Quandle homology and group homology

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Introduction

Quandle : a set X with a binary operation ($x_1 * x_2$ for $x_i \in X$)

$H_n^Q(X)$, $H_Q^n(X)$: quandle homology and cohomology

$H_n^Q(X)$ is defined similar to group homology

Question

- Any relation to group homology?
- Do important quandle cohomology classes (e.g. Mochizuki cocycles) come from group cocycles?

Quandle

Quandle X is a set with $* : X \times X \to X$ satisfying the axioms

(Q1) $x * x = x$ for $x \in X$

(Q2) For any $y \in X$, $*y : x \mapsto x * y$ is a bijection

(Q3) $(x * y) * z = (x * z) * (y * z)$ for $x, y, z \in X$

Example

G : a group, $S \subset G$: a subset closed under conjugation.

S is a quandle with $x * y = y^{-1}xy$ $(x, y \in S)$.

(Q1) and (Q2) are clearly satisfied, and we have

$$\begin{aligned}(x * y) * z &= z^{-1}(y^{-1}xy)z = z^{-1}y^{-1}zz^{-1}xzz^{-1}yz \\ &= (y * z)^{-1}(x * z)(y * z) = (x * z) * (y * z).\end{aligned}$$

Adjoint group

For a quandle X, define the *adjoint group* by

$$\mathrm{Ad}(X) = \langle x \in X \mid x * y = y^{-1}xy \rangle.$$

(also known as the *associated group* or *enveloping group*)

Remark

For a Lie algebra L (a vector space with $[,] : V \otimes V \to V$), the universal enveloping algebra is defined by

$$U(L) = \left(\bigoplus_{n \geq 0} L^{\otimes n} \right) \Big/ \{[v_1, v_2] = v_1 \otimes v_2 - v_2 \otimes v_1\}.$$

$\mathrm{Ad}(X)$ satisfies some universal property as $U(L)$ does.

Remark

Lie algebra (co)homology is defined as the (co)homology of the associative algebra $U(L)$. But quandle (co)homology is NOT isomorphic to the (co)homology of the group Ad(X).

Quandle homology (Carter-Jelsovsky-Kamada-Langford-Saito, Fenn-Rourke-Sanderson)

For a quandle X, let

$$C_n^R(X) = \mathsf{span}_{\mathbb{Z}[\mathsf{Ad}(X)]}\{(x_1, \ldots, x_n) \mid x_i \in X\}.$$

Define the boundary operator $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

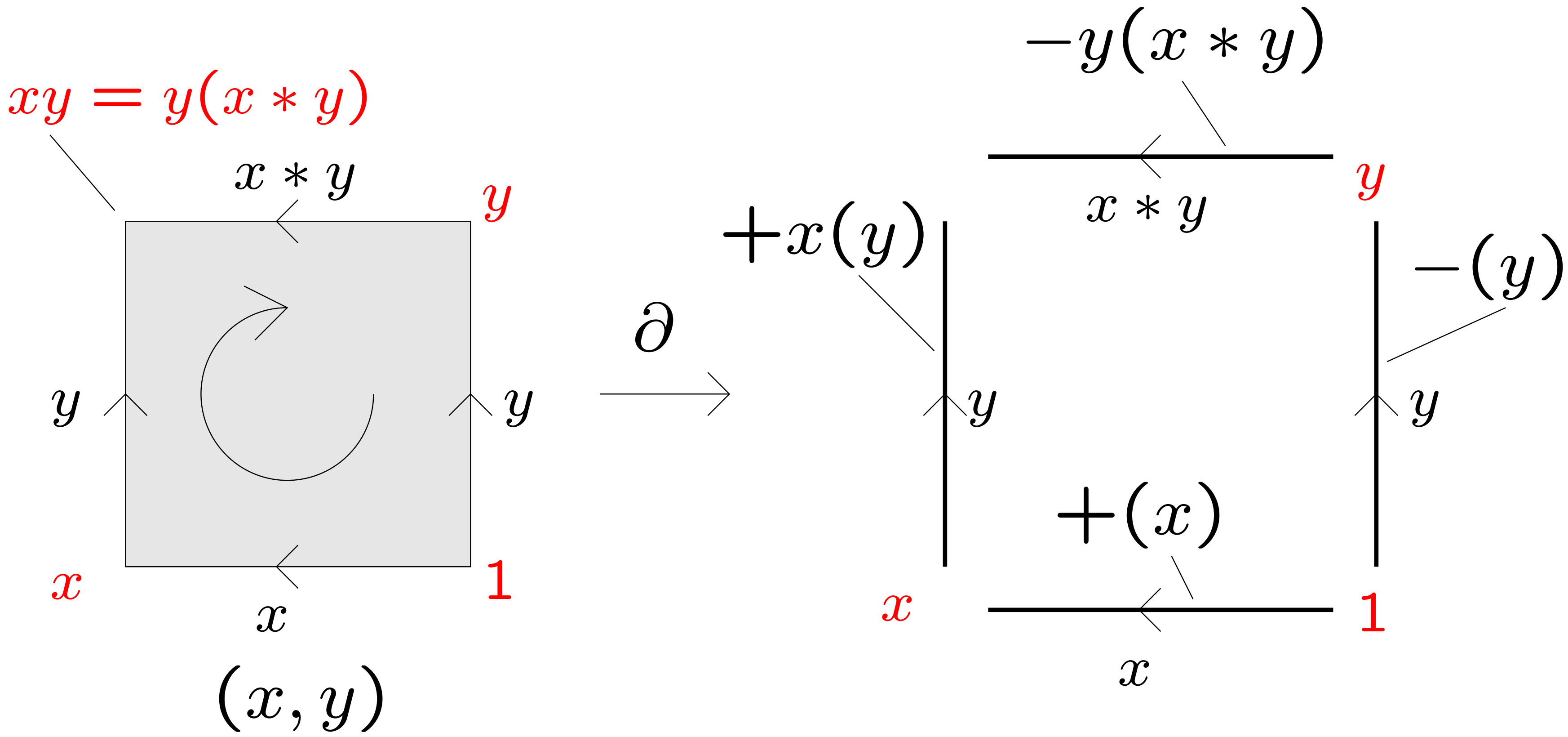
$$\partial(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i \{(x_1, \ldots, \widehat{x_i}, \ldots, x_n) - x_i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}.$$

For examples:

$$\partial(x, y) = -((y) - x(y)) + ((x) - y(x * y)),$$

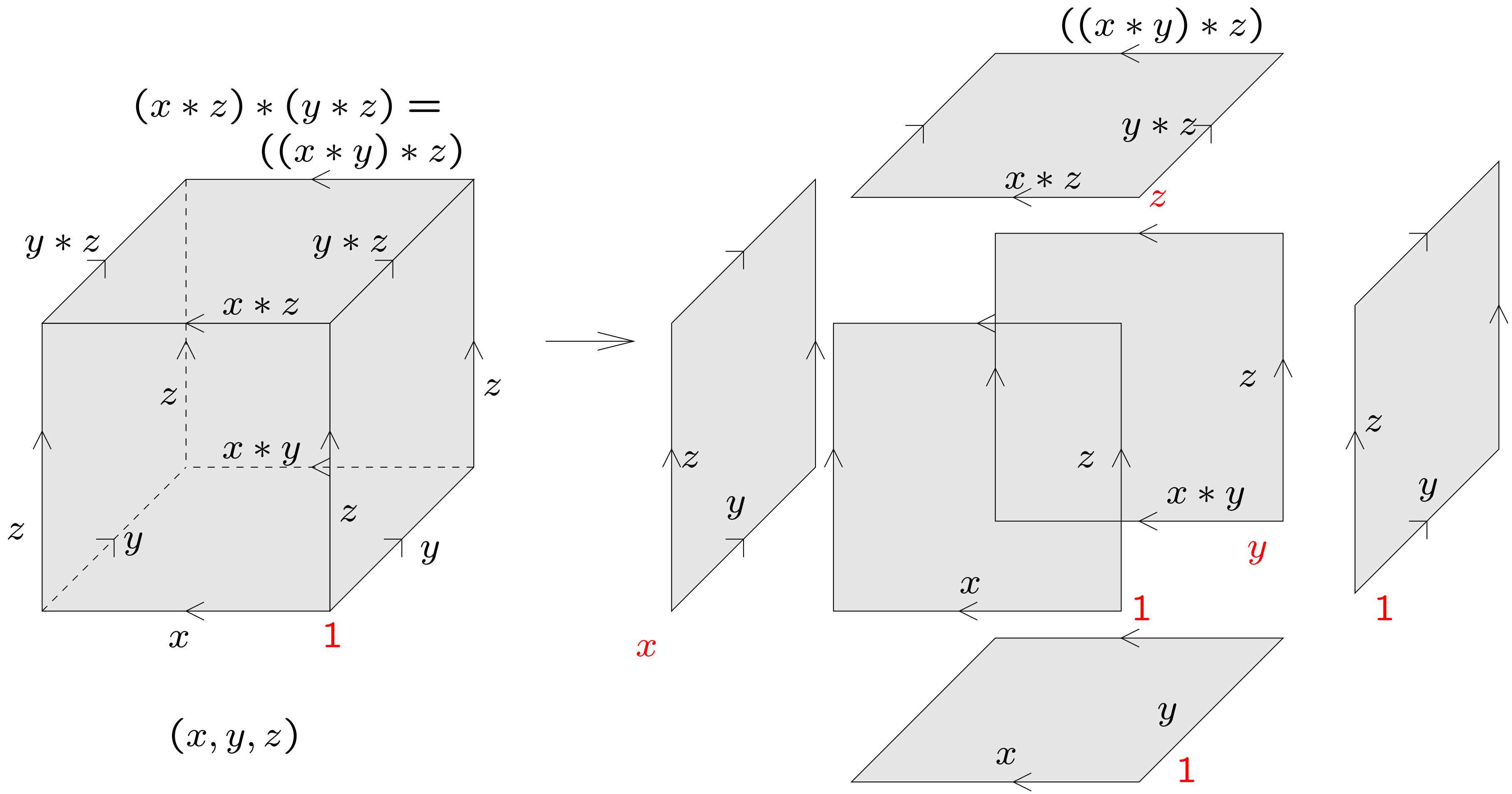
$$\partial(x, y, z) = -((y, z) - x(y, z)) + ((x, z) - y(x * y, z)) - ((x, y) - z(x * z, y * z)).$$

