

Let S be a surface of genus g $(g > 1)$. A *pants decomposition* C is a disjoint union of s.c.c. such that $S - C$ is a collection of three holed spheres (pairs of pants).

The number of simple closed curves of C is equal to $3g - 3$ and $S - C$ consists of $2g - 2$ pairs of pants .



We assume that each simple closed curve is oriented (denote as $\overrightarrow{c}$ to emphasize). $(C = \overrightarrow{c}_1 \cup \cdots \cup \overrightarrow{c}_{3g-3})$

Parametrizations of pairs of pants

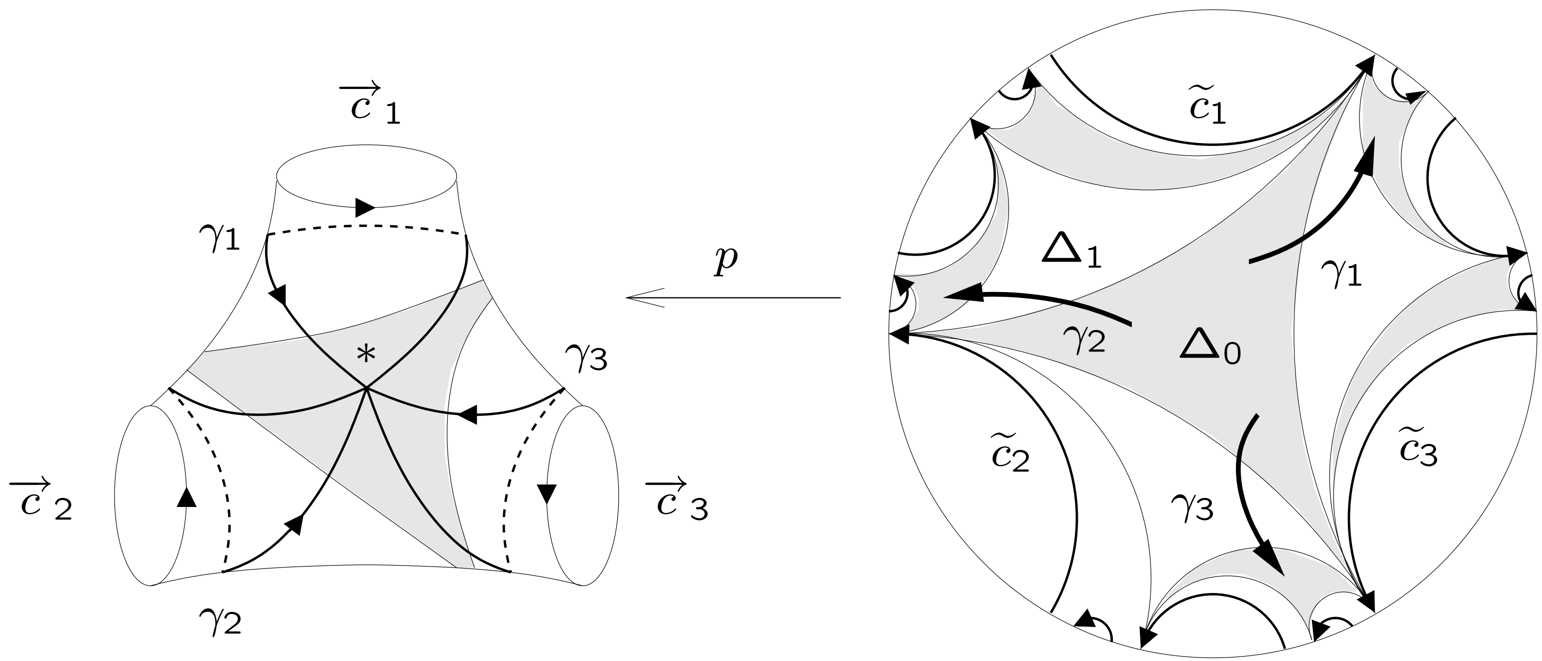
We parametrize the irreducible representations of a pair of pants P.

Let $\partial P = \overrightarrow{c}_1 \cup \overrightarrow{c}_2 \cup \overrightarrow{c}_3$ and ρ be an irreducible representation of $\pi_1(P)$. Assume that $\rho(\overrightarrow{c}_i)$'s are hyperbolic (have two fixed points in $\mathbb{C}P^1$).

We construct a developing map $\widetilde{P} \to \mathbb{H}^3$ and its holonomy representation.

Parametrizations of pairs of pants

Fix a hyperbolic metric on P so that $\overrightarrow{c}_i$ are geodesics and ideally triangulation P by spinning two triangles along $\overrightarrow{c}_i$.



Parametrizations of pairs of pants

Let x_i and y_i be the fixed points of $\rho(\gamma_i)$. By irreducibility $x_1, x_2, x_3, y_1, y_2, y_3$ are distinct points of $\mathbb{C}P^1$.

Let e_i be the one of eigenvalues of $\rho(\gamma_i)$. Assume that x_i is the attractive point of $\rho(\gamma_i)$ if $|e_i| > 1$.

If we assume that $x_1 = 0$, $x_2 = \infty$ and $x_3 = 1$, then

$$\rho(\gamma_1) = \begin{pmatrix} e_1^{-1} & 0 \\ \frac{e_1^{-1} - e_1}{y_1} & e_1 \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} e_2 & (e_2^{-1} - e_2)y_2 \\ 0 & e_2^{-1} \end{pmatrix},$$

$$\rho(\gamma_3) = \frac{1}{y_3 - 1} \begin{pmatrix} e_3^{-1}y_3 - e_3 & (e_3 - e_3^{-1})y_3 \\ e_3^{-1} - e_3 & e_3 y_3 - e_3^{-1} \end{pmatrix}$$

By the relation $\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3) = I$, we obtain

$$y_1 = \frac{e_1 - e_1^{-1}}{e_1 - e_2 e_3^{-1}}, \quad y_2 = \frac{e_2^{-1} - e_1^{-1} e_3}{e_2^{-1} - e_2}, \quad y_3 = \frac{e_1 - e_2 e_3}{e_1 - e_2 e_3^{-1}}.$$

So the representation is uniquely determined by (x_1, x_2, x_3) and (e_1, e_2, e_3). Conversely we can construct such representation for given x_i and e_i. Up to conjugation, this is uniquely determined by $(e_1^{\pm 1}, e_2^{\pm 1}, e_3^{\pm 1})$.

This gives a lift to a SL$(2, \mathbb{C})$-representation. Other lifts are obtained by the action of $H^1(P; \mathbb{Z}/2\mathbb{Z})$. For example, $(e_1, e_2, e_3) \mapsto (-e_1, -e_2, e_3)$ gives another lift to SL$(2, \mathbb{C})$.

