Parametrization of PSL(n,C)-representations of surface group I, II

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Outline

S : a compact orientable surface (genus g, $|\partial S| = b$, $\chi(S) < 0$)

$X_{PSL}(S)$: the PSL$(2, \mathbb{C})$-character variety of S

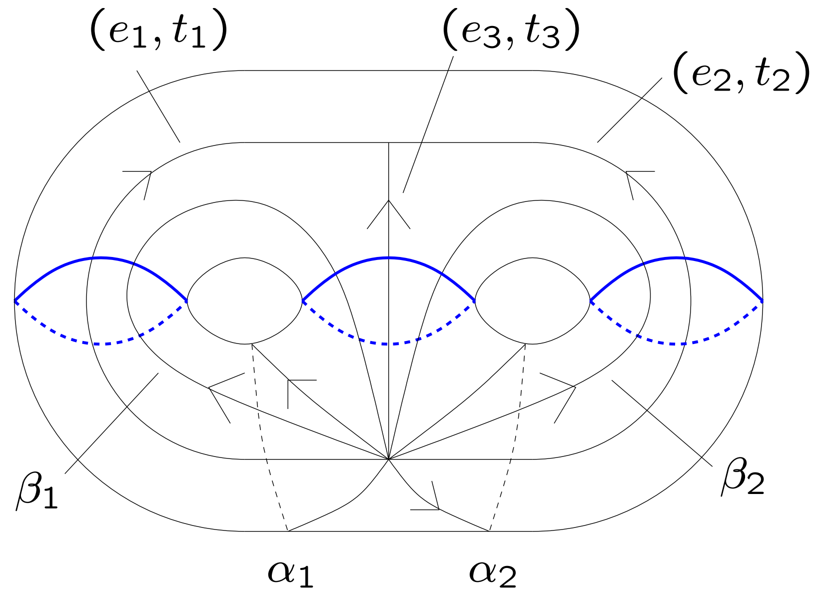
We will construct a rational map

$$\mathbb{C}^{6g-6+2b} \to X_{PSL}(S)$$

generically 2^{4g-3+b} to 1 as an analogue of the (complex) Fenchel-Nielsen coordinates. – Part I

We generalize this construction to PGL$(n, \mathbb{C})$-representations using Fock-Goncharov's work (joint with Xin Nie). – Part II

The construction is quite explicit. Actually for a closed surface of genus 2, we have six parameters e_1, e_2, e_3, t_1, t_2, t_3 and the rep. is given by



$$\rho(\alpha_1) = \begin{pmatrix} e_1^{-1} & 0 \\ -e_1 + e_2^{-1} e_3 & e_1 \end{pmatrix}, \quad \rho(\alpha_2) = \begin{pmatrix} e_1 e_3^{-1} & e_2 - e_1 e_3^{-1} \\ -e_2^{-1} + e_1 e_3^{-1} & e_2 + e_2^{-1} - e_1 e_3^{-1} \end{pmatrix},$$

$$\rho(\beta_1) = \frac{1}{\sqrt{t_1 t_3}} \begin{pmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \rho(\beta_2) = \frac{1}{(e_2{}^2 - 1) e_3 \sqrt{t_2 t_3}} \begin{pmatrix} (e_1 e_2 - e_3) t_2 - e_2 (e_2 e_3 - e_1) & b_{12} \\ (e_1 e_2 - e_3)(t_2 + 1) & b_{22} \end{pmatrix},$$

$$a_{12} = -\frac{(e_2 e_3 - e_1)(t_3 + 1)}{e_1 (e_3{}^2 - 1)}, \quad a_{21} = \frac{e_1 (t_1 + 1)(e_1 e_2 - e_3)}{(e_1{}^2 - 1) e_2},$$

$$a_{22} = \frac{(e_1 e_2 e_3 - 1)(e_1 e_3 - e_2) t_1 t_3 - (e_1 e_2 - e_3)(e_2 e_3 - e_1)(t_1 + t_3 + 1)}{(e_1{}^2 - 1) e_2 (e_3{}^2 - 1)},$$

$$b_{12} = -(e_2 e_3 - e_1)(e_3 (e_1 e_2 e_3 - 1) t_2 t_3 + (e_3 - e_1 e_2) t_2$$
$$+ e_2 e_3 (e_2 - e_1 e_3) t_3 - e_2 (e_1 - e_2 e_3)) / (e_1 (e_3{}^2 - 1)),$$

$$b_{22} = -(e_3 (e_1 e_2 e_3 - 1)(e_2 e_3 - e_1) t_2 t_3 - e_3 (e_1 e_2 - e_3)(e_1 e_3 - e_2) t_3$$
$$+ (e_1 e_2 - e_3)(e_2 e_3 - e_1)(1 + t_2)) / (e_1 (e_3{}^2 - 1)).$$

Introduction

$\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$ is

- isomorphic to $\mathrm{Isom}^+(\mathbb{H}^3)$ (orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$),
- the group of conformal transformations of $\mathbb{C}P^1$.

So $\mathrm{PSL}(2,\mathbb{C})$-representations of surface groups are important in the study of

- Kleinian surface groups,
- $\mathbb{C}P^1$-structures on a surface,
- Teichmüller space ($\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$), stable holomorphic rank 2 vector bundles ($\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$).

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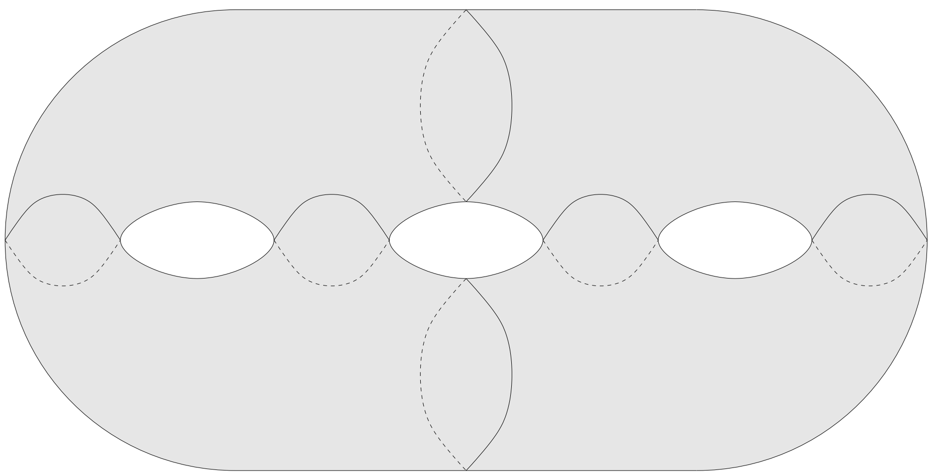
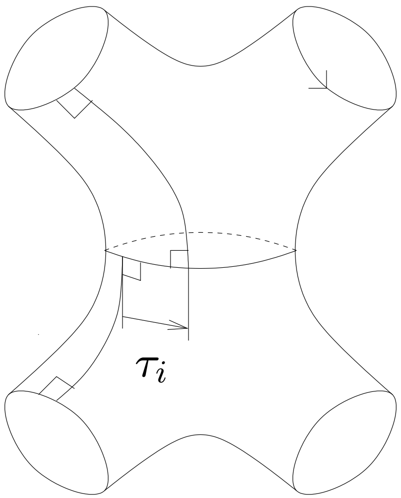
Introduction

Fenchel-Nielsen coordinates

$\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$: discrete faithful	$\longleftrightarrow$	hyperbolic metric on S

Give a pants decomposition. Length and twist parameters give coordinates for these representations.





$$\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$$
$$\Cup$$
$$\{(l_i, \tau_i)\}$$

Length l_i of each scc

Twist parameter τ_i

Introduction

We will construct an analogue of F-N coordinates for $\mathrm{PSL}(2,\mathbb{C})$-reps using eigenvalues instead of length (or trace) functions.

Remark

The complex F-N coordinates for quasi-Fuchsian representations have already been defined by Tan, Kourouniotis.

Advantages of our coordinates

- Cover a much larger class of $\mathrm{PSL}(2,\mathbb{C})$-representations,
- Only use elementary facts on 2×2 matrices,
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Basics of PSL$(2,\mathbb{C})$

- SL$(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\}$
- PSL$(2,\mathbb{C}) =$ SL$(2,\mathbb{C})/\{\pm I\}$
- PGL$(2,\mathbb{C}) \cong$ PSL$(2,\mathbb{C})$ by $A \mapsto \frac{1}{\sqrt{\det A}} A$
- PSL$(2,\mathbb{C})$ acts on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ by linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

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Lem A

(x_1, x_2, x_3) : distinct 3 points of $\mathbb{C}P^1$

(x'_1, x'_2, x'_3) : other distinct 3 points of $\mathbb{C}P^1$

There exists a unique $A \in \mathrm{PSL}(2, \mathbb{C})$ s.t. $A \cdot x_i = x'_i$.