Pictorial description $\partial : C_2^R(X) \to C_1^R(X)$



$$\partial(x, y) = -(y) + x(y) + (x) - y(x * y)$$

Pictorial description $\partial : C_3^R(X) \to C_2^R(X)$



$$\partial(x, y, z) = -(y, z) + x(y, z) + (x, z) - y(x * y, z)$$
$$-(x, y) + z(x * z, y * z)$$

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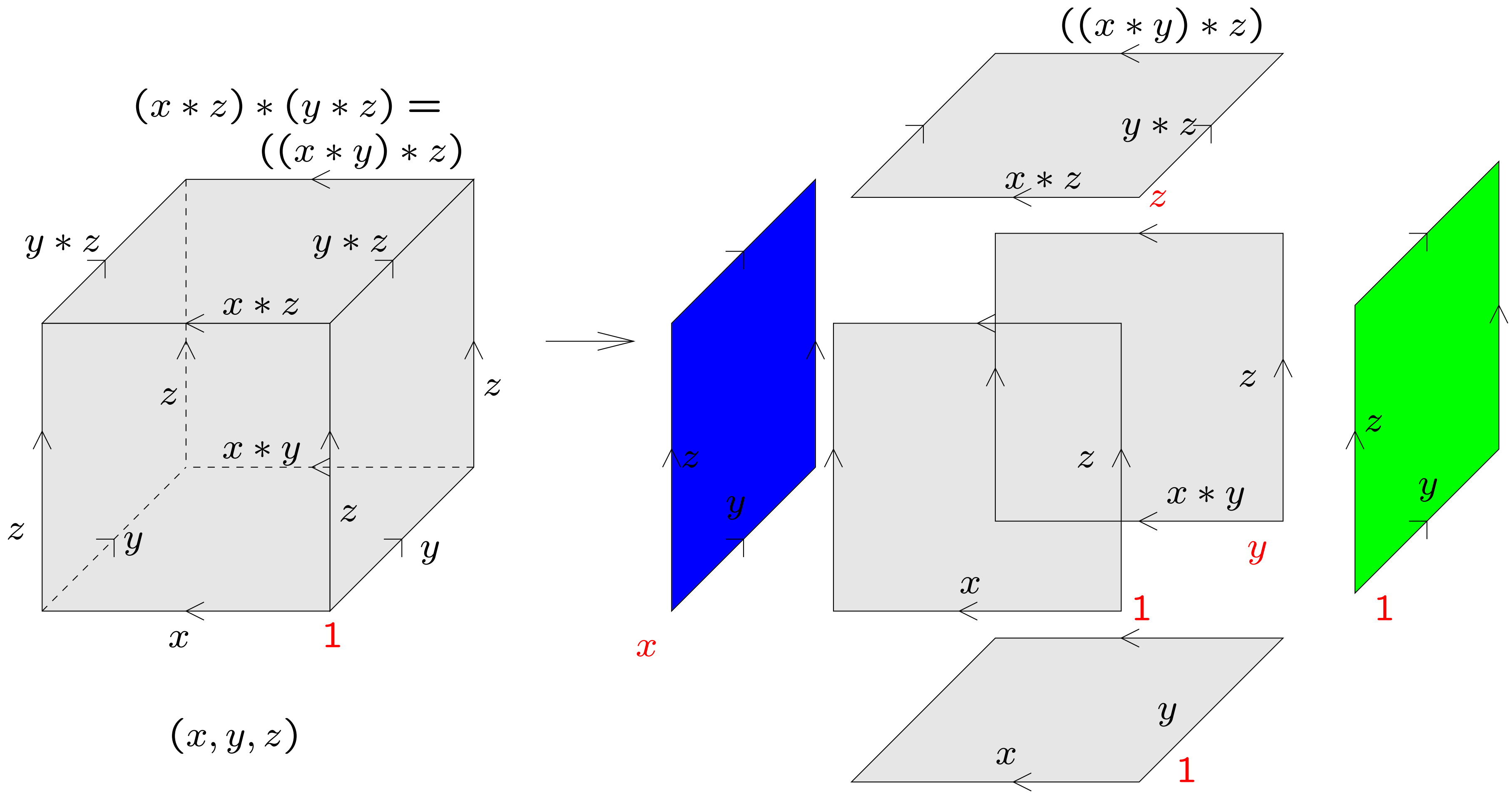

(x, y, z)

$$\partial(x,y,z) = -{\color{green}(y,z)} + {\color{blue}x(y,z)} + (x,z) - y(x*y,z)$$
$$-(x,y) + z(x*z,y*z)$$

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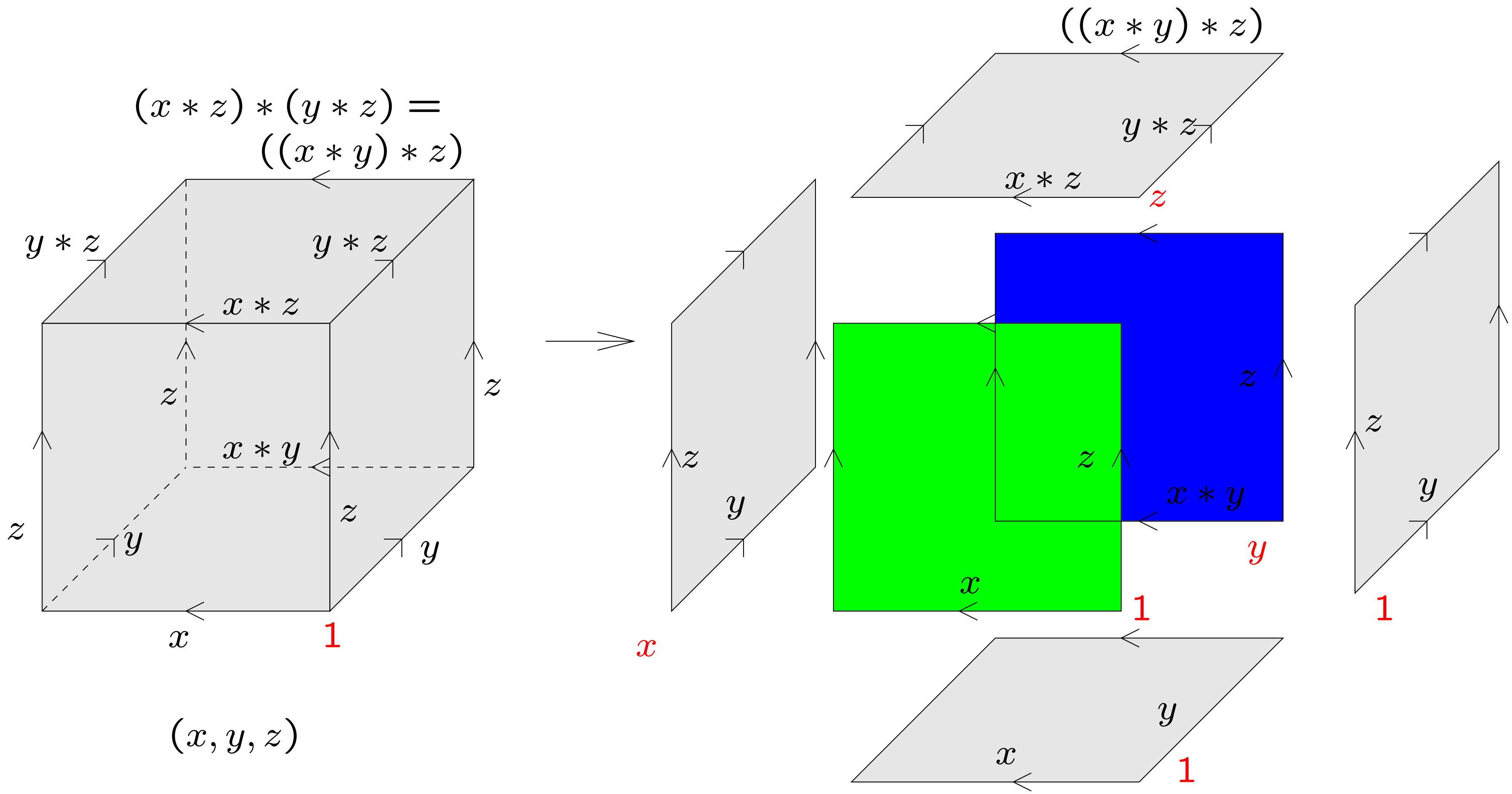

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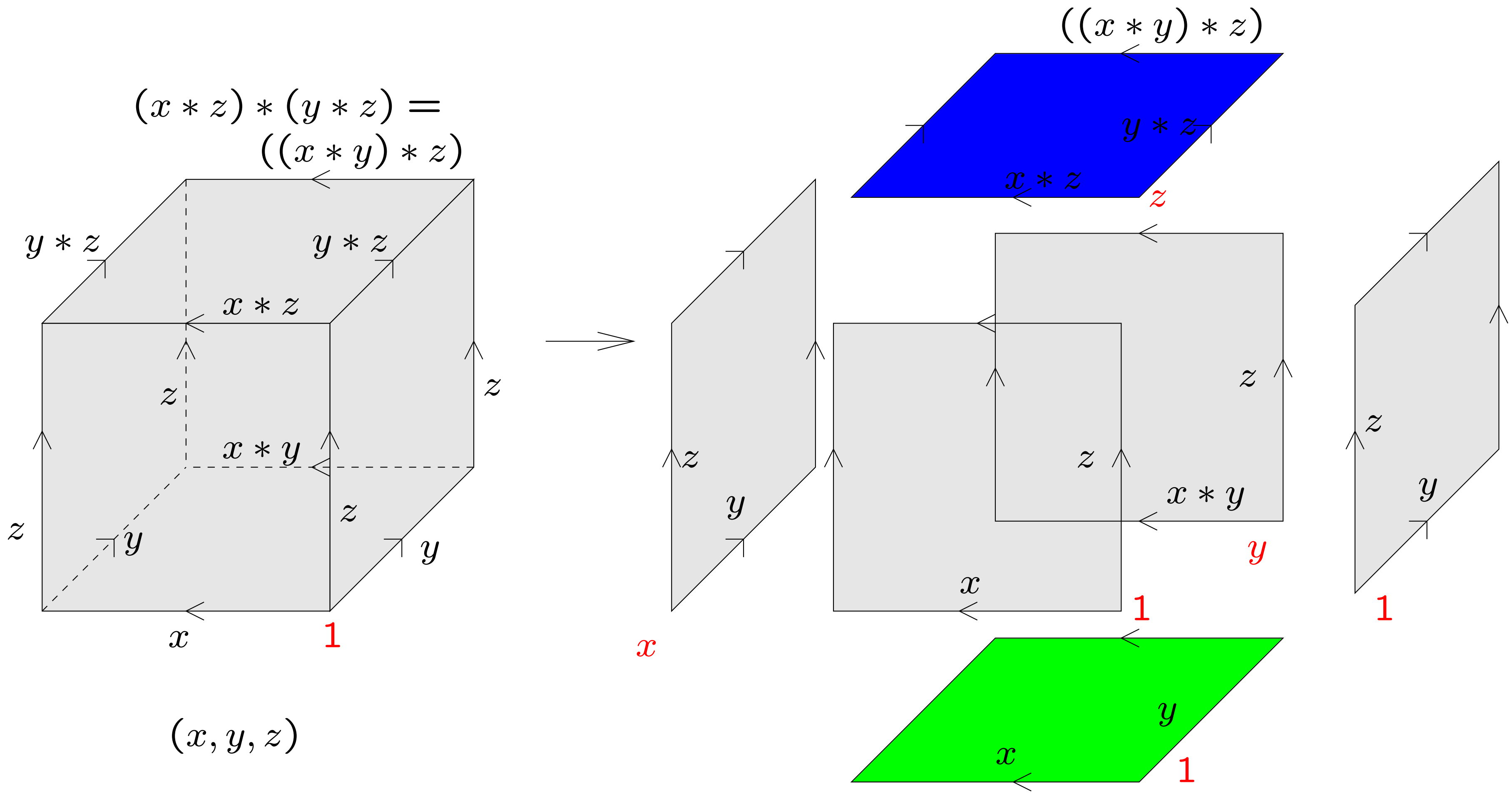