For general (x_1, x_2, x_3), $\rho(\gamma_i)$ are given by

$$\rho(\gamma_i) = \frac{1}{e_i e_{i+2}(x_{i+1} - x_i)(x_{i+2} - x_i)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$a_{11} = e_i^2 e_{i+2} x_i (x_i - x_{i+1}) + e_{i+2} x_{i+1} (x_{i+2} - x_i) + e_i e_{i+1} x_i (x_{i+1} - x_{i+2}),$$

$$a_{12} = x_i (e_i^2 e_{i+2} x_{i+2} (x_{i+1} - x_i) + e_{i+2} x_{i+1} (x_i - x_{i+2}) + e_i e_{i+1} x_i (x_{i+2} - x_{i+1})),$$

$$a_{21} = e_i^2 e_{i+2} (x_i - x_{i+1}) + e_{i+2} (x_{i+2} - x_i) + e_i e_{i+1} (x_{i+1} - x_{i+2}),$$

$$a_{22} = e_i^2 e_{i+2} x_{i+2} (x_{i+1} - x_i) + e_{i+2} x_i (x_i - x_{i+2}) + e_i e_{i+1} x_i (x_{i+2} - x_{i+1}),$$

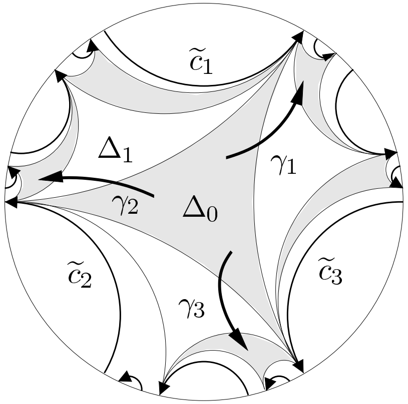
and

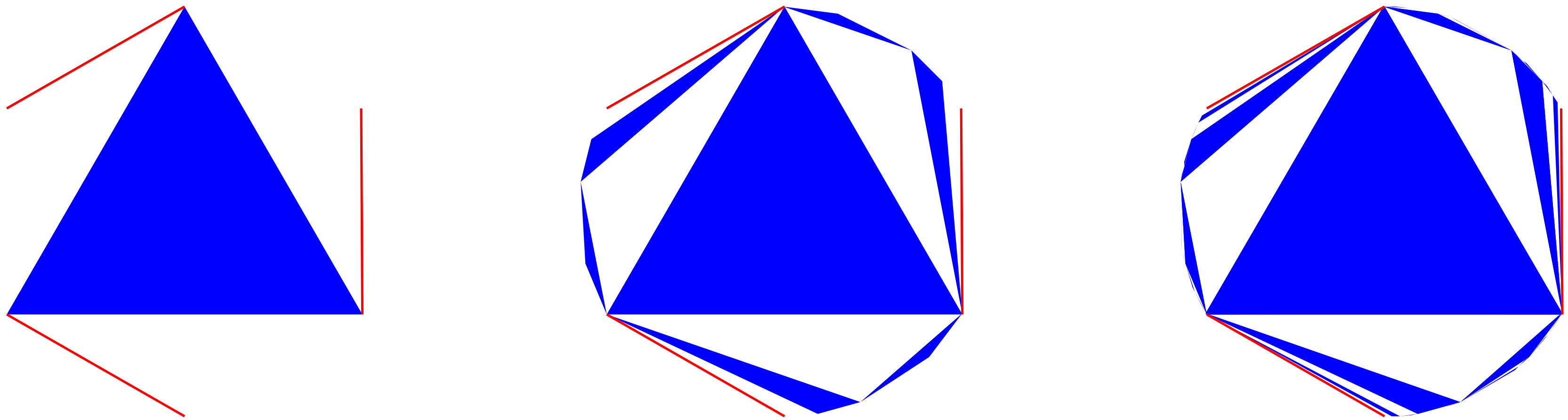
$$y_i = \frac{e_i^2 e_{i+2} x_{i+2} (x_i - x_{i+1}) + e_{i+2} x_{i+1} (x_{i+2} - x_i) + e_i e_{i+1} x_i (x_{i+1} - x_{i+2})}{e_i^2 e_{i+2} (x_i - x_{i+1}) + e_{i+2} (x_{i+2} - x_i) + e_i e_{i+1} (x_{i+1} - x_{i+2})}$$

for $i = 1, 2, 3 \pmod 3$.

Developing map of a pair of pants

Develop the ideal triangle (x_1, x_2, x_3) by using $\{\rho(\gamma_i)\}_i$.

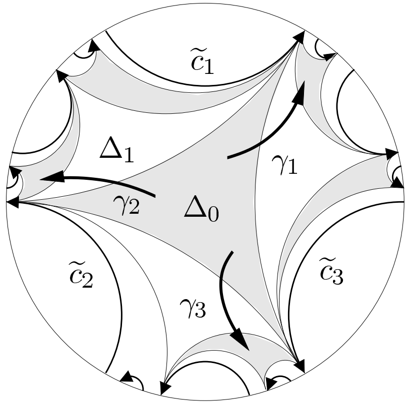


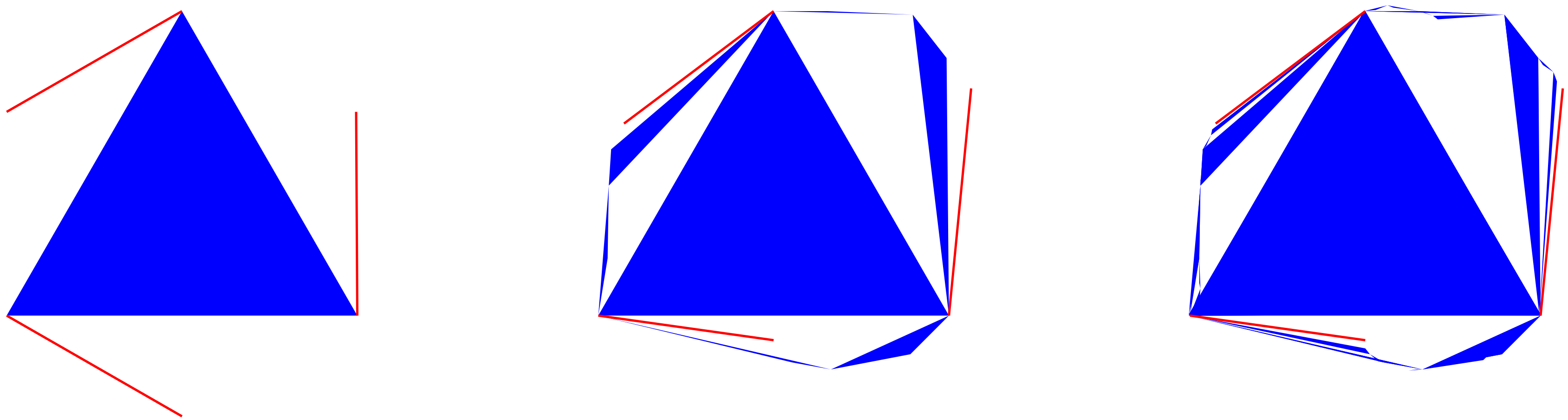


$$(e_1, e_2, e_3) = (-0.5, -0.5, -0.5)$$

Developing map of a pair of pants

Develop the ideal triangle (x_1, x_2, x_3) by using $\{\rho(\gamma_i)\}_i$.