In fact, A is explicitly given by

$$A = \frac{1}{\sqrt{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)(x'_1 - x'_2)(x'_2 - x'_3)(x'_3 - x'_1)}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

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Proof of Lem A

Consider a linear fractional transformation which sends (x_1, x_2, x_3) to $(\infty, 0, 1)$. This is given by

$$z \longmapsto \frac{z - x_2}{z - x_1} \cdot \frac{x_3 - x_1}{x_3 - x_2}$$

uniquely. Thus the matrix is given by

$$\begin{pmatrix} (x_3 - x_1) & -(x_3 - x_1)x_2 \\ (x_3 - x_2) & -(x_3 - x_2)x_1 \end{pmatrix}$$

For general case, consider the composition $A_2^{-1}A_1$:

$$(x_1, x_2, x_3) \xrightarrow[A_1]{\cong} (\infty, 0, 1) \xleftarrow[A_2]{\cong} (x'_1, x'_2, x'_3) \quad \square$$

Fact

$A \in \mathsf{SL}(2, \mathbb{C})$

$e \in \mathbb{C} \setminus \{0, \pm 1\}$: one of the eigenvalues of A

$x, y \in \mathbb{C}P^1$: fixed points of A. (Assume $x \neq y$.)

Suppose x corresponds to the eigenvector of e.

(Equivalently, assume x is the attractive fixed point if $|e| > 1$.)

Then A is uniquely determined by e, x, y.

$$\begin{aligned} A &= \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{x-y} \begin{pmatrix} ex - e^{-1}y & -(e-e^{-1})xy \\ e - e^{-1} & -ey + e^{-1}x \end{pmatrix} \\ &=: M(e; x, y) \end{aligned}$$

E.g.

- $M(e;\infty,0) = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$.
  As a linear fractional transformation $t \mapsto e^2 t$.
- $M(e;0,\infty) = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}$.
  As a linear fractional transformation $t \mapsto e^{-2} t$.

$$\left(M(e;x,y) = \frac{1}{x-y} \begin{pmatrix} ex - e^{-1}y & -(e-e^{-1})xy \\ e - e^{-1} & -ey + e^{-1}x \end{pmatrix} \right)$$

Lem B

x, y : distinct points on $\mathbb{C}P^1$.

z_1, z_2 : points on $\mathbb{C}P^1$ different from x and y. (z_1 and z_2 may coincide.)

Then there exists a unique $t \in \mathbb{C}^*$ up to sign such that $M(t; x, y)$ sends z_1 to z_2.

Proof Since

$$M(t; x, y) \cdot z_1 = \frac{(tx - t^{-1}y)z_1 - (t - t^{-1})xy}{(t - t^{-1})z_1 - ty + t^{-1}x} = z_2,$$

we have

$$t^2 = \frac{(x - z_1)(y - z_2)}{(x - z_2)(y - z_1)} = [y : x : z_1 : z_2].$$

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SL$(2,\mathbb{C})$- and PSL$(2,\mathbb{C})$- character variety

- S : a compact orientable surface
- $\rho : \pi_1(S) \to$ SL$(2,\mathbb{C})$ (or PSL$(2,\mathbb{C})$) is reducible if $\rho(\pi_1(S))$ fixes a point on $\mathbb{C}P^1$. Otherwise ρ is called irreducible.
- SL$(2,\mathbb{C})$ acts on SL$(2,\mathbb{C})$-representations by conjugation.
- $\{\rho : \pi_1(S) \to \mathrm{SL}(2,\mathbb{C}) \mid \text{irred. reps}\}/\sim_{\text{conj}}$ can be regarded as a subset of the SL$(2,\mathbb{C})$-character variety $X_{SL}(S)$.
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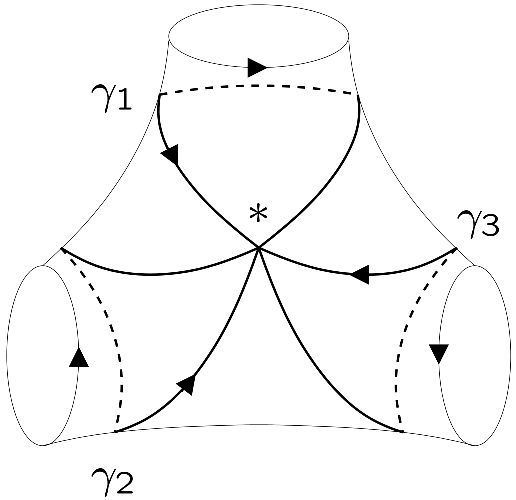
Representations of $\pi_1(P)$

Let P be a pair of pants (3 holed sphere).

Take generators γ_i of $\pi_1(P)$ as:



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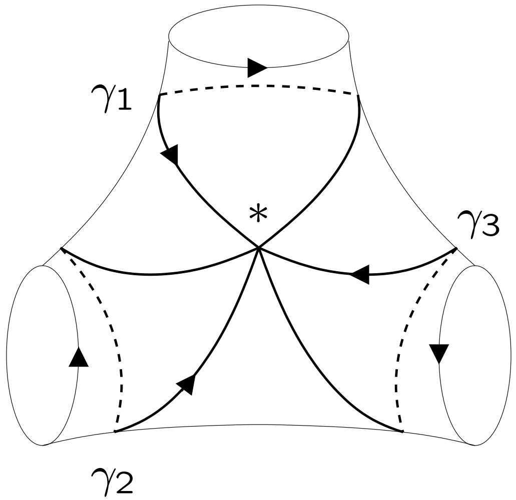

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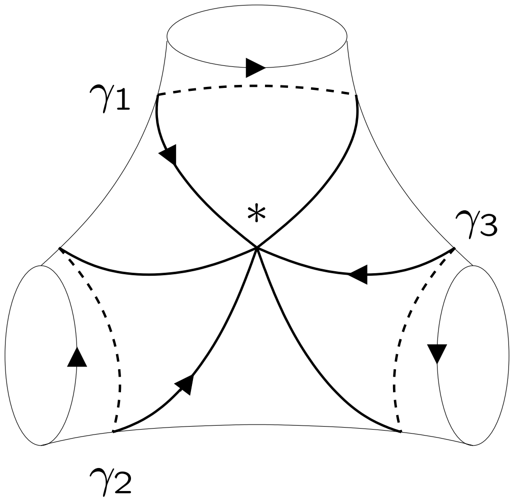

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Proof of Prop 1

Assume that $(x_1, x_2, x_3) = (0, \infty, 1)$. Then $\rho(\gamma_i)$ are uniquely determined by **Fact** $\left(M(e; x, y) = \frac{1}{x-y}\begin{pmatrix} ex - e^{-1}y & -(e-e^{-1})xy \\ e - e^{-1} & -ey + e^{-1}x \end{pmatrix}\right)$

$$\rho(\gamma_1) = \begin{pmatrix} e_1^{-1} & 0 \\ \frac{e_1^{-1}-e_1}{y_1} & e_1 \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} e_2 & (e_2^{-1} - e_2)y_2 \\ 0 & e_2^{-1} \end{pmatrix},$$

$$\rho(\gamma_3) = \frac{1}{y_3 - 1}\begin{pmatrix} e_3^{-1}y_3 - e_3 & (e_3 - e_3^{-1})y_3 \\ e_3^{-1} - e_3 & e_3y_3 - e_3^{-1} \end{pmatrix}.$$

From the identity $\rho(\gamma_1)\rho(\gamma_2) = \rho(\gamma_3)^{-1}$, we have

$$y_1 = \frac{e_1 - e_1^{-1}}{e_2^{-1}e_3 - e_1^{-1}}, \quad y_2 = \frac{e_2 - e_1e_3^{-1}}{e_2 - e_2^{-1}}, \quad y_3 = \frac{e_2 - e_1e_3^{-1}}{e_2 - e_1e_3}.$$

For general case, use **Lem A**.

Proof of Prop 1

Conversely, we can show that the above ρ is a homomorphism for any $e_i \neq 0$ and distinct triple x_1, x_2, x_3.

To make sure that ρ is irreducible, we have to check that y_i's are different from x_i's and distinct each other.

We can show that

$$e_1 = e_2 e_3 \text{ iff } x_1 = y_2 = y_3,$$

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- ρ is completely determined (not up to conjugacy) by e_i and x_i $(i = 1, 2, 3)$.