$$\partial(x, y, z) = -(y, z) + x(y, z) + (x, z) - y(x*y, z)$$

$$-(x, y) + z(x*z, y*z)$$

Let M be a right $\mathbb{Z}[\mathrm{Ad}(X)]$-module. The homology group of $C_n^R(X; M) = M \otimes_{\mathbb{Z}[\mathrm{Ad}(X)]} C_n^R(X)$ is called the *rack homology* $H_n^R(X; M)$. (Fenn-Rourke-Sanderson)

Let

$$C_n^D(X) = \mathrm{span}_{\mathbb{Z}[\mathrm{Ad}(X)]}\{(x_1, \dots, x_n) \mid x_i \in X, \\ x_i = x_{i+1} (\text{for some } i)\}.$$

This is a subcomplex of $C_n^R(X)$. Let $C_n^Q(X)$ be the quotient $C_n^R(X)/C_n^D(X)$. The homology of $M \otimes_{\mathrm{Ad}(X)} C_n^Q(X)$ is called the *quandle homology* $H_n^Q(X; M)$.

Cf. Group homology

For a group G, let

$$C_n(G) = \text{span}_{\mathbb{Z}[G]}\{[g_1|\ldots|g_n] \mid g_i \in G\}.$$

The boundary operator is defined by

$$\begin{aligned}\partial[g_1|\ldots|g_n] =& g_1[g_2|\ldots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\ldots|g_ig_{i+1}|\ldots|g_n] \\ &+ (-1)^n[g_1|\ldots|g_{n-1}].\end{aligned}$$

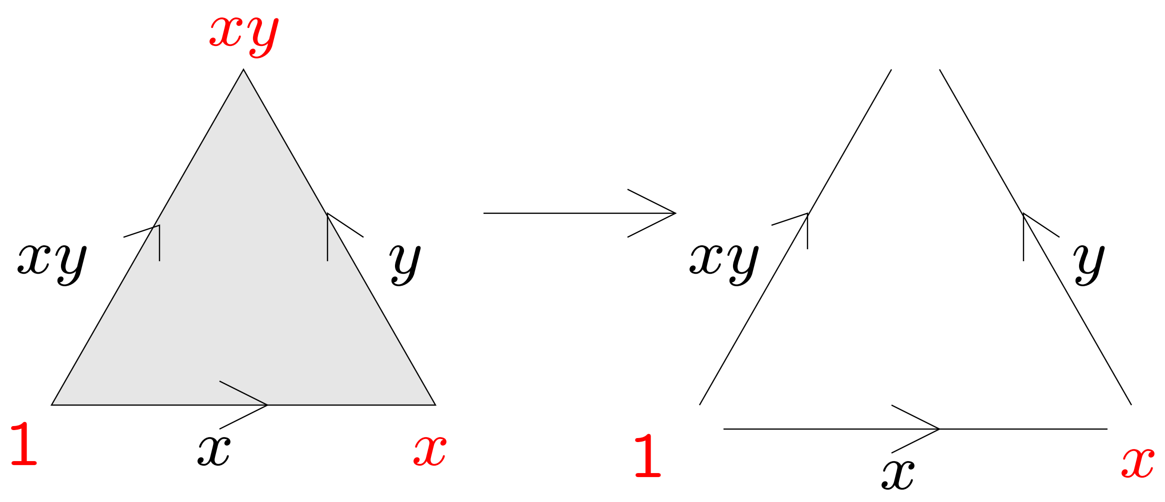
For example:

$$\partial[x|y] = x[y] - [xy] + [x],$$

$$\partial[x|y|z] = x[y|z] - [xy|z] + [x|yz] - [x|y].$$

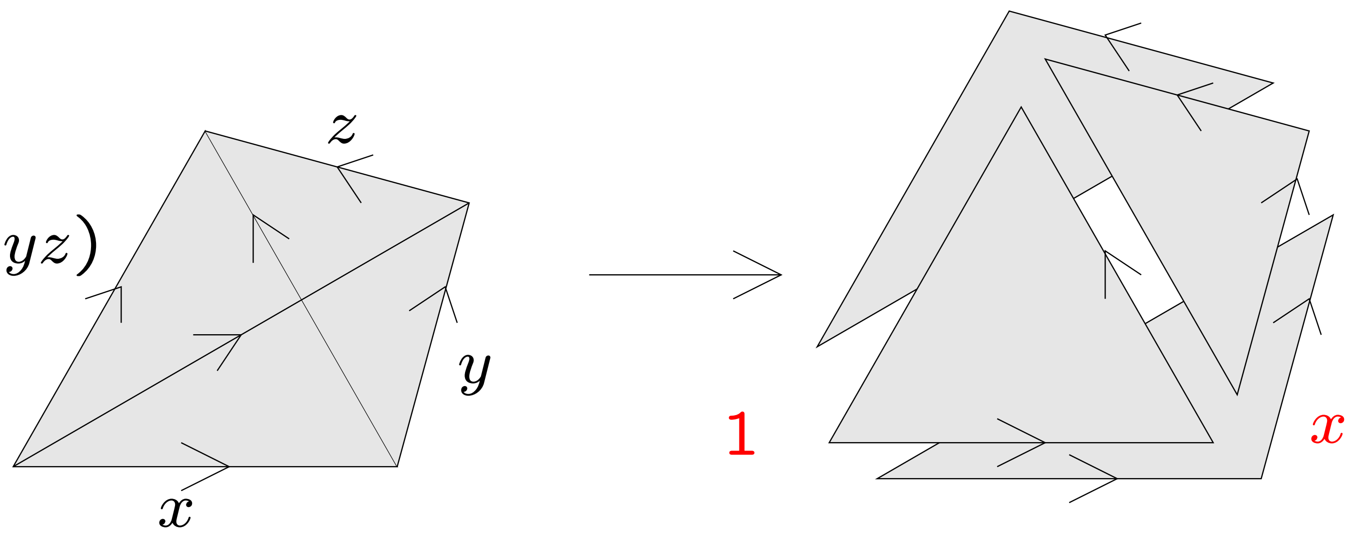
Let M be a right $\mathbb{Z}[G]$-module. The homology group of $M \otimes_{\mathbb{Z}[G]} C_n(G)$ is called the *group homology* $H_n(G; M)$.

Pictorial description



$$\partial[x|y] = x[y] - [xy] + [x]$$

$$(xy)z = x(yz)$$



$$\partial[x|y|z] = x[y|z] - [xy|z] + [x|yz] - [x|y]$$

Remark

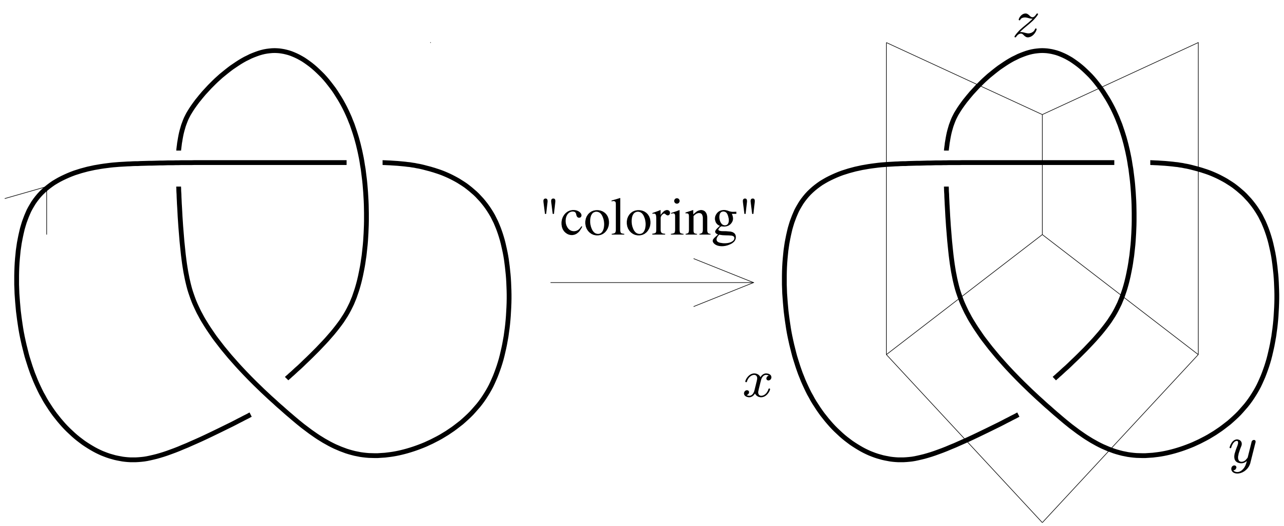
$C_n(G)$ is $\mathbb{Z}[G]$-free and acyclic. $C_n^R(X)$ is $\mathbb{Z}[\mathrm{Ad}(X)]$-free but NOT acyclic. If $C_n^R(X)$ is acyclic, $H_n^R(X; M)$ is isomorphic to the group homology $H_n(\mathrm{Ad}(X); M)$.