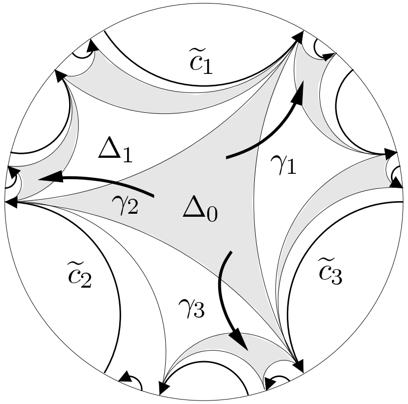


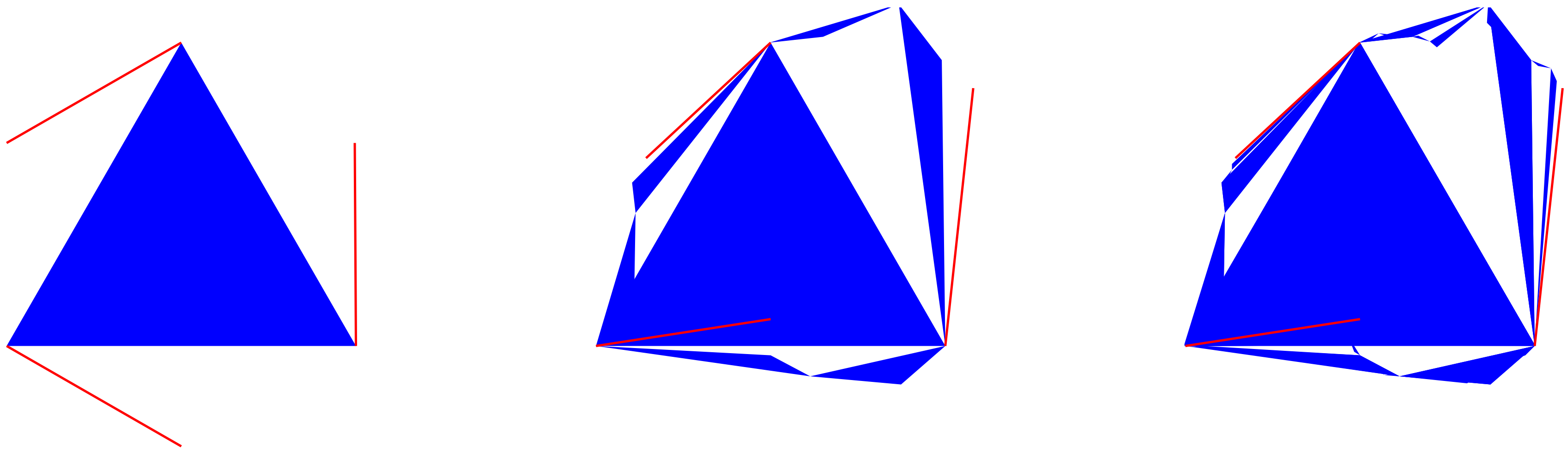


$$(e_1, e_2, e_3) = (-0.5 + 0.2i, -0.5, -0.5)$$

Developing map of a pair of pants

Develop the ideal triangle (x_1, x_2, x_3) by using $\{\rho(\gamma_i)\}_i$.