- The conjugacy class of ρ is determined by e_i $(i = 1, 2, 3)$, since it is determined by $\mathrm{tr}(\gamma_i) = e_i + e_i^{-1}$.

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$$(e_1, e_2, e_3) \mapsto (\varepsilon_1 e_1, \varepsilon_2 e_2 . \varepsilon_3 e_3)$$

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Observation

Let

$$\mathbb{C}^3 \supset E = \{(e_1, e_2, e_3) \mid e_i \neq 0, \pm 1, \\ e_1^{s_1} e_2^{s_2} e_3^{s_3} \neq 1 \text{ for any } s_i = \pm 1\}.$$

Then we have

$$E/(\mathbb{Z}_2)^3 \xrightarrow{\text{injective}} X_{SL}(P)$$

where $(\mathbb{Z}_2)^3$ acts on E as $e_i \mapsto e_i{}^{-1}$. We also have

$$(E/(\mathbb{Z}_2)^3)/(\mathbb{Z}_2)^2 \xrightarrow{\text{injective}} X_{PSL}(P)$$

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Observation

Let $X = \{(x_1, x_2, x_3) \mid x_i \neq x_j (i \neq j)\} \subset \mathbb{C}^3$, then

$$
\begin{array}{ccc}
E \times X & \xrightarrow{\text{inj}} & \mathrm{Hom}(\pi_1(P), \mathrm{SL}(2, \mathbb{C})) \\
\downarrow \text{pr} & & \\
E & & \downarrow \\
\downarrow & & \\
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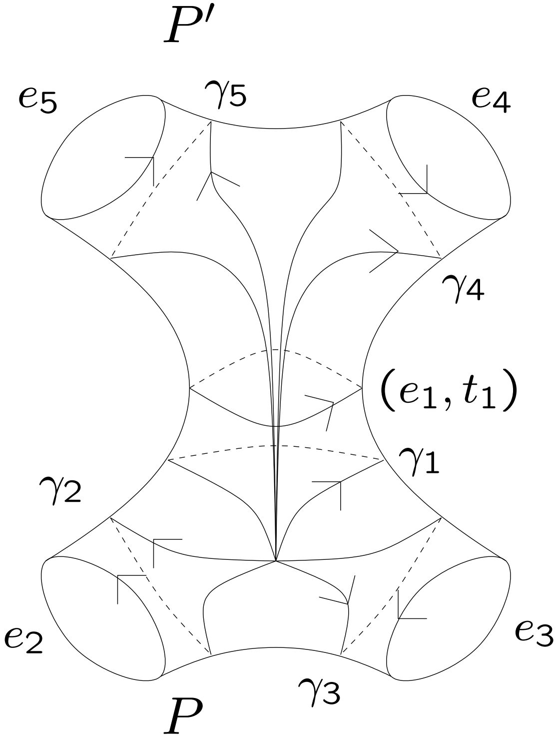
The images of the horizontal maps (irreducible representations s.t. $\rho(\gamma_i)$'s have two fixed points) are open and dense.

Twist parameter

$S = P \cup P'$: a four holed sphere

$\rho : \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$: a rep whose restrictions to $\pi_1(P)$ and $\pi_1(P')$ satisfies (i), (ii).



Let e_i be one of the eigenvalues of $\rho(\gamma_i)$ for $i = 1, \ldots, 5$ and x_i (resp. y_i) be the fixed point corresponding to e_i (resp. e_i^{-1}).

By **Lem B**, there exists a unique t_1 satisfying

$$M(\sqrt{-t_1}; x_1, y_1) \cdot x_2 = x_5.$$

We call t_1 the twist parameter. To define t_1, we assign an (oriented) graph dual to the pants decomposition.

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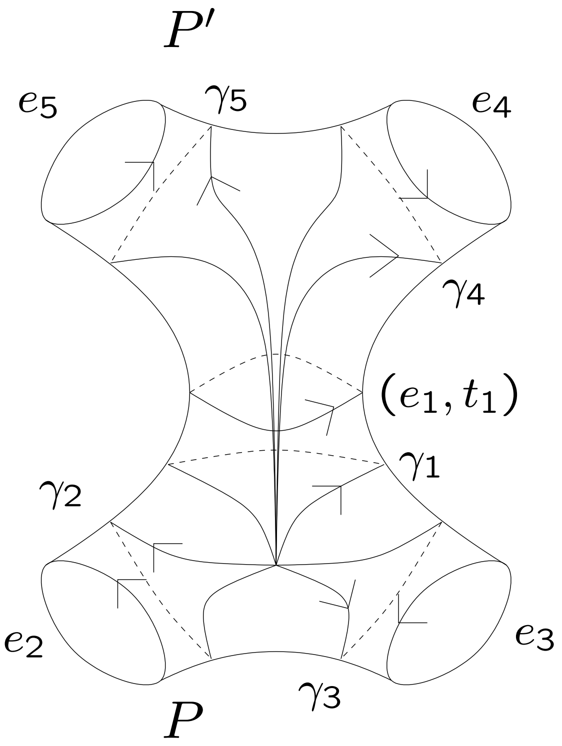

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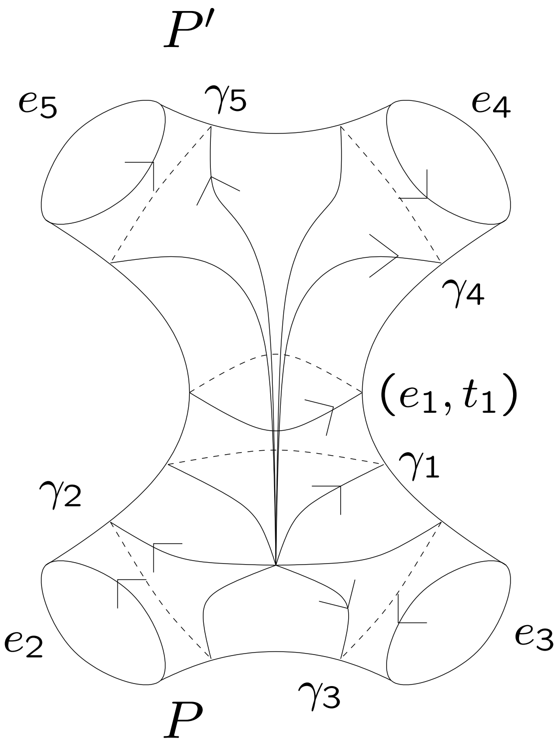

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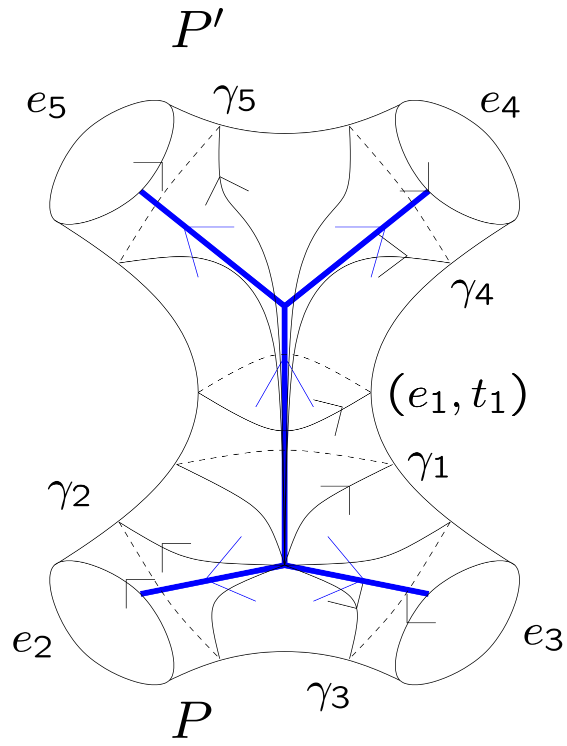

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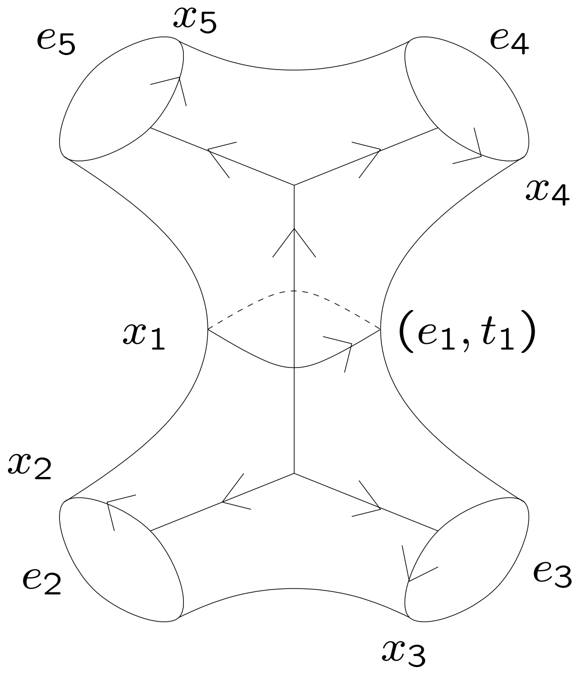
Remark

Once we fix choices of the eigenvalues $e_1, \ldots, e_5$, the twist parameter t_1 is a conjugacy invariant. It is also well-defined for $\mathrm{PSL}(2, \mathbb{C})$-representations.