$C_n(G)$ is acyclic because we can define a cone map $h : C_n(G) \to C_{n+1}(G)$ satisfying $\partial h + h\partial = \mathrm{id}$.

We can not construct such h for $C_n^R(X)$ since the cone of an n-dim cube is not an $(n+1)$-dim cube!

Application to knot theory

Let $K \subset S^3$ be an (oriented) knot i.e. an embedding of S^1 into S^3. Remove one point from S^3, we can assume that $K \subset S^2 \times \mathbb{R}$. Project K to S^2 , we obtain a diagram of knot on S^2.



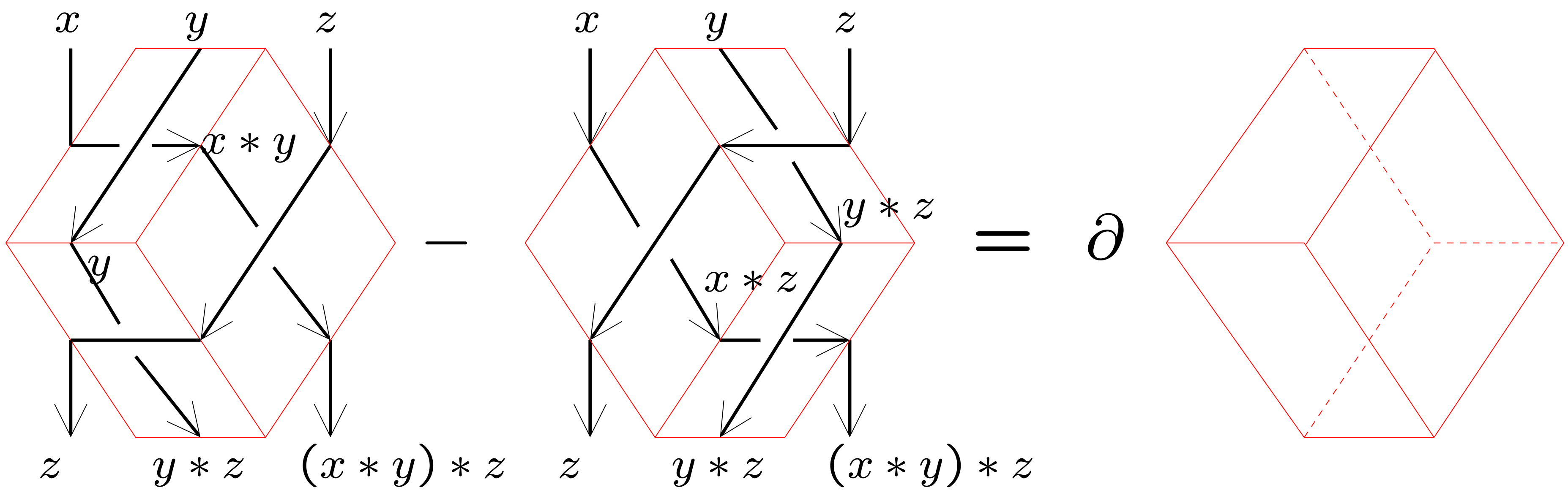
A cycle

$(x, z) + (z, y) + (y, x)$

where $x * y = z$.

A "coloring" of the arcs by a quandle X gives a homology class in $H_2^Q(X; \mathbb{Z})$.

This homology class in $H_2^Q(X;\mathbb{Z})$ does not depend on the choice of the diagram. The invariance under the Reidemeister III move is shown in the following figure.



$$((x,y)+y(x*y,z)+(y,z))-((x,z)+x(y,z)+z(x*z,y*z))$$
$$=\partial(x,y,z)$$

Let BX be a 'geometric realization' of $\{C_n^R(X)\}$. A knot diagram (with framing) also define an element of $\pi_2(BX)$.

More generally, codim-2 framed knot ($S^n \subset S^{n+2}$) with a coloring defines an element of $\pi_{n+1}(BX)$ (Fenn-Rourke-Sanderson).

Since $\pi_1(BX) \cong \mathrm{Ad}(X)$, we have a natural homomorphism

$$H_n^R(X;\mathbb{Z}) \cong H_n(BX) \to H_n(K(\pi_1(BX),1)) \cong H_n(\mathrm{Ad}(X);\mathbb{Z}).$$

But this map may be trivial in many cases.
(Clauwens, arXiv:1004.4423)

Simplicial quandle homology $H_n^{\triangle}(X)$ **(Inoue-K.)**

For a quandle X, let $C_n^{\triangle}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \ldots, x_n) \mid x_i \in X\}$.

Define the boundary operator $\partial : C_n^{\triangle}(X) \to C_{n-1}^{\triangle}(X)$ by

$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_n).$$

Since $\text{Ad}(X)$ acts on X from the right by

$$x * (x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}) = (\cdots((x *^{\epsilon_1} x_1) *^{\epsilon_2} x_2) \cdots) *^{\epsilon_n} x_n$$

where $*^{-1}x_i$ is the inverse of $*x_i$, $C_n^{\triangle}(X)$ is a right $\mathbb{Z}[\text{Ad}(X)]$-module.

Define $H_n^{\triangle}(X)$ by the homology of $C_n^{\triangle}(X) \otimes_{\text{Ad}(X)} \mathbb{Z}$.

A map $H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$

Let $\mathbb{Z}[X] = \text{span}_{\mathbb{Z}} X$, then $\mathbb{Z}[X]$ is a $\mathbb{Z}[\text{Ad}(X)]$-module. We remark that

$$H_n^R(X;\mathbb{Z}[X]) \cong H_{n+1}^R(X;\mathbb{Z}).$$

We will construct a chain map

$$\varphi : \mathbb{Z}[X] \bigotimes_{\text{Ad}(X)} C_n^R(X) \to C_{n+1}^{\Delta}(X) \bigotimes_{\text{Ad}(X)} \mathbb{Z}$$

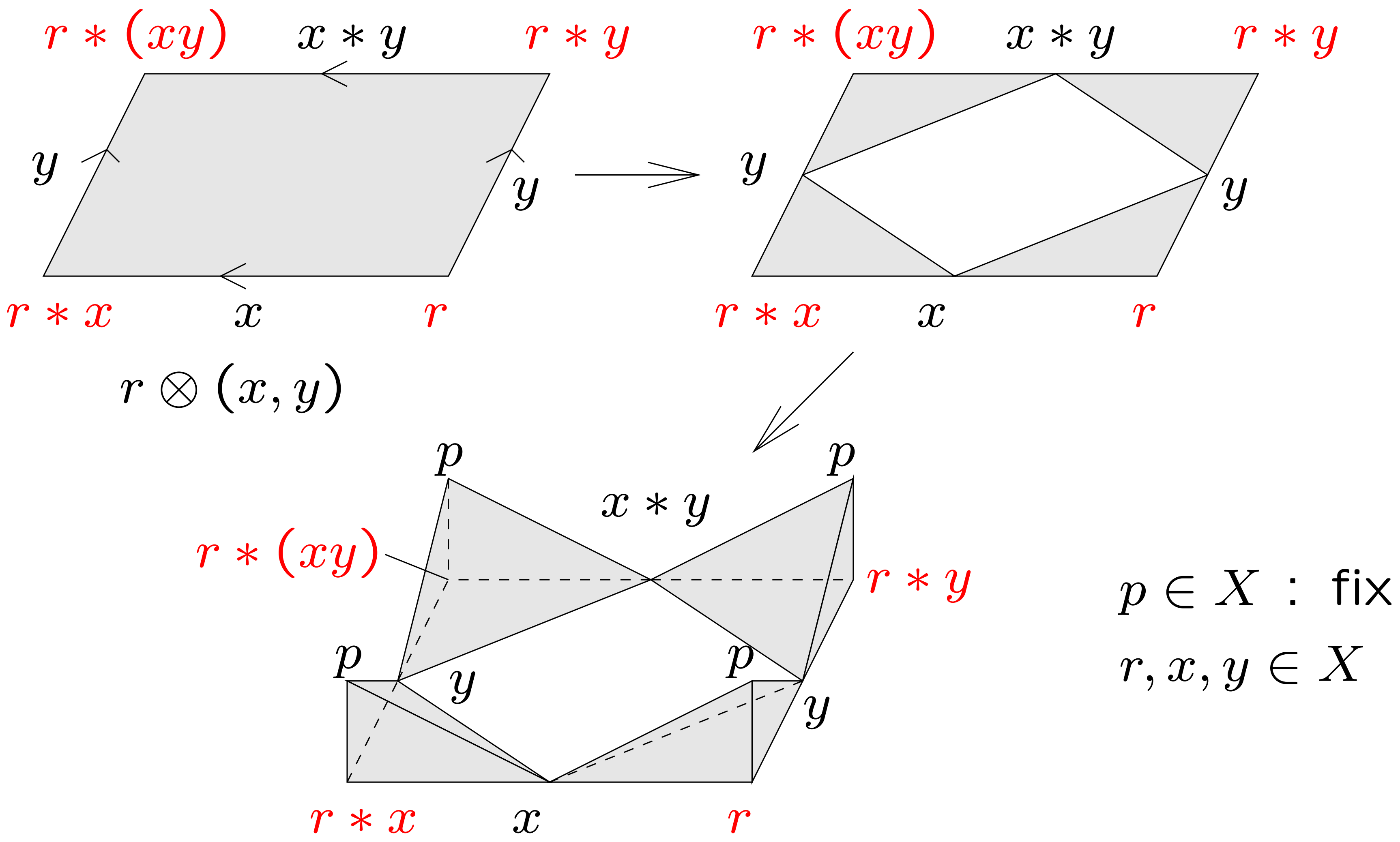
and thus a homomorphism

$$\varphi_* : H_n^R(X;\mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$$

in the following way:

$n = 2$

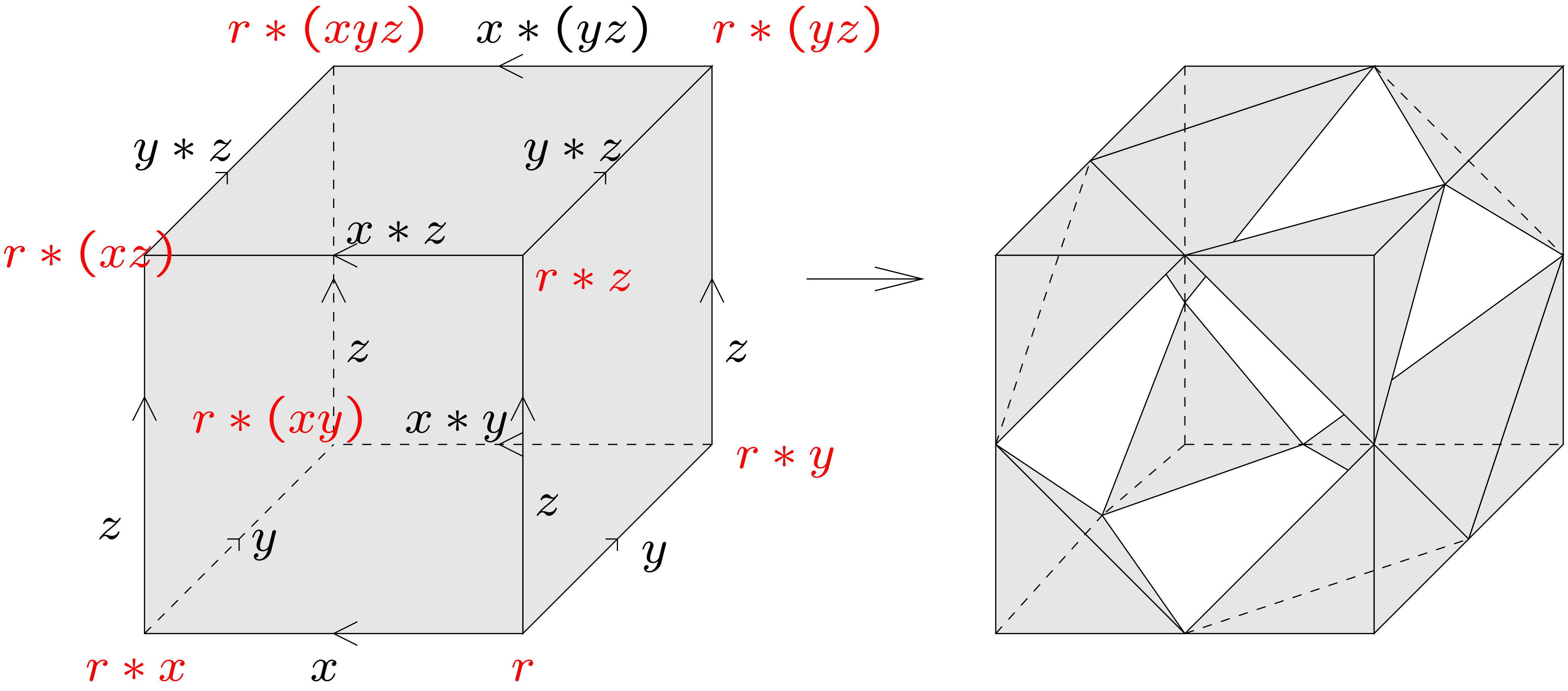
$$\varphi : \mathbb{Z}[X] \otimes_{\mathsf{Ad}(X)} C_2^R(X) \to C_3^{\Delta}(X) \otimes_{\mathsf{Ad}(X)} \mathbb{Z}$$



$$\varphi(r \otimes (x, y)) = (p, r, x, y) - (p, r * x, x, y)$$
$$-(p, r * y, x * y, y) + (p, r * (xy), x * y, y)$$

$\underline{n = 3}$

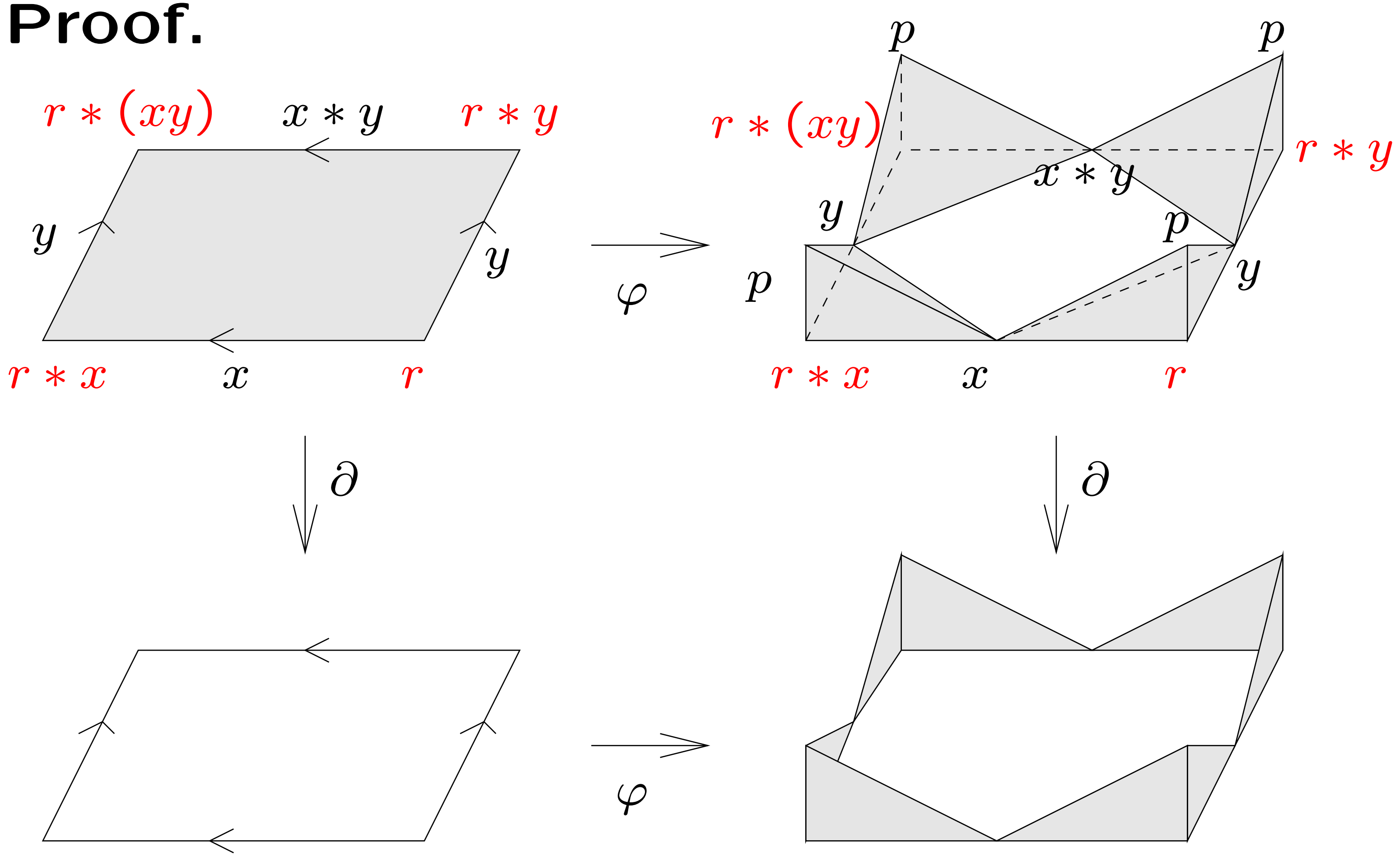
$$\varphi : \mathbb{Z}[X] \otimes_{\mathrm{Ad}(X)} C_3^R(X) \to C_4^{\Delta}(X) \otimes_{\mathrm{Ad}(X)} \mathbb{Z}$$



$$\varphi(r \otimes (x,y,z)) = (p,r,x,y,z) - (p,r*x,x,y,z) - (p,r*y,x,x*y,z)$$
$$-(p,r*z,x*z,y*z,z) + (p,r*(xy),x*y,y,z)$$
$$+(p,r*(xz),x*z,y*z,z) + (p,r*(yz),x*(yz),y*z,z)$$
$$-(p,r*(xyz),x*(yz),y*z,z)$$

Thm $\varphi : C^R_n(X;\mathbb{Z}[X]) \longrightarrow C^{\Delta}_{n+1}(X) \otimes_{\mathsf{Ad}(X)} \mathbb{Z}$ *is a chain map.*

Proof.



∎

Thm $\varphi : C_n^R(X; \mathbb{Z}[X]) \longrightarrow C_{n+1}^{\Delta}(X) \otimes_{\mathrm{Ad}(X)} \mathbb{Z}$ *is a chain map.*

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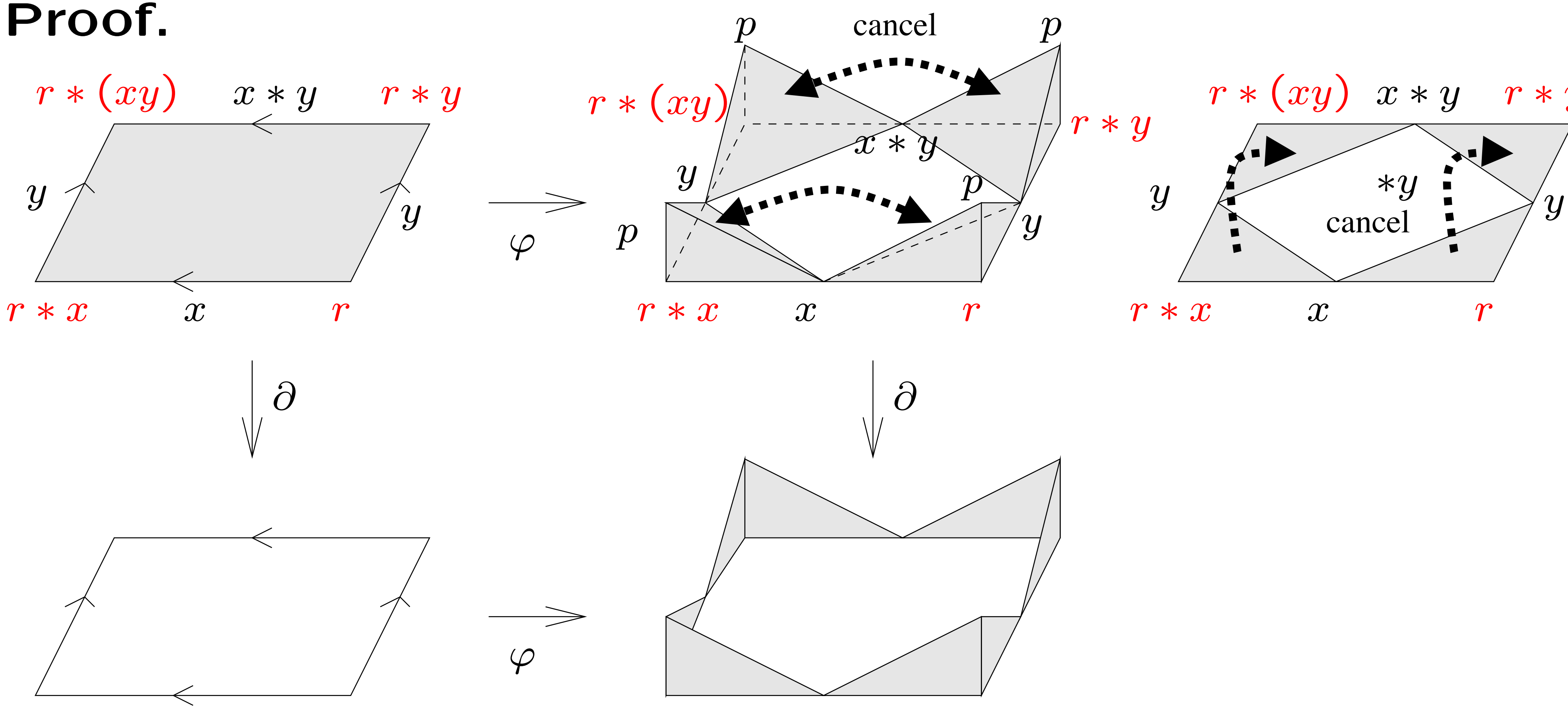