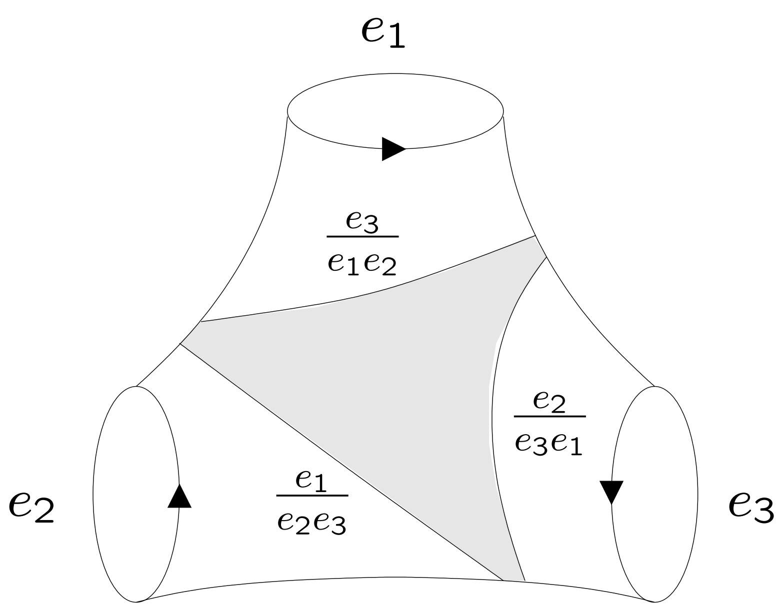


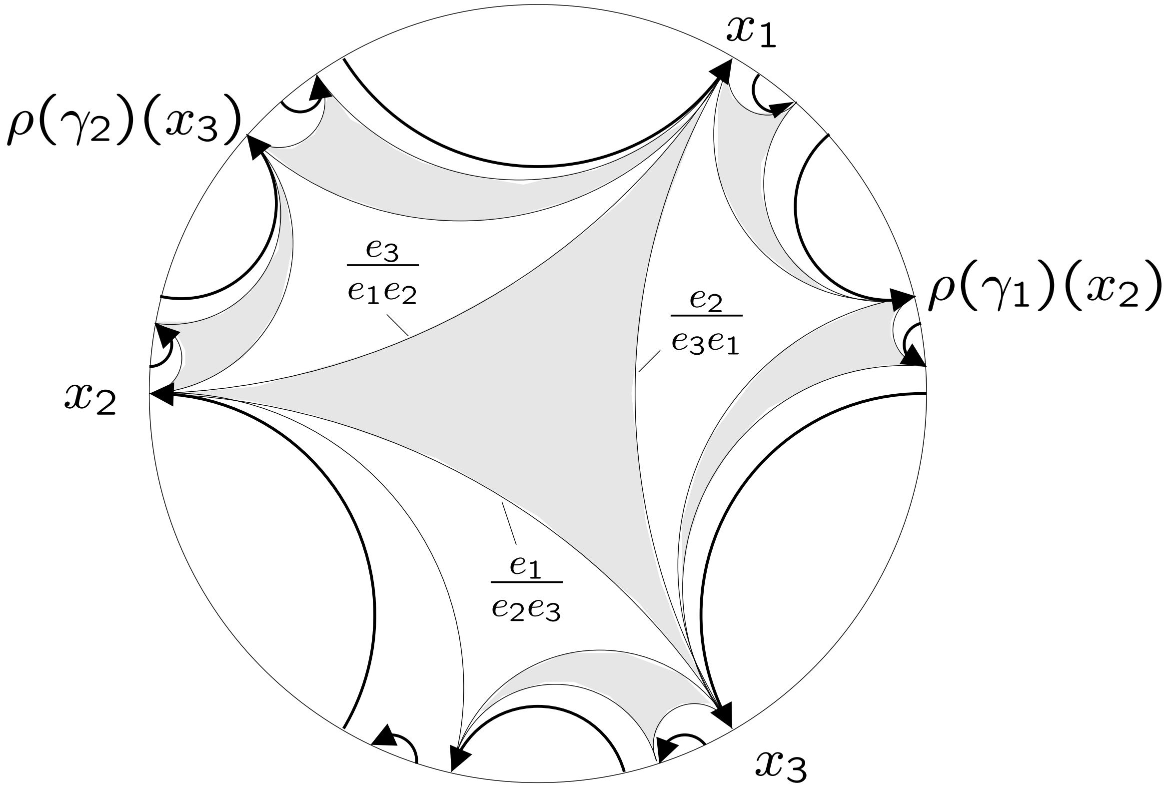


$$(e_1, e_2, e_3) = (-0.5 + 0.4i, -0.5, -0.5)$$

Developing map of a pair of pants

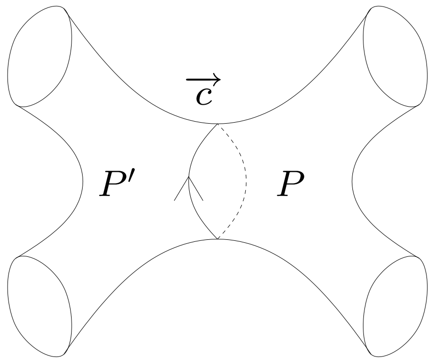
Conversely the eigenvalues are computed from the cross ratios associated with the developing map.





Gluing developing maps of pairs of pants

We give how to glue two representations of pairs of pants along a common curve.

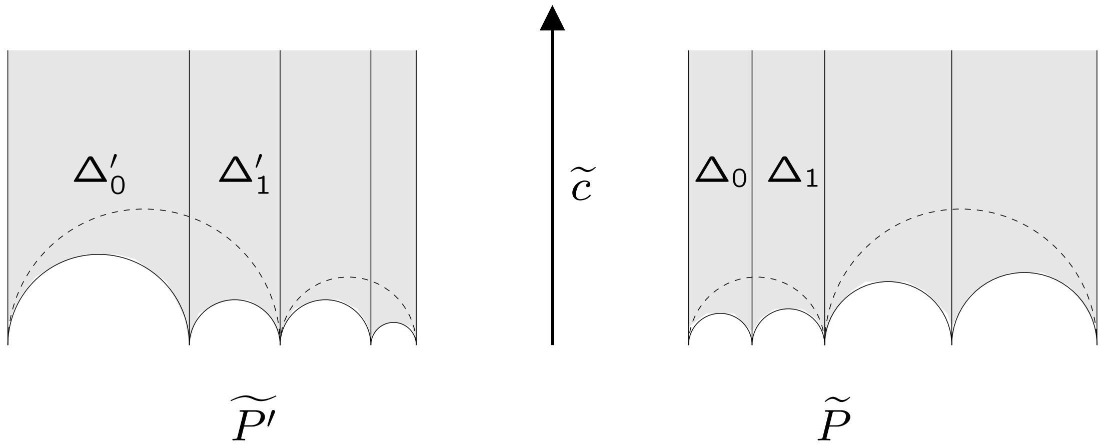


$P \cup_{\overrightarrow{c}} P'$: Pairs of pants with a common curve $\overrightarrow{c}$

$\tilde{c}$: a lift of $\overrightarrow{c}$

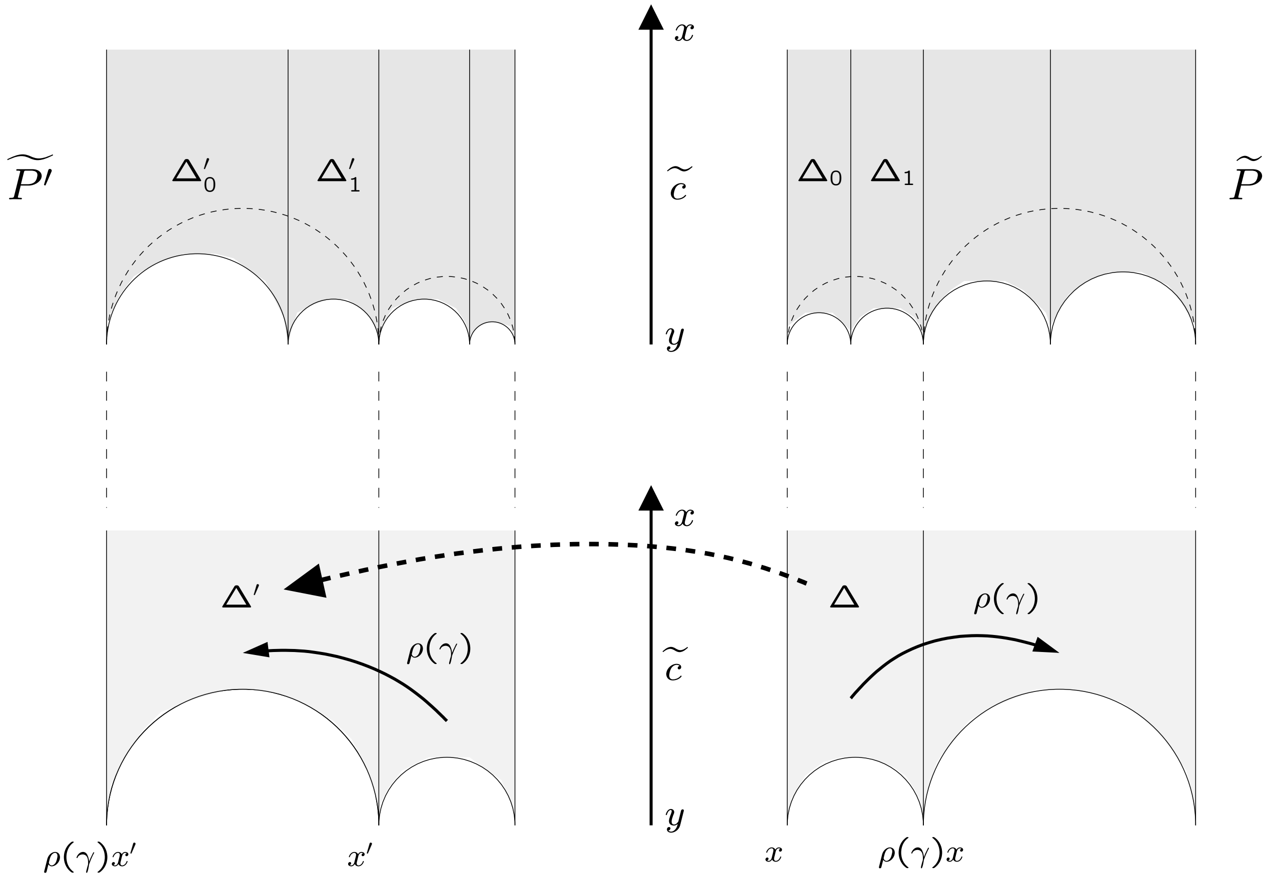
Δ_0 and Δ_1 : an adjacent pair of ideal triangles in $\widetilde{P}$ sharing ∞ as a common ideal vertex.