Prop 2



e_i : one of the eigenvalues of $\rho(\gamma_i)$

x_i : the fixed point corresponding to e_i

t_1 : twist parameter

Then we have

$$\begin{aligned} x_4 = \{&e_1(-(e_2 - e_1e_3)(e_5 - e_1e_4)t_1 + e_3(e_1e_5 - e_4))x_1(x_2 - x_3) \\ &+{e_1}^2e_2(e_1e_5 - e_4)x_2(x_3 - x_1) + e_2(e_1e_5 - e_4)x_3(x_1 - x_2)\}/ \\ &\{e_1(-(e_2 - e_1e_3)(e_5 - e_1e_4)t_1 + e_3(e_1e_5 - e_4))(x_2 - x_3) \\ &+{e_1}^2e_2(e_1e_5 - e_4)(x_3 - x_1) + e_2(e_1e_5 - e_4)(x_1 - x_2)\} \end{aligned}$$

$$x_5 = \frac{(-(e_2 - e_1e_3)t_1 + e_1e_3)x_1(x_2 - x_3) + {e_1}^2e_2x_2(x_3 - x_1) + e_2x_3(x_1 - x_2)}{(-(e_2 - e_1e_3)t_1 + e_1e_3)(x_2 - x_3) + {e_1}^2e_2(x_3 - x_1) + e_2(x_1 - x_2)}$$

This means that x_4 and x_5 are uniquely determined by $x_1, x_2, x_3 \in \mathbb{C}P^1$, $e_1, \ldots, e_5$ and t_1.

Remark

- Conversely x_2 and x_3 are determined by $x_1, x_4, x_5 \in \mathbb{C}P^1$, $e_1, \ldots, e_5$ and t_1:

$$x_2 = \frac{(-(e_4 - e_1e_5){t_1}^{-1} + e_1e_5)x_1(x_5 - x_4) + {e_1}^2e_4x_5(x_4 - x_1) + e_4x_4(x_1 - x_5)}{(-(e_4 - e_1e_5){t_1}^{-1} + e_1e_5)(x_5 - x_4) + {e_1}^2e_4(x_4 - x_1) + e_4(x_1 - x_5)}$$

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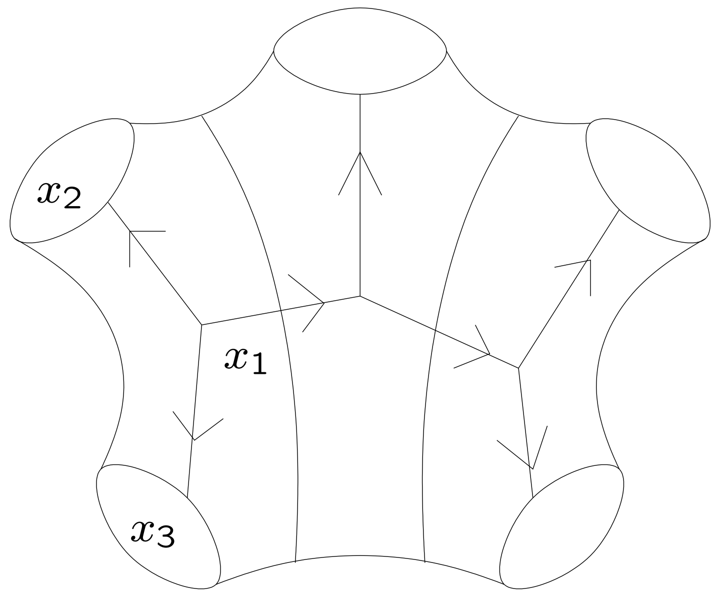
- From **Prop 1** and **Prop 2**, ρ is uniquely determined by $x_1, x_2, x_3 \in \mathbb{C}P^1$, $e_1, \ldots, e_5$ and t_1. The conjugacy class of ρ is uniquely determined by $e_1, \ldots, e_5$ and t_1.

Remark

- Moreover a conjugacy class of π_1(b-holed sphere) $\to$ SL$(2,\mathbb{C})$ is completely determined by these eigenvalue parameters e_i and twist parameters t_i by **Prop 1** and **Prop 2**.



(We assign (e_i, t_i) for each 'interior' edge and e_i for each 'boundary' edge of the dual graph.)

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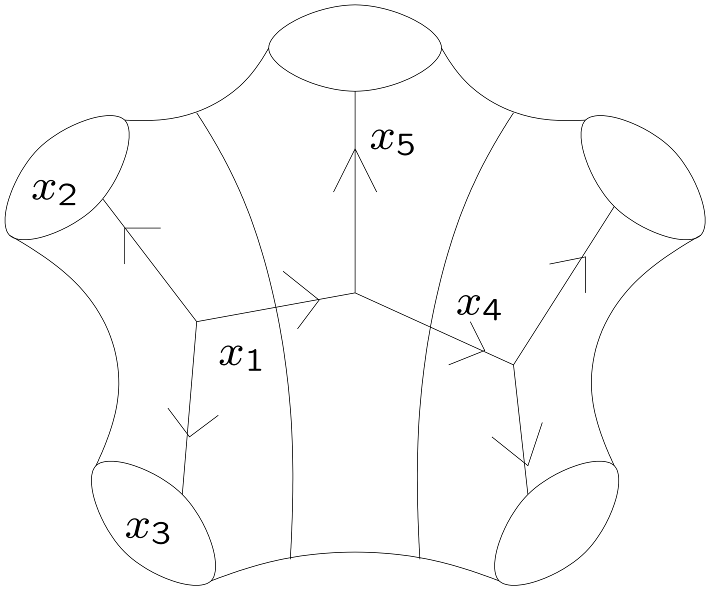

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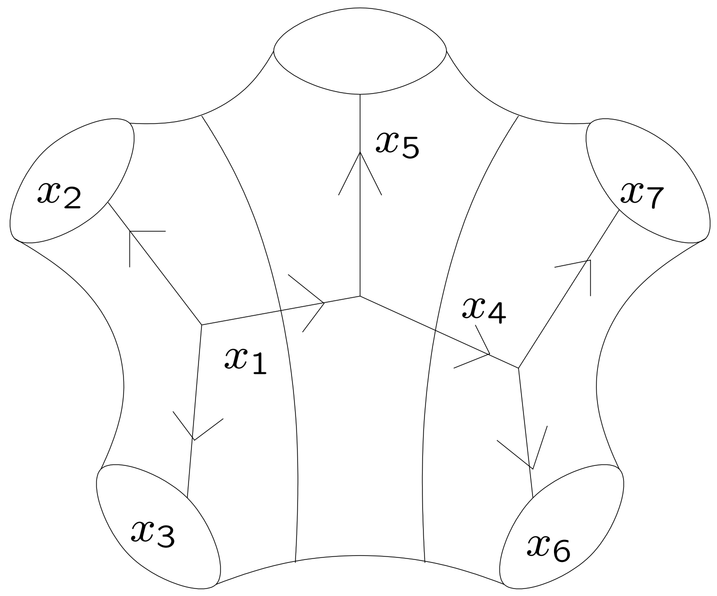
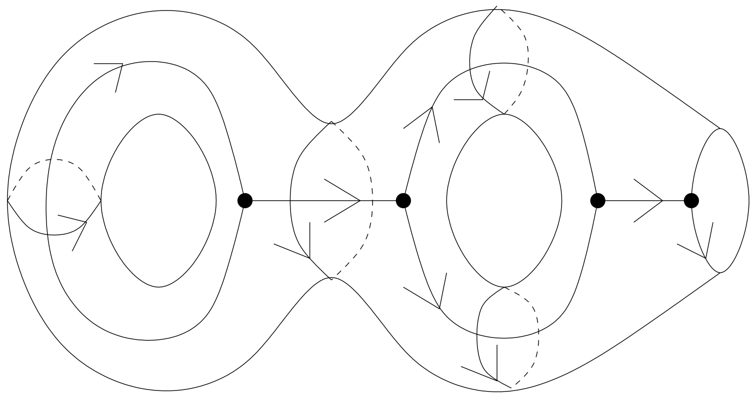
Remark

- Moreover a conjugacy class of π_1(b-holed sphere) $\to$ SL$(2, \mathbb{C})$ is completely determined by these eigenvalue parameters e_i and twist parameters t_i by **Prop 1** and **Prop 2**.