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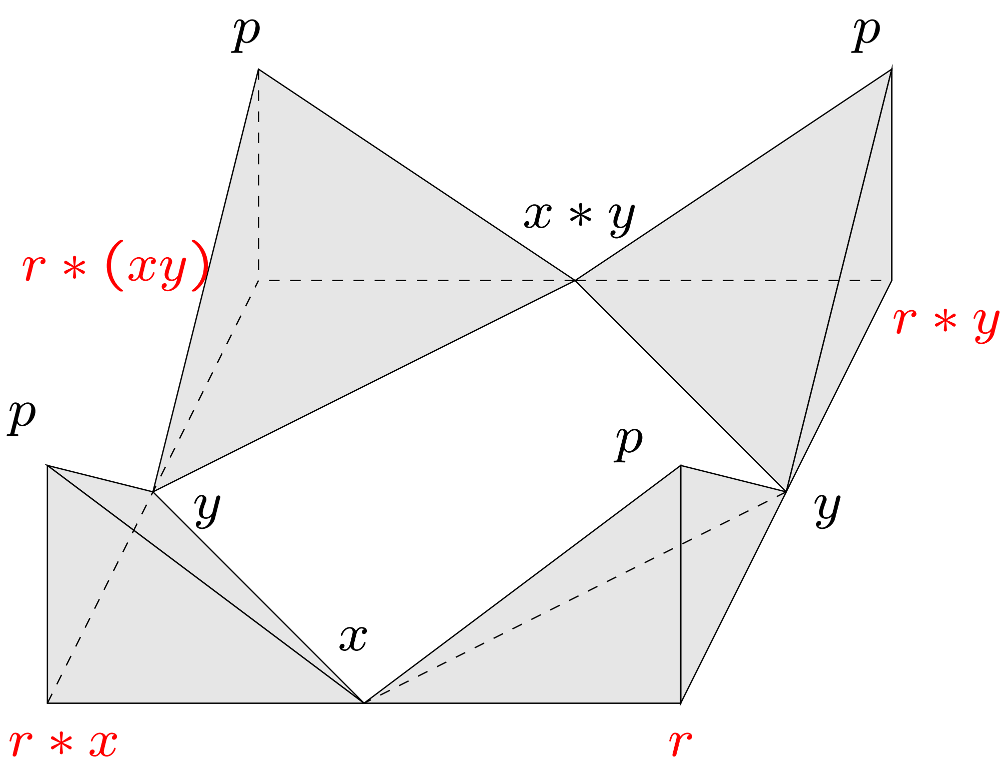
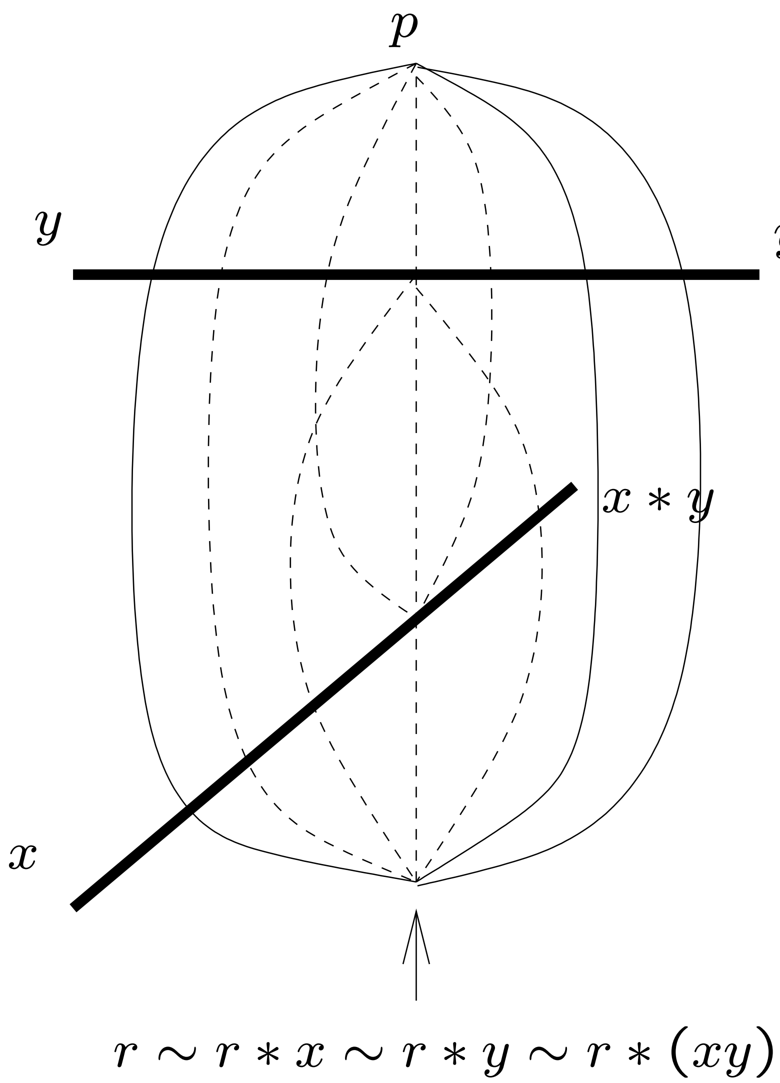
Remark

The definition of φ is motivated by a triangulation of $S^3 \setminus K$.





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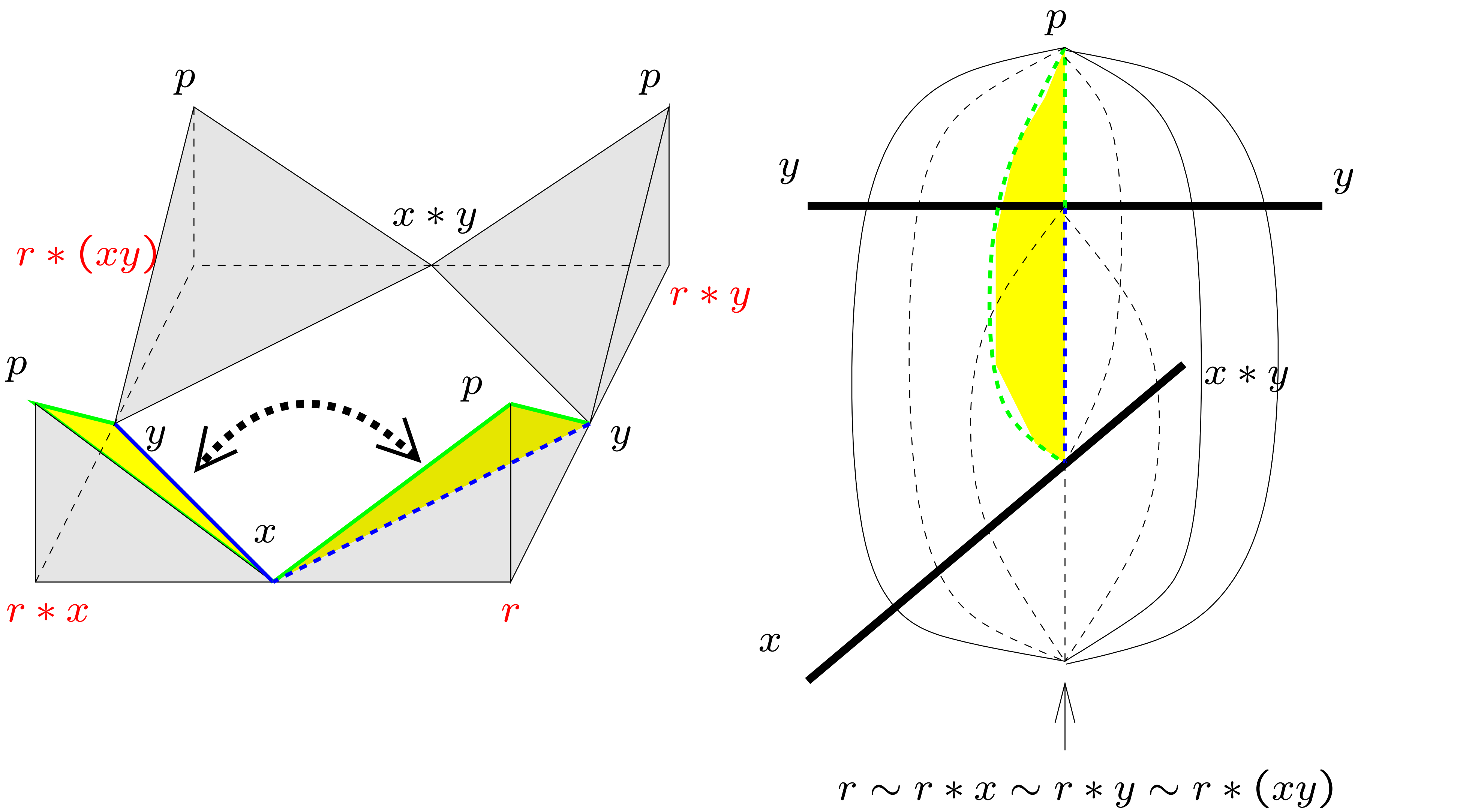