Take Δ'_0 and Δ'_1 in $\widetilde{P'}$ similarly.



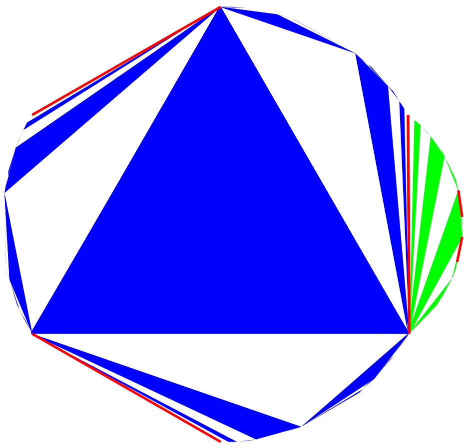
We can not glue Δ_i and Δ_i' in equivariant way.

Define $\Delta \subset \widetilde{P}$ and $\Delta' \subset \widetilde{P'}$ as follows:

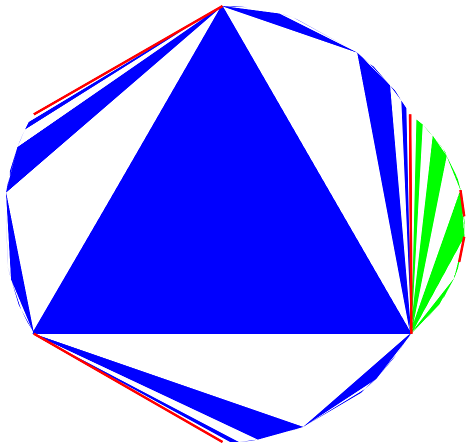


Glue Δ and Δ' by

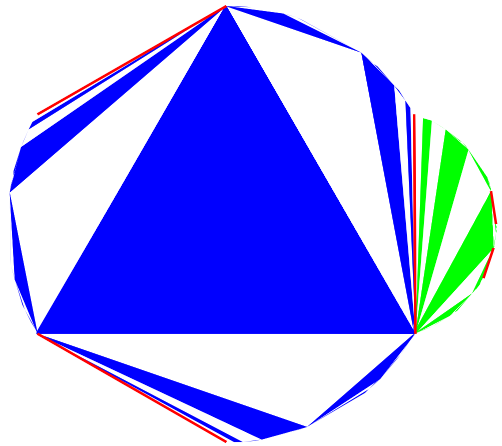
$$\begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}^{-1}.$$



$t = i$



$t = i - 0.2$



$t = i - 0.4$

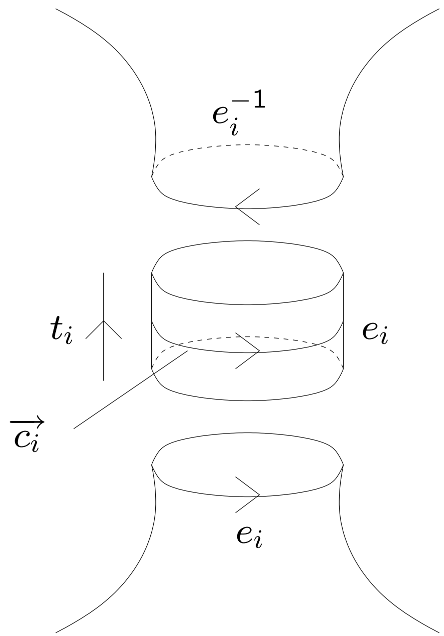
Coordinates of surface representations

S : a surface of genus g

$C = \overrightarrow{c}_1 \cup \cdots \cup \overrightarrow{c}_{3g-3}$: a pants decomposition.

For each curve $\overrightarrow{c}_i$, assign two complex parameters, the eigenvalue e_i and the twist parameter t_i.

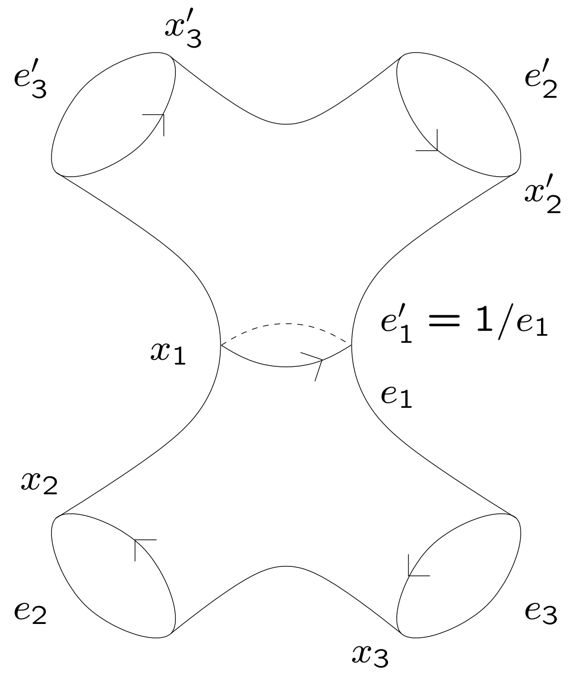
Each boundary curve of a pair of pants inherits the eigenvalue parameter e_i or e_i^{-1}.