(We assign (e_i, t_i) for each ‘interior’ edge and e_i for each ‘boundary’ edge of the dual graph.)

Parametrization



S : a surface of genus g.

Given

$C \subset S$: a pants decomposition

(set of maximal disjoint scc.)

G : an (oriented) graph dual to C

We will parametrize the reps into PSL$(2, \mathbb{C})$ satisfying,

(i) $\rho(c)$ has two fixed points for each scc $c \subset C$,

(ii) restriction to each pair of pants is irreducible.

Parameters

Assign an eigenvalue parameter e_i and a twist parameter t_i to each edge of G.

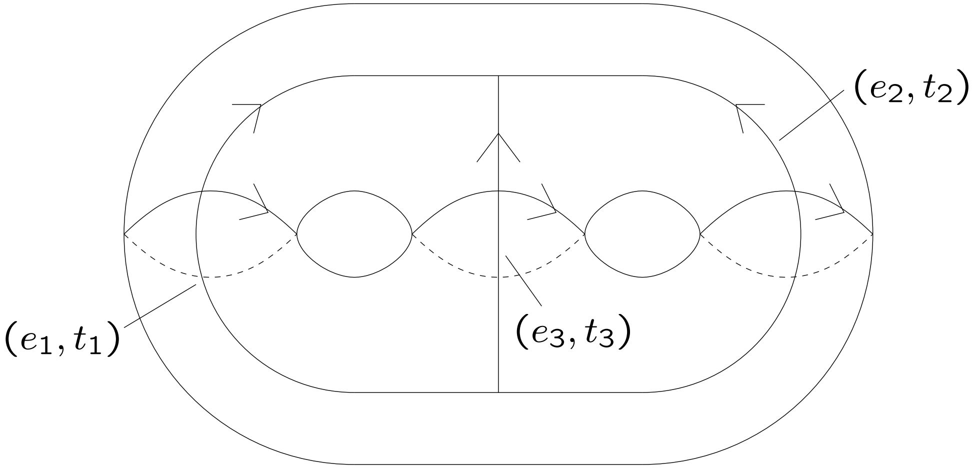
Let $\mathcal{P}$ be the set of triples of scc's of C which are the boundary of a component of $S \setminus C$. We let

$$E(S,C) = \{(e_1, \ldots, e_{3g-3}) \,|e_i \neq 0, \pm 1, \\ e_i^{s_1} e_j^{s_2} e_k^{s_3} \neq 1 \text{ for } (i,j,k) \in \mathcal{P}\}.$$

We will reconstruct a representation from $(e_1, \ldots, e_{3g-3}) \in E(S,C)$ and $t_i \in \mathbb{C} \setminus \{0\}$.

Matrix generators

For a dual graph, we give a presentation of $\pi_1(S)$.



Matrix generators

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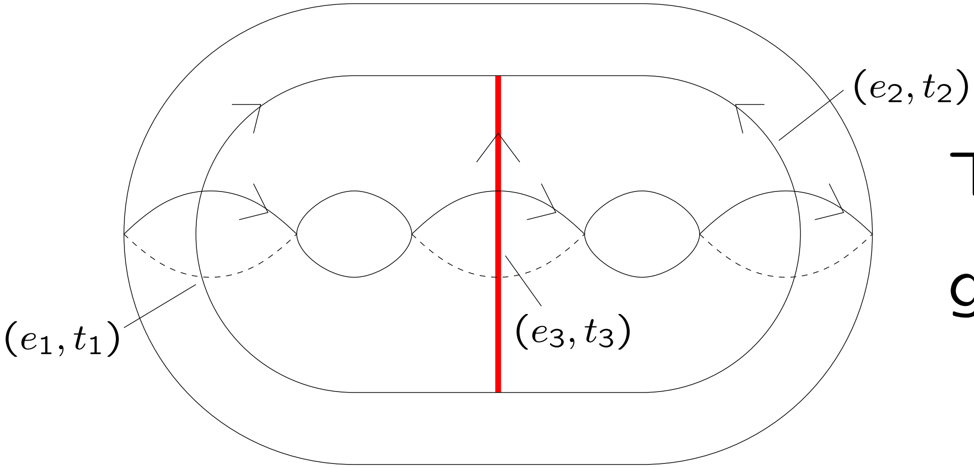
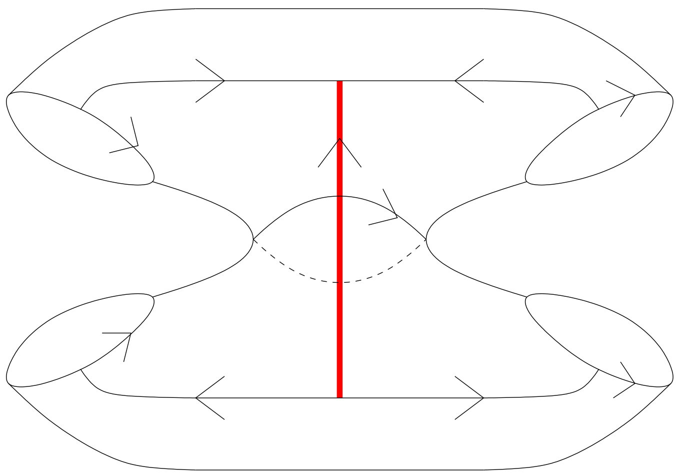


Take a maximal tree T in the dual graph G.

Matrix generators

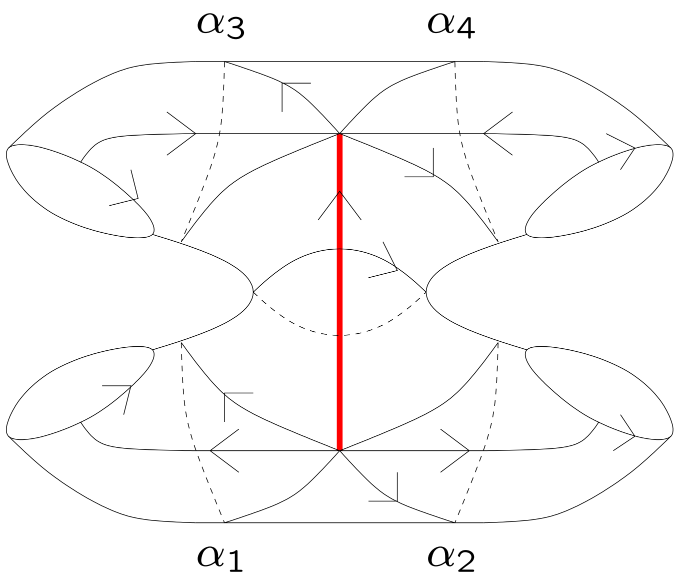
For a dual graph, we give a presentation of $\pi_1(S)$.



Cut S along the scc's corresponding to the edges $G \setminus T$, we obtain a $2g$-holed sphere S_0.

Matrix generators

For a dual graph, we give a presentation of $\pi_1(S)$.