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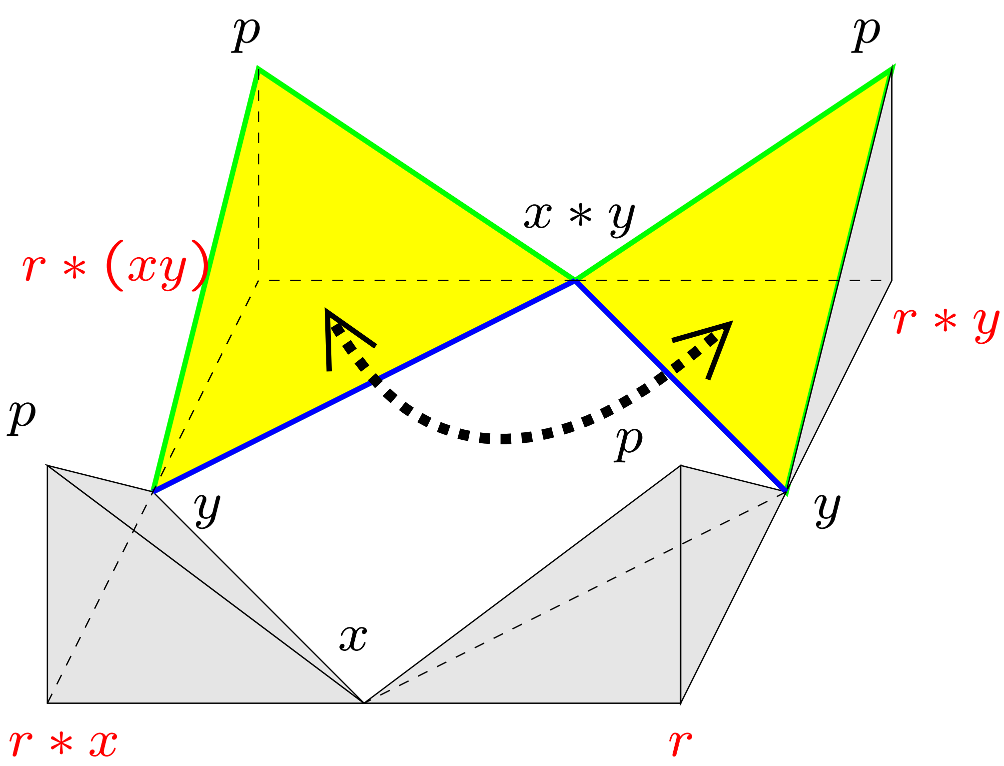

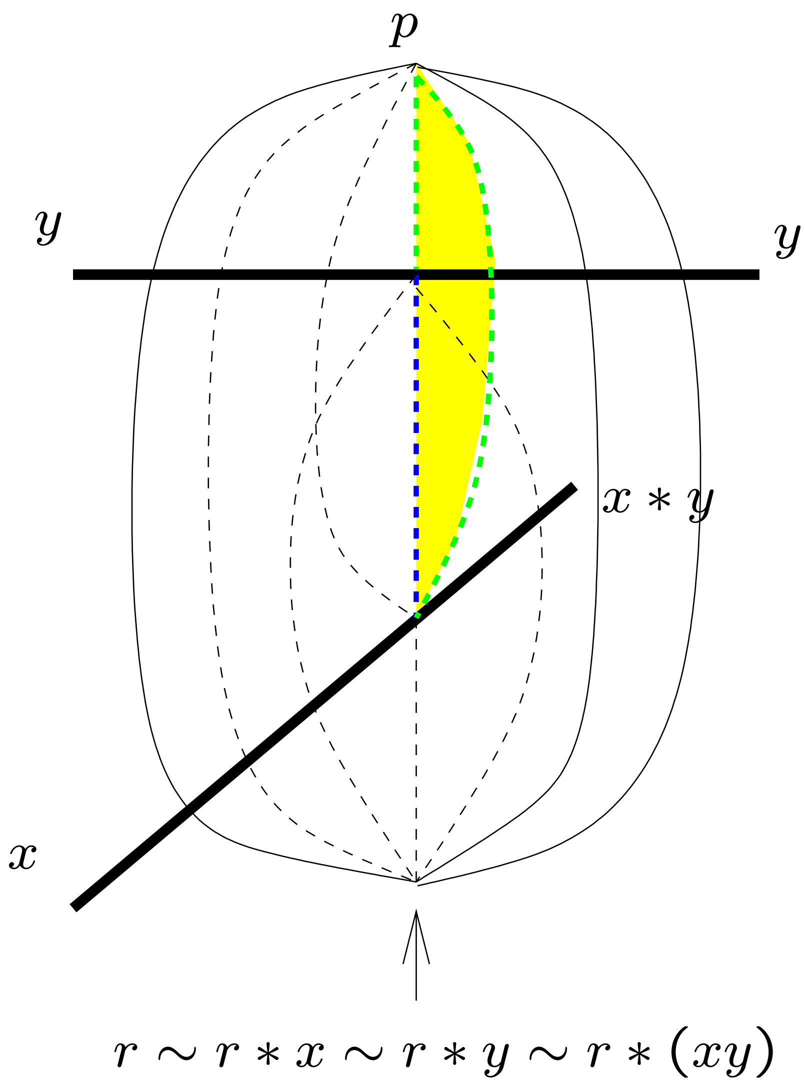

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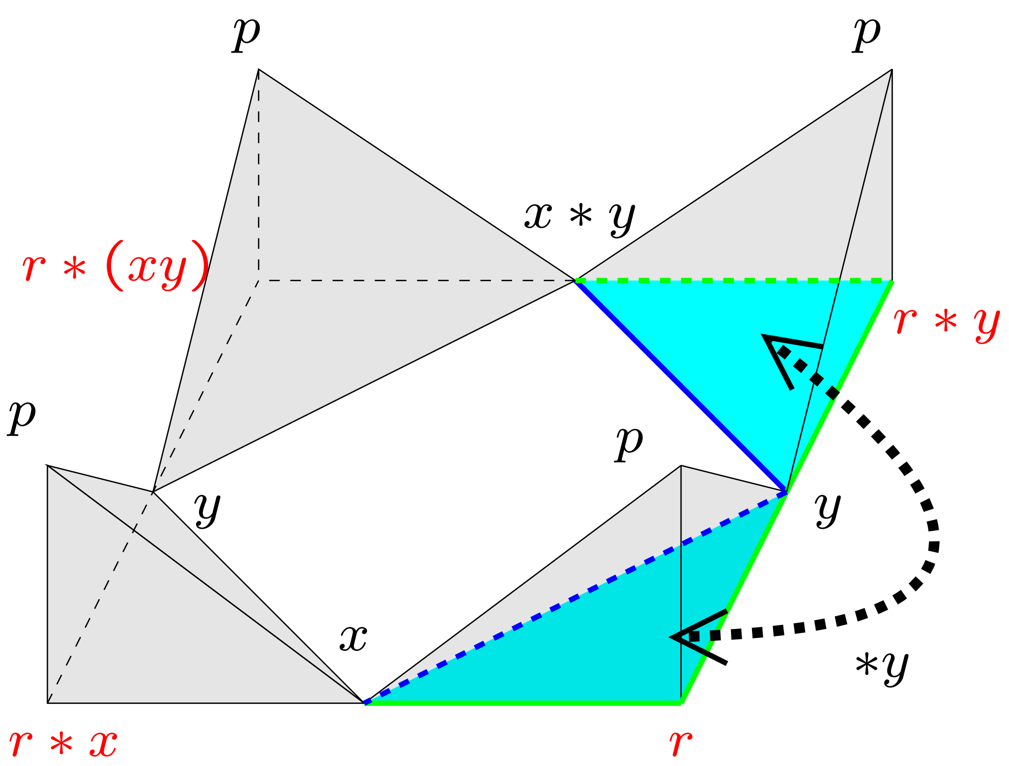