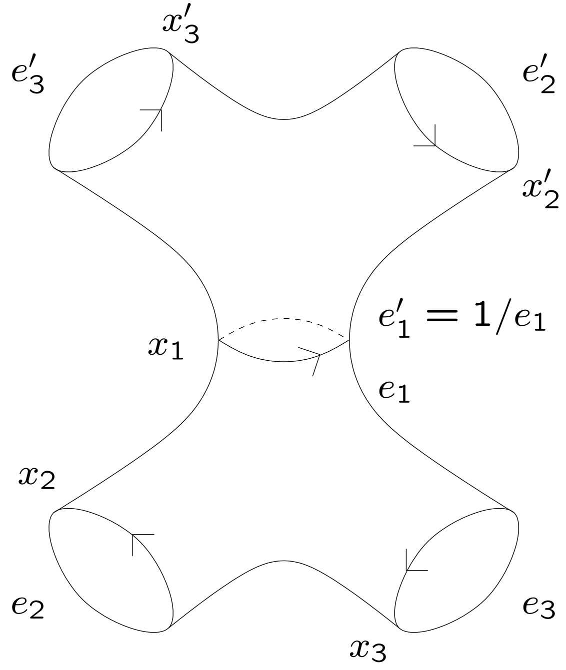
Transition of the fixed points

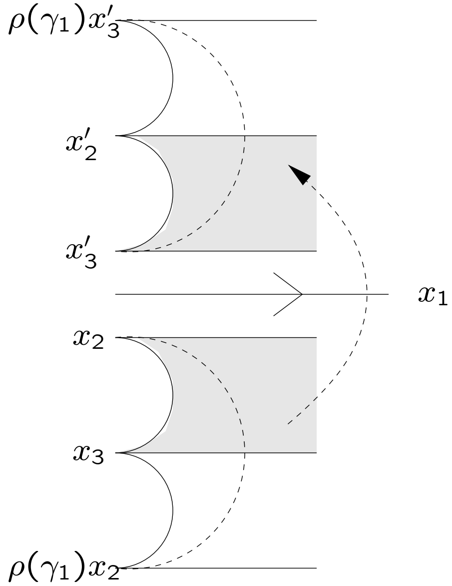
Fix a pair of pants $P \subset S - C$. Let $\partial P = \overrightarrow{c}_1 \cup \overrightarrow{c}_2 \cup \overrightarrow{c}_3$ and e_i be the eigenvalue and x_i be the fixed point of $\rho(\gamma_i)$.

The fixed points of the adjacent pair of pants sharing $\overrightarrow{c}_1$ can be expressed by the twist parameter t_1 and the eigenvalues.



x'_1, x'_2 and x'_3 are given by





$$x'_1 = x_1,$$
$$x'_2 = \frac{a_1}{a_2},$$
$$\begin{aligned} a_1 = {} & e_1((e_2 - e_1e_3)(e'_3 - e_1e'_2){t_1}^2 + e_3(e_1e'_3 - e'_2))x_1(x_2 - x_3) \\ & + {e_1}^2e_2(e_1e'_3 - e'_2)x_2(x_3 - x_1) + e_2(e_1e'_3 - e'_2)x_3(x_1 - x_2), \\ a_2 = {} & e_1((e_2 - e_1e_3)(e'_3 - e_1e'_2){t_1}^2 + e_3(e_1e'_3 - e'_2))(x_2 - x_3) \\ & + {e_1}^2e_2(e_1e'_3 - e'_2)(x_3 - x_1) + e_2(e_1e'_3 - e'_2)(x_1 - x_2), \end{aligned}$$
$$x'_3 = \frac{((e_2 - e_1e_3){t_1}^2 + e_1e_3)x_1(x_2 - x_3) + {e_1}^2e_2x_2(x_3 - x_1) + e_2x_3(x_1 - x_2)}{((e_2 - e_1e_3){t_1}^2 + e_1e_3)(x_2 - x_3) + {e_1}^2e_2(x_3 - x_1) + e_2(x_1 - x_2)}.$$

Matrices parametrized by (e_i, t_i)'s

Using these formulas, we can compute the developed image of the ideal triangles of S.

Take two lifts of ideal triangle Δ_0 and $\gamma\Delta_0$ of $\widetilde{P}$. Since the ideal vertices of $D(\Delta_0)$ and $D(\gamma\Delta_0)$ consist of three points of $\mathbb{C}P^1$, there exists a unique element $\rho(\gamma)$ such that $D(\gamma\Delta_0) = \rho(\gamma)D(\Delta_0)$.

So we obtain an explicit matrix representatives of the representation corresponding to (e_i, t_i)'s.

Remarks

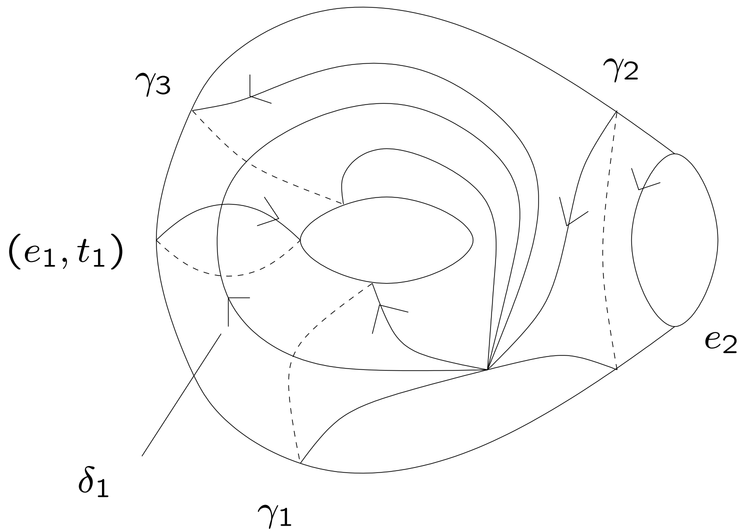
- Bonahon gave a parametrization of $\mathrm{PSL}(2,\mathbb{C})$ representations of a surface group by using the *shear-bend cocycle* of maximal geodesic lamination λ. Our parametrization closely related to the shear-bend cocycle.

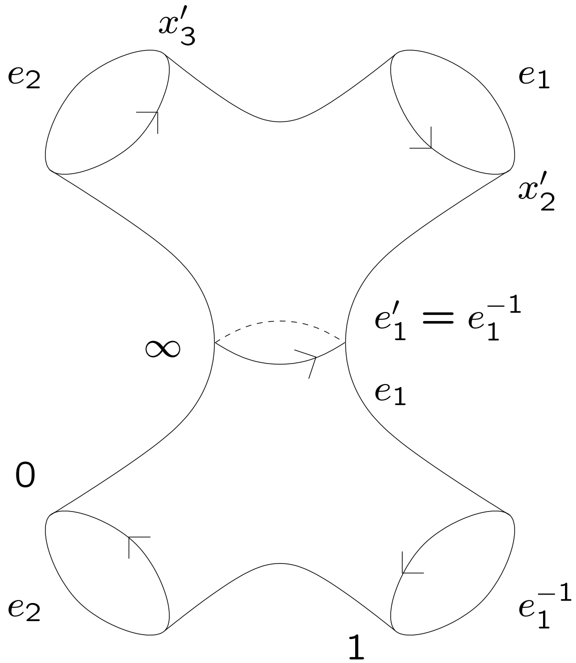
- Maskit gave an explicit construction of matrix generators of Fuchsian groups parametrized by Fenchel-Nielsen coordinates.

One holed torus

Give pointed loops and the parameters as in the Figure:

Let $(x_1, x_2, x_3) = (\infty, 0, 1)$.





Then compute the transition of the fixed points using previous formula.