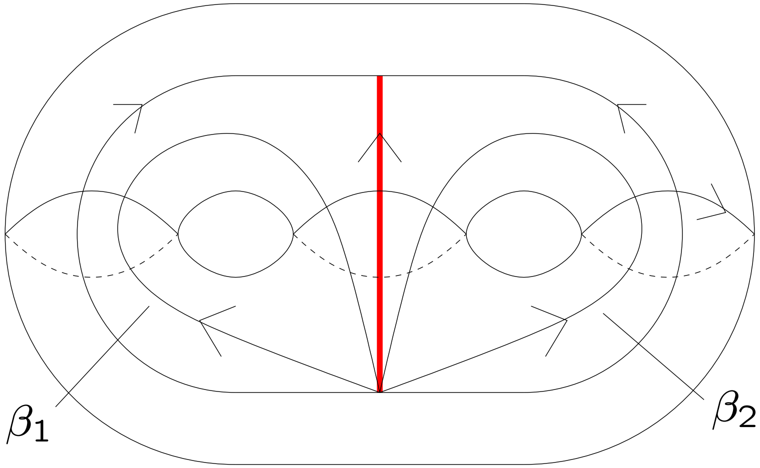
Take $\alpha_i, \alpha_{i+g} \in \pi_1(S_0)$ for each edge of $G \setminus T$. They satisfy

$$\alpha_{i_1} \ldots \alpha_{i_{2g}} = 1.$$

(Eg. $\alpha_3\alpha_1\alpha_2\alpha_4 = 1$ on the left Figure)

Matrix generators

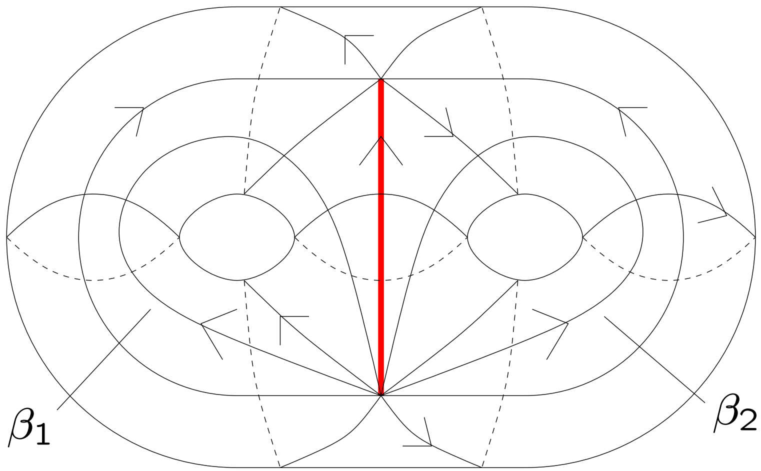
For a dual graph, we give a presentation of $\pi_1(S)$.



Take $\beta_1, \ldots, \beta_g \in \pi_1(S)$ for each edge of $G \setminus T$.

Matrix generators

For a dual graph, we give a presentation of $\pi_1(S)$.



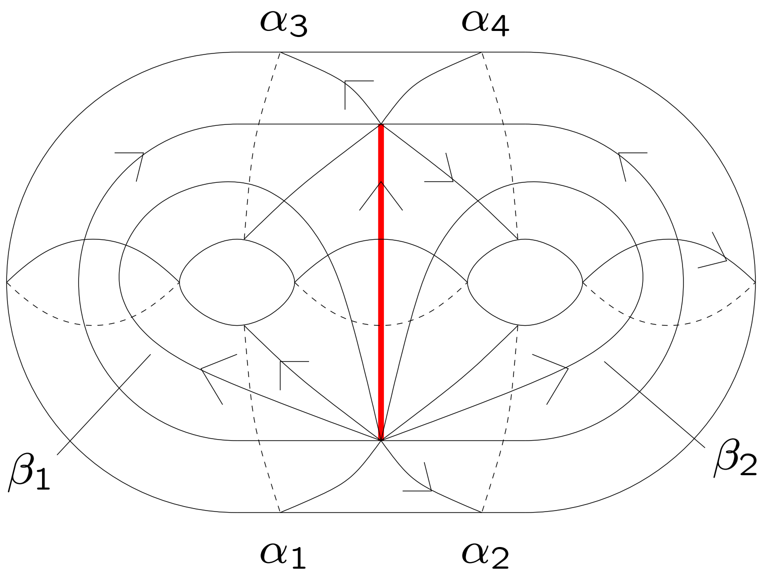
$\pi_1(S)$ has the following presentation:

$$\langle \alpha_1, \ldots, \alpha_2 g, \beta_1 \ldots \beta_g \mid \alpha_{i_1} \ldots \alpha_{i_{2g}} = 1, \quad {\alpha_{g+i}}^{-1} = {\beta_i}^{-1} \alpha_i \beta_i \rangle$$

$$= \langle \alpha_1, \ldots, \alpha_g, \beta_1 \ldots, \beta_g \mid \prod_k^g [{\alpha_{i_k}}^{\pm 1}, {\beta_{i_k}}^{\pm 1}] = 1 \rangle$$

Matrix generators

For a dual graph, we give a presentation of $\pi_1(S)$.



$\pi_1(S)$ has the following presentation:

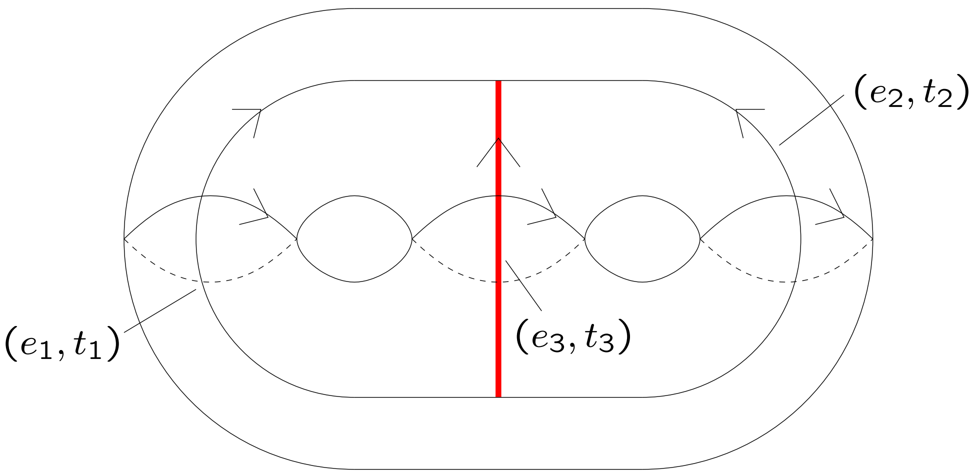
(Eg. on the above Figure,

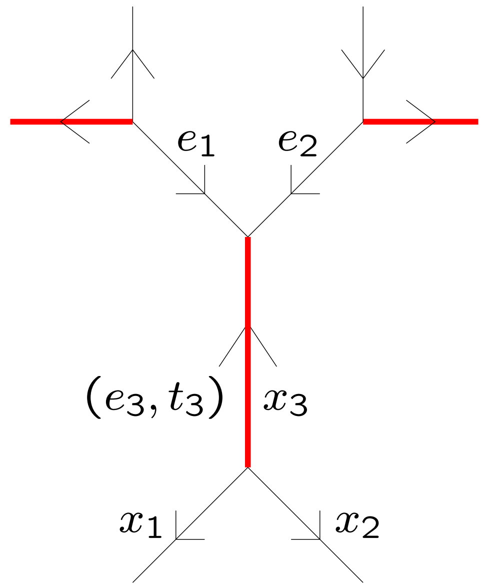
$$\langle \alpha_1, \ldots, \alpha_4, \beta_1, \beta_2 \mid \alpha_3\alpha_1\alpha_2\alpha_4 = 1, \quad {\alpha_3}^{-1} = {\beta_1}^{-1}\alpha_1\beta_1, \quad {\alpha_4}^{-1} = {\beta_2}^{-1}\alpha_2\beta_2 \rangle$$

$$= \langle \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [{\beta_1}^{-1}, \alpha_1^{-1}][\alpha_2, {\beta_2}^{-1}] = 1 \rangle \qquad)$$

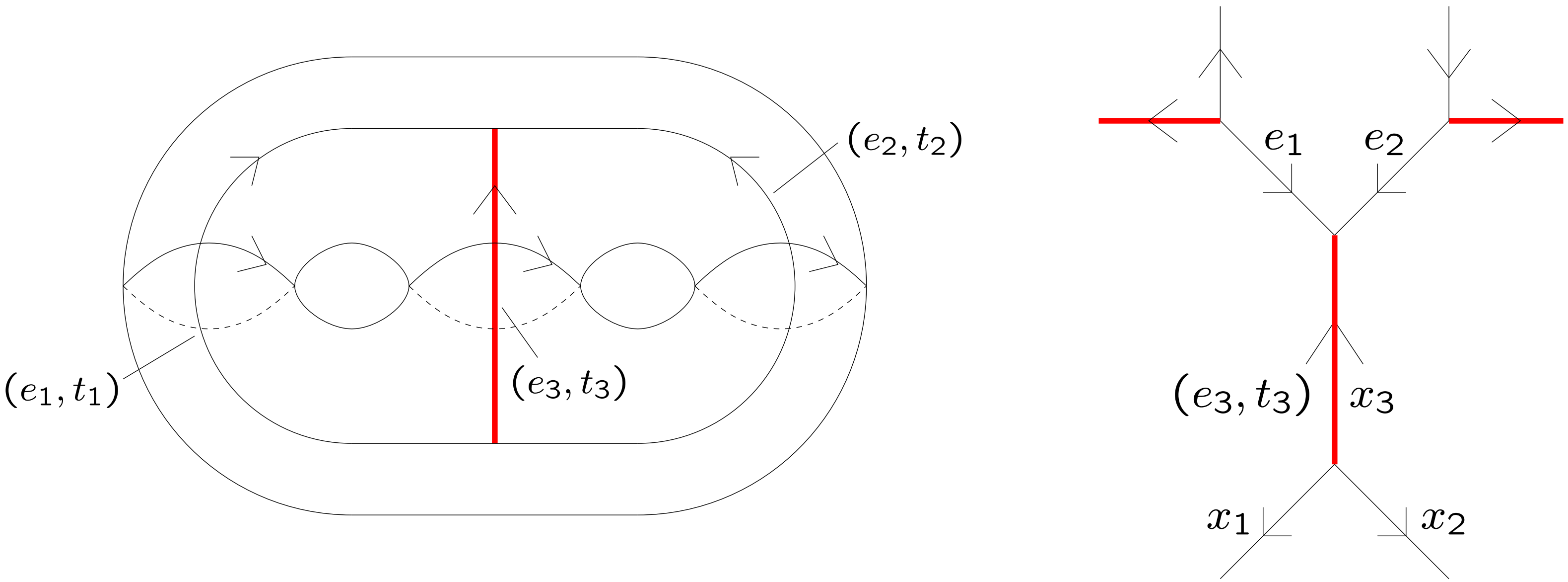
We give matrices corresponding to these generators.