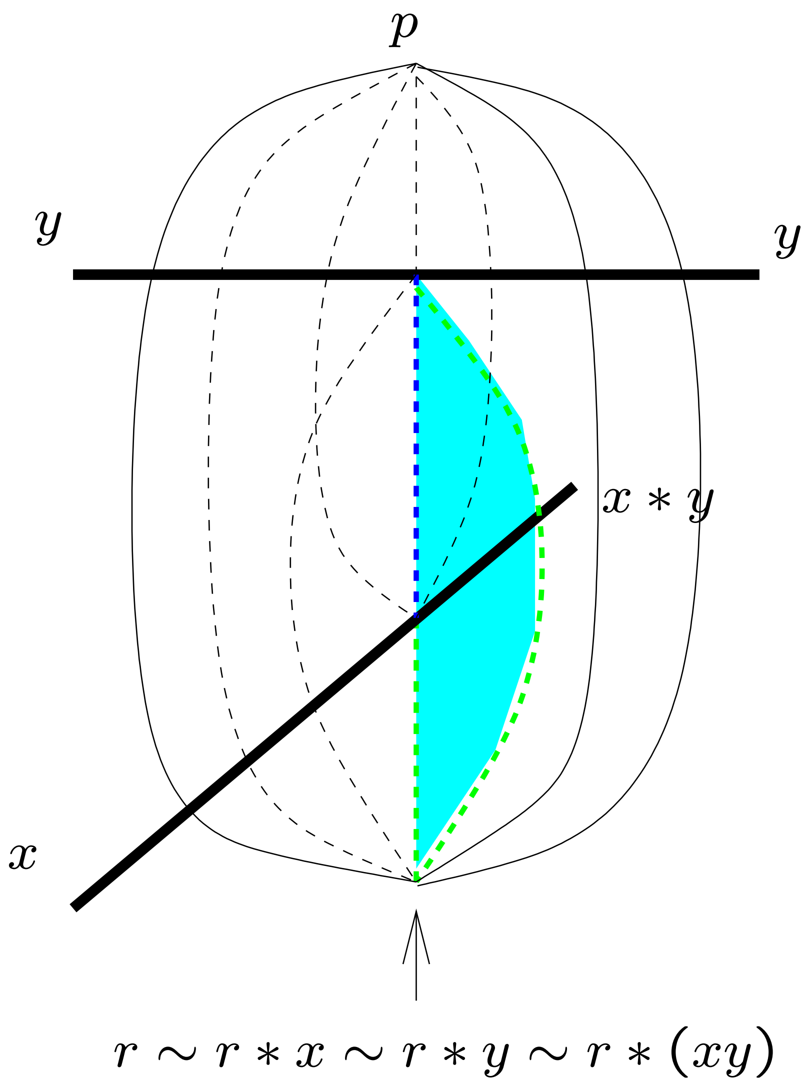

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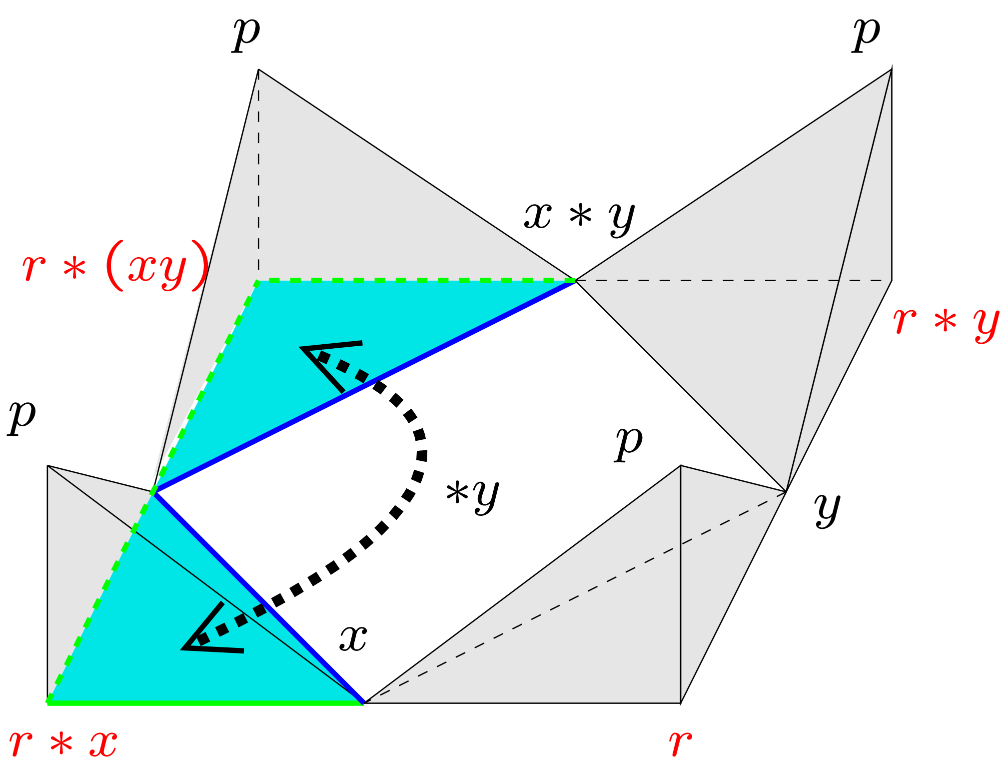

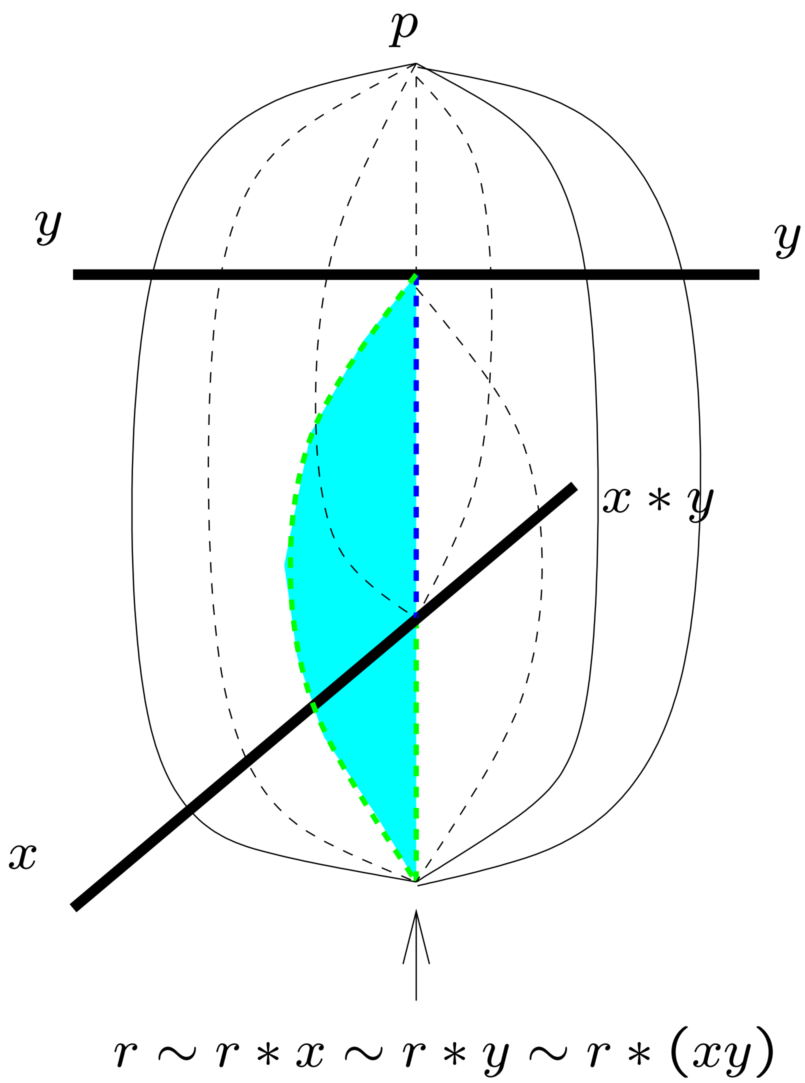

The map φ induces a homomorphism

$$H^R_{n+1}(X;\mathbb{Z}) \cong H^R_n(X;\mathbb{Z}[X]) \to H^\Delta_{n+1}(X).$$

Roughly, we will construct a map

$$H^\Delta_{n+1}(X) \to H_{n+1}(\mathrm{Aut}(X);\mathbb{Z})$$

where $\mathrm{Aut}(X) = \{f : X \to X \mid f(x*y) = f(x)*f(y)\}$ is a group by composition.

Dually, we can construct a quandle cocycle from a group cocycle.

Quandle Conj(h)

G : group, Fix an element $h \in G$.

$$\mathrm{Conj}(h) = \{g^{-1}hg \mid g \in G\}$$

has a quandle structure by $x * y = y^{-1}xy$. Let

$$Z(h) = \{g \in G \mid gh = hg\}. \quad \text{(centralizer)}$$

Lemma *As a set* $\mathrm{Conj}(h) \cong Z(h)\backslash G$ *by*

$$g^{-1}hg \leftrightarrow Z(h)g \quad \textit{(right coset).}$$

Remark *Conversely, under some conditions, we can find such G from a quandle X, e.g. $G = \mathrm{Aut}(X)$. Then we have*

$$X \cong Z(h)\backslash G = \mathrm{Stab}(x)\backslash\mathrm{Aut}(X).$$

The quandle structure on Conj(h) induces a quandle operation on $Z(h)\backslash G$, which is given by

$$Z(h)h_1 * Z(h)h_2 = Z(h)g_1(g_2^{-1}hg_2).$$

Define $* : G \times G \to G$ by

$$g_1 * g_2 := h^{-1}g_1(g_2^{-1}hg_2) \quad (g_1, g_2 \in G).$$

The projection $\pi : G \to Z(h)\backslash G$ is a quandle homomorphism.

Let $s : Z(h)\backslash G \to G$ be a section of π ($\pi \circ s = \mathrm{Id}$). For simplicity, denote Conj(h) by X. Define $c : X \times X \to Z(h)$ by

$$s(x * y) = c(x, y)(s(x) * s(y)).$$

Lemma *If $Z(h)$ is abelian, $c : X \times X \to Z(h)$ is a quandle 2-cocycle. If c is cohomologous to zero, we can modify the section s so that $s(x * y) = s(x) * s(y)$.*

If $h^l = 1$ for some integer $l > 1$, $s : X \to G$ induces a chain map $C_n^{\Delta}(X) \otimes_{\mathsf{Ad}(X)} \mathbb{Z} \to \mathbb{Z} \otimes_G C_n(G)$ by

$$(x_0, x_1, \ldots, x_n) \mapsto \sum_{i=0}^{l-1} (h^i s(x_0), h^i s(x_1), \ldots, h^i s(x_n)).$$

Thus we obtain a homomorphism

$$H_{n+1}^R(X; \mathbb{Z}) \cong H_n^R(X; \mathbb{Z}[X]) \xrightarrow{\varphi_*} H_{n+1}^{\Delta}(X) \to H_{n+1}(G; \mathbb{Z}).$$

There are examples that this map is non-trivial.

Example: dihedral group D_{2p} $(p > 2)$

$G = D_{2p} = \langle h, x \mid h^2 = x^p = hxhx = 1 \rangle$: dihedral group $(p > 2)$

Then

$$\mathrm{Conj}(h) = \{x^{-i} h x^i \mid i = 0, 1, \ldots, p-1\}$$

If we regard $\mathrm{Conj}(h)$ as $\mathbb{Z}/p\mathbb{Z}$, we have

$$i * j \equiv 2j - i \mod p.$$

This is called the *dihedral quandle*. For simplicity, assume p is prime. Since

$$H^3(D_{2p}; \mathbb{F}_p) \cong H^3(\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p)^{\mathbb{Z}/2\mathbb{Z}},$$

we will construct a quandle 3-cocycle from a group 3-cocycle of $\mathbb{Z}/p\mathbb{Z}$.

$H^3(\mathbb{Z}/p\mathbb{Z};\mathbb{F}_p)^{\mathbb{Z}/2\mathbb{Z}}$ is generated by $[x|y|z] \mapsto x \cdot d(y,z)$ where

$$d(y,z) = \begin{cases} 1 & \text{if } \bar{y} + \bar{z} > p \\ -1 & \text{if } \bar{y} + \bar{z} < p \text{ and } yz \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{x}$ is an integer $0 \leq \bar{x} < p$ with $x = \bar{x} \mod p$.

Prop *The quandle 3-cocycle obtained from this cocycle is given by*

$$(x,y,z) \mapsto 2z(d(y-x,z-y) + d(y-x,y-z)).$$

This is a non-trivial cocycle.

Remark *The non-triviality follows from the fact that the image of φ gives a cycle represented by a cyclic branched covering of S^3 branched along a knot.*

Thank you