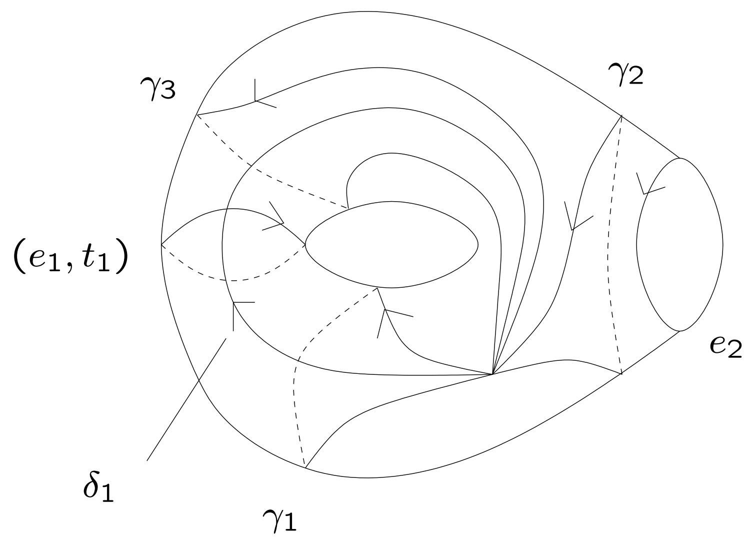
$$x_2' = \left(\frac{e_2 - e_1^2}{e_2(e_1^2 - 1)}\right) t_1^2 + \frac{1 - e_2}{e_2(e_1^2 - 1)},$$

$$x_3' = \frac{(t_1^2 - 1)(e_2 - 1)}{e_2(e_1^2 - 1)}.$$

One holed torus

We have

$$\rho(\gamma_1) = \begin{pmatrix} e_1 & e_1^{-1} - e_1^{-1} e_2^{-1} \\ 0 & e_1^{-1} \end{pmatrix}.$$



Because $\rho(\delta_1)$ is the matrix which sends $(\infty, 0, 1)$ to (x_3', x_2', ∞), we have

$$\rho(\delta_1) = \begin{pmatrix} (e_1^2 - e_2)t_1^2 + (e_2 - 1) & (t_1^2 - 1)(e_2 - 1) \\ -e_2(e_1^2 - 1) & e_2(e_1^2 - 1) \end{pmatrix}$$

in $\mathrm{PGL}(2, \mathbb{C})$. Actually these matrices satisfy the equality

$$\rho(\delta_1)^{-1}\rho(\gamma_1)^{-1}\rho(\delta_1)\rho(\gamma_1) = \rho(\gamma_2)^{-1}.$$

Once punctured torus

When $e_2 = -1$, replace t_1 with $\sqrt{-1}t_1$, we have

$\rho(\gamma_1) = \begin{pmatrix} e_1 & 2e_1^{-1} \\ 0 & e_1^{-1} \end{pmatrix}$,

$\rho(\delta_1) = \dfrac{1}{t_1(1-e_1^2)} \begin{pmatrix} -(e_1^2+1)t_1^2 - 2 & 2(t_1^2+1) \\ e_1^2 - 1 & -(e_1^2-1) \end{pmatrix} = \begin{pmatrix} \frac{-(e_1^2+1)t_1^2-2}{t_1(1-e_1^2)} & \frac{2(t_1^2+1)}{t_1(1-e_1^2)} \\ -t_1^{-1} & t_1^{-1} \end{pmatrix}$.

Let $A = \rho(\gamma_1)$ and $B = \rho(\delta_1)$. Then

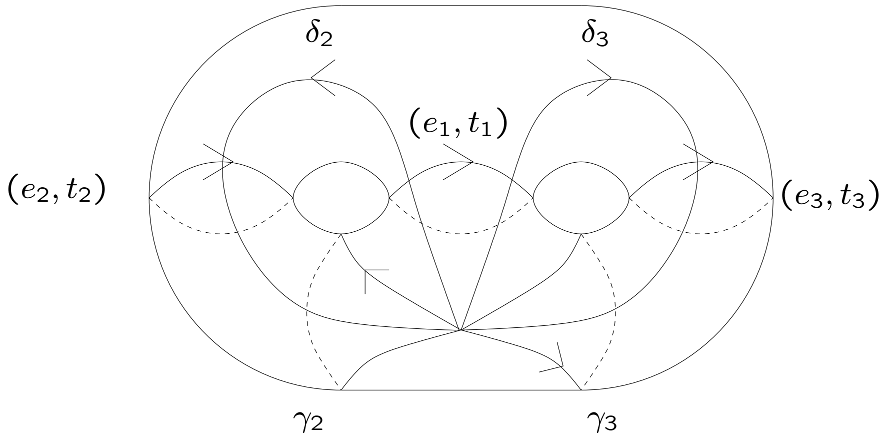
$$\mathrm{tr}(A) = e_1 + e_1^{-1}, \quad \mathrm{tr}(B) = \frac{(e_1+e_1^{-1})(t_1+t_1^{-1})}{(e_1-e_1^{-1})},$$
$$\mathrm{tr}(AB) = \frac{(e_1+e_1^{-1})(e_1t_1+e_1^{-1}t_1^{-1})}{(e_1-e_1^{-1})}.$$

These satisfy the Markov identity

$$\mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(AB)^2 - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) = 0.$$

Closed surface of genus 2

Give pointed loops and the parameters as in the Figure:



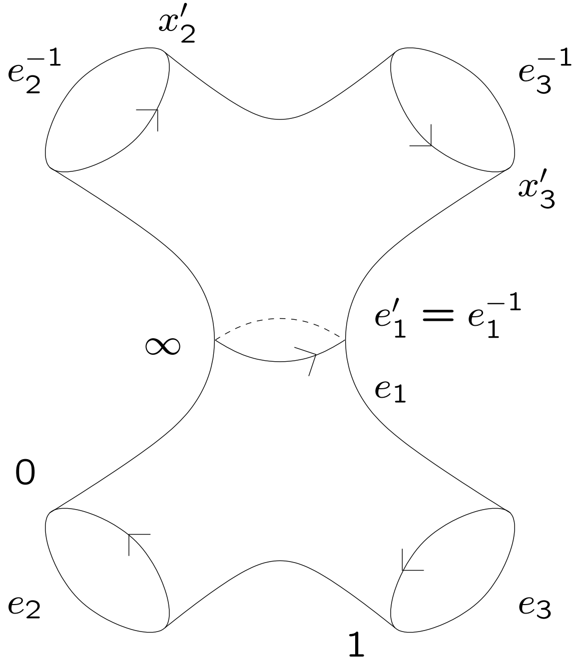
Let $(x_1,x_2,x_3)=(\infty,0,1)$ be fixed points of the lower pants.