Step 1 Compute sufficiently many number of fixed points for $\widetilde{G}$ (universal cover of G) by using **Prop 2**.

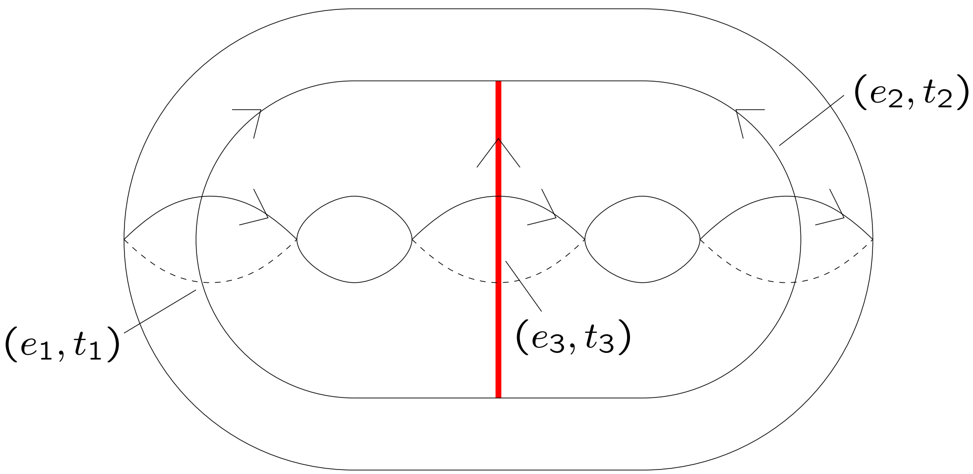


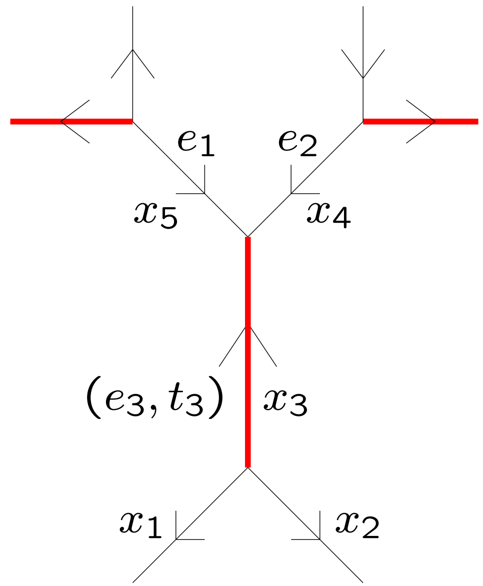


Step 1 Compute sufficiently many number of fixed points for $\widetilde{G}$ by using **Prop 2**.



Step 1 Compute sufficiently many number of fixed points for $\widetilde{G}$ by using **Prop 2**.





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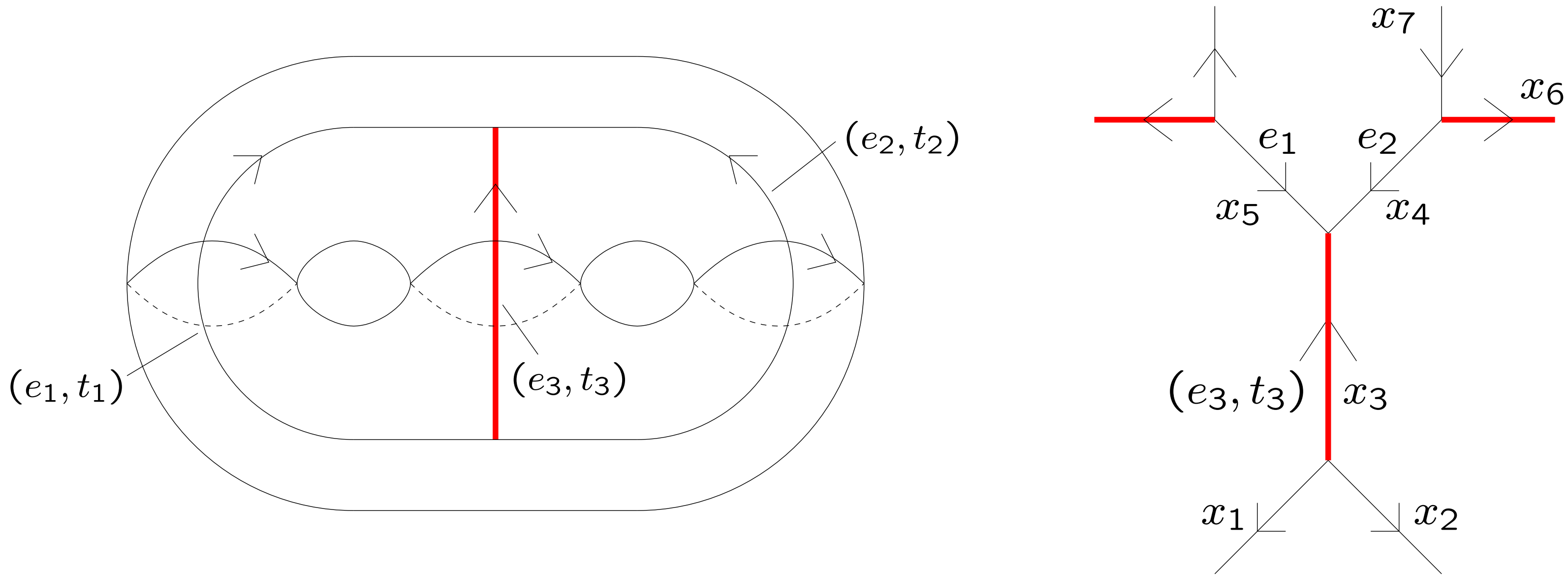

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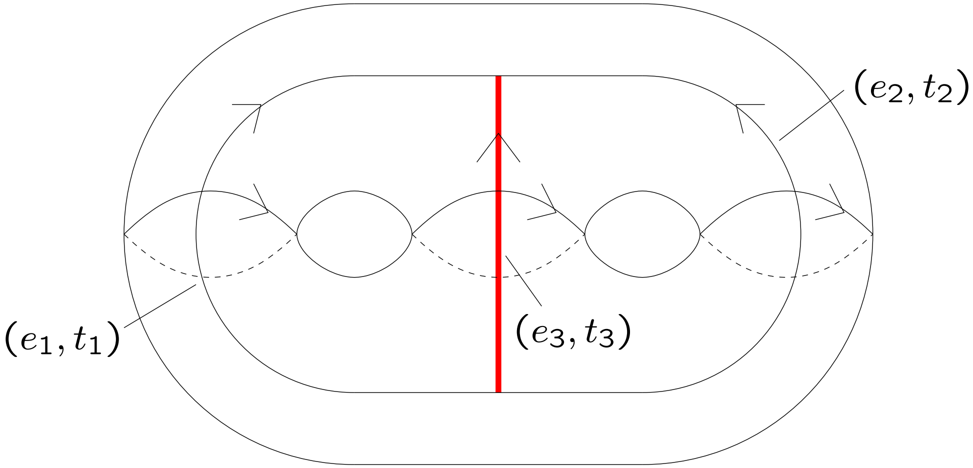
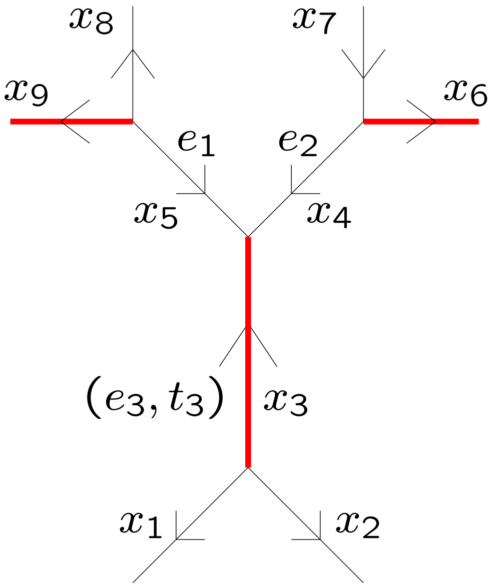




Step 2 Using **Prop 1**, compute the matrices for α_i's.