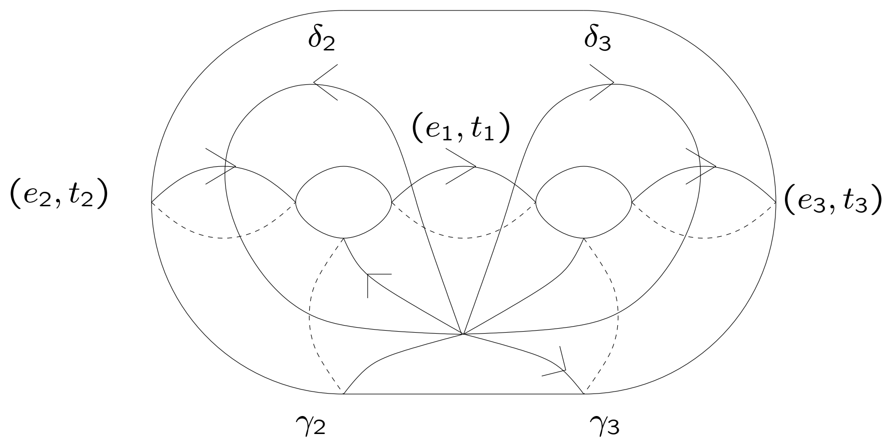
Then we have

$$\rho(\gamma_2)=\begin{pmatrix} e_2^{-1} & 0 \\ -e_2+e_1e_3^{-1} & e_2 \end{pmatrix},$$

$$\rho(\gamma_3)=\begin{pmatrix} e_1^{-1}e_2 & e_3-e_1^{-1}e_2 \\ -e_3^{-1}+e_1^{-1}e_2 & e_3+e_3^{-1}-e_1^{-1}e_2 \end{pmatrix}.$$

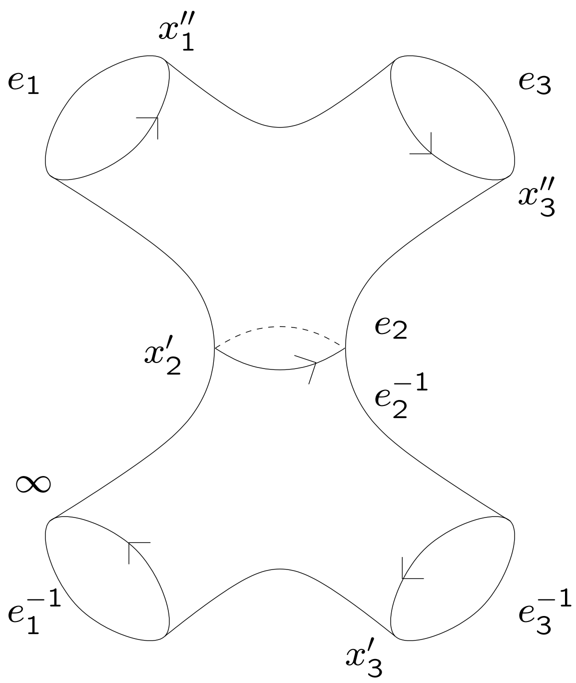
Closed surface of genus 2





Then compute the fixed points (x_1, x_2', x_3') of the pair of pants adjacent along $\overrightarrow{c}_1$.

Closed surface of genus 2



Then compute the fixed points (x_1'', x_2', x_3'') of the pair of pants adjacent along $\overrightarrow{c}_2$. We can compute $\rho(\delta_2)$ as the matrix which sends $(\infty, 0, 1)$ to (x_1'', x_2', x_3'').

$$\rho(\delta_2) = \frac{1}{t_1 t_2} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$a_{11} = \frac{(e_1e_2e_3 - 1)(e_1e_2 - e_3){t_1}^2{t_2}^2 + (e_2e_3 - e_1)(e_1e_3 - e_2)({t_1}^2 + {t_2}^2 - 1)}{({e_1}^2 - 1)({e_2}^2 - 1)e_3},$$

$$a_{12} = -\frac{(e_1e_3 - e_2)({t_1}^2 - 1)}{({e_1}^2 - 1)e_2},$$

$$a_{21} = \frac{e_2({t_2}^2 - 1)(e_2e_3 - e_1)}{({e_2}^2 - 1)e_3},$$

$$a_{22} = 1,$$

Closed surface of genus 2

Similarly we have

$$
\begin{aligned}
\rho(\delta_3) =& \frac{-1}{e_1({e_3}^2-1)t_1t_3}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \\
b_{11} &= -(e_1(e_1e_2e_3-1)(e_1e_3-e_2){t_1}^2{t_3}^2 + e_1(e_2e_3-e_1)(e_1e_2-e_3){t_1}^2 \\
&\qquad + (e_2e_3-e_1)(e_1e_3-e_2)(1-{t_3}^2))/(({e_1}^2-1)e_2), \\
b_{12} &= (e_1e_3-e_2)(e_1(e_1e_2e_3-1){t_1}^2{t_3}^2 + (e_1-e_2e_3){t_3}^2 \\
&\qquad + e_1e_3(e_3-e_1e_2){t_1}^2 + e_3(e_2-e_1e_3))/(({e_1}^2-1)e_2), \\
b_{21} &= (e_2e_3-e_1)({t_3}^2-1), \\
b_{22} &= (-e_2e_3+e_1){t_3}^2 - e_1{e_3}^2 + e_2e_3.
\end{aligned}
$$

We can check that these matrices satisfy the equality

$$\rho(\delta_2)\rho(\gamma_2)^{-1}\rho(\delta_2)^{-1}\rho(\gamma_2)\rho(\gamma_3)\rho(\delta_3)\rho(\gamma_3)^{-1}\rho(\delta_3)^{-1} = I,$$

Closed surface of genus 2

The traces of $\rho(\gamma_i)$ and $\rho(\delta_i)$ are

$$\mathrm{tr}(\rho(\gamma_2)) = e_2 + e_2^{-1}, \quad \mathrm{tr}(\rho(\gamma_3)) = e_3 + e_3^{-1},$$

$$\mathrm{tr}(\rho(\delta_2)) = \frac{(e_1e_2 - e_3)(e_1e_2e_3 - 1)({t_1}^2{t_2}^2 + 1) + (e_1e_3 - e_2)(e_2e_3 - e_1)({t_1}^2 + {t_2}^2)}{({e_1}^2 - 1)({e_2}^2 - 1)e_3t_1t_2},$$

$$\mathrm{tr}(\rho(\delta_3)) = \frac{(e_1e_3 - e_2)(e_1e_2e_3 - 1)({t_1}^2{t_3}^2 + 1) + (e_1e_2 - e_3)(e_2e_3 - e_1)({t_1}^2 + {t_3}^2)}{({e_1}^2 - 1)e_2({e_3}^2 - 1)t_1t_3},$$

Remark *Maskit gave an explicit set of matrix generators parametrized by Fenchel-Nielsen coordinates.*

Thank you for your attention.