Recall **Prop 1**:

$$\rho(\gamma_i) = \frac{1}{e_i e_{i+2}(x_{i+1}-x_i)(x_{i+2}-x_i)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

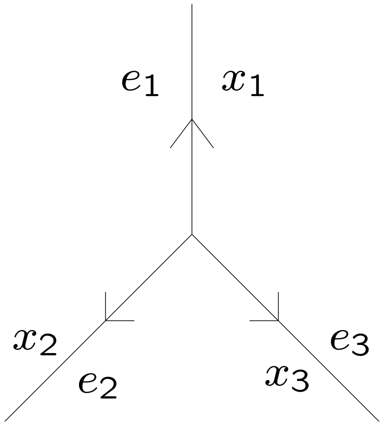
$$a_{11} = e_i^2 e_{i+2} x_i (x_i - x_{i+1}) + e_{i+2} x_{i+1}(x_{i+2} - x_i) + e_i e_{i+1} x_i (x_{i+1} - x_{i+2}),$$

$$a_{12} = x_i (e_i^2 e_{i+2} x_{i+2}(x_{i+1} - x_i) + e_{i+2} x_{i+1}(x_i - x_{i+2}) + e_i e_{i+1} x_i (x_{i+2} - x_{i+1})),$$

$$a_{21} = e_i^2 e_{i+2}(x_i - x_{i+1}) + e_{i+2}(x_{i+2} - x_i) + e_i e_{i+1}(x_{i+1} - x_{i+2}),$$

$$a_{22} = e_i^2 e_{i+2} x_{i+2}(x_{i+1} - x_i) + e_{i+2} x_i (x_i - x_{i+2}) + e_i e_{i+1} x_i (x_{i+2} - x_{i+1}),$$

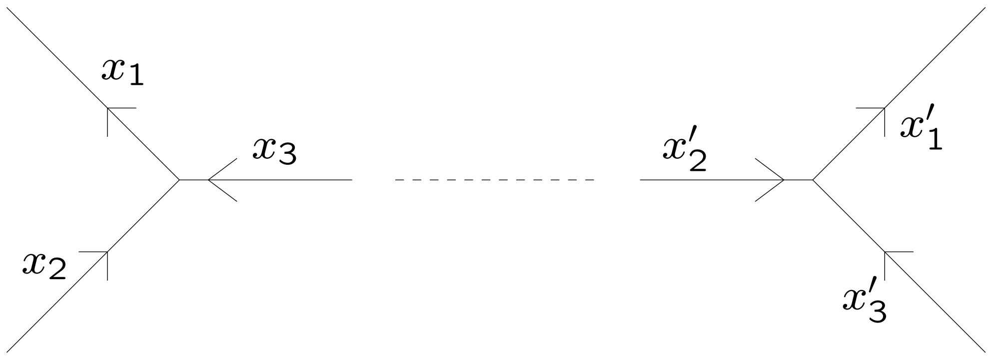
($SL(2,\mathbb{C})$ matrix from eigs e_1, e_2, e_3 and fixed pts x_1, x_2, x_3.)



Step 3 Using **Lem A**, compute the matrices for β_i's.

Recall **Lem A**:

There exists a unique matrix in PSL$(2,\mathbb{C})$ s.t. $(x_1, x_2, x_3) \mapsto (x'_1, x'_2, x'_3)$



This kind of matrix conjugating $\rho(\alpha_i)$ to $\rho(\alpha_{g+i})^{-1}$.

Thus we can reconstruct a PSL$(2, \mathbb{C})$-representation from the eigenvalue and twist parameters. In other words, we have obtained a map

$$E(S, C) \times (\mathbb{C} \setminus \{0\})^{3g-3} \to X_{PSL}(S).$$

If we take a covering space Y corresponding to the signs of $\rho(\beta_i)$, we obtain the following diagram

$$\begin{array}{ccc} Y & \longrightarrow & X_{SL}(S) \\ \Big\downarrow H^1(G;\mathbb{Z}_2) & & \Big\downarrow H^1(S;\mathbb{Z}_2) \\ E(S,C) \times (\mathbb{C} \setminus \{0\})^{3g-3} & \longrightarrow & X_{PSL}(S) \end{array}$$

(Here the covering group $H^1(G; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^g$.)

Thus we can reconstruct a $\mathrm{PSL}(2,\mathbb{C})$-representation from the eigenvalue and twist parameters. In other words, we have obtained a map

$$E(S,C) \times (\mathbb{C} \setminus \{0\})^{3g-3} \to X_{PSL}(S).$$

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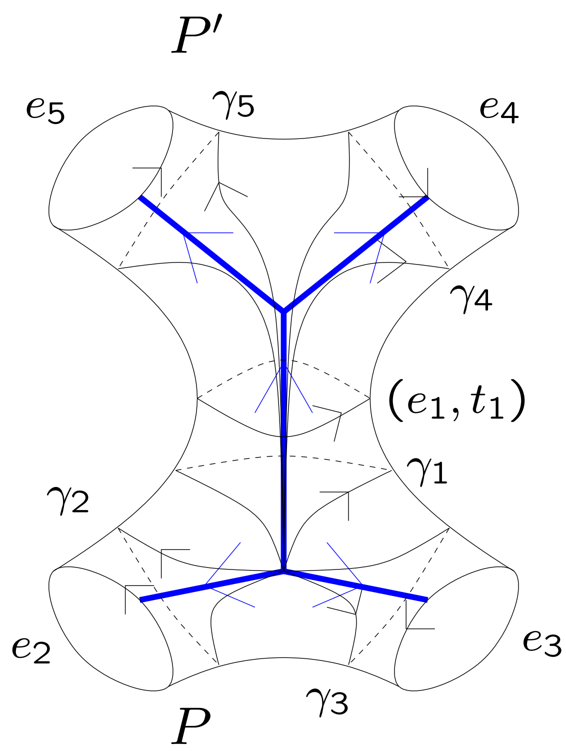
(Here the covering group $H^1(G;\mathbb{Z}_2) \cong (\mathbb{Z}_2)^g$.)

Examples:

4 holed sphere



We apply **Prop 2** for $(x_1, x_2, x_3) = (\infty, 1, 0)$, then we have

$$x_4 = \frac{e_1(e_2 - e_1e_3)(e_5 - e_1e_4)t_1 - e_1(e_3 - e_1e_2)(e_1e_5 - e_4)}{({e_1}^2 - 1)e_2(e_1e_5 - e_4)},$$

$$x_5 = \frac{(e_2 - e_1e_3)t_1 - e_1(e_3 - e_1e_2)}{({e_1}^2 - 1)e_2}.$$

By **Prop 1**, we have,

$$\rho(\gamma_1) = \begin{pmatrix} e_1 & \frac{e_3}{e_2} - e_1 \\ 0 & \frac{1}{e_1} \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} -\frac{e_1}{e_3} + e_2 + \frac{1}{e_2} & \frac{e_1}{e_3} - \frac{1}{e_2} \\ e_2 - \frac{e_1}{e_3} & \frac{e_1}{e_3} \end{pmatrix}$$

$$\rho(\gamma_3) = \begin{pmatrix} \frac{1}{e_3} & 0 \\ \frac{1}{e_3} - \frac{e_2}{e_1} & e_3 \end{pmatrix}, \quad \rho(\gamma_4) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$a_{11} = \frac{{e_1}^2(e_4 + {e_4}^{-1})}{({e_1}^2 - 1)} - \frac{e_1({e_5}^2 + 1)}{({e_1}^2 - 1)e_5} + \frac{(e_1e_4e_5 - 1)(e_1e_2 - e_3)(e_1e_5 - e_4)}{({e_1}^2 - 1)(e_1e_3 - e_2)e_4e_5t_1},$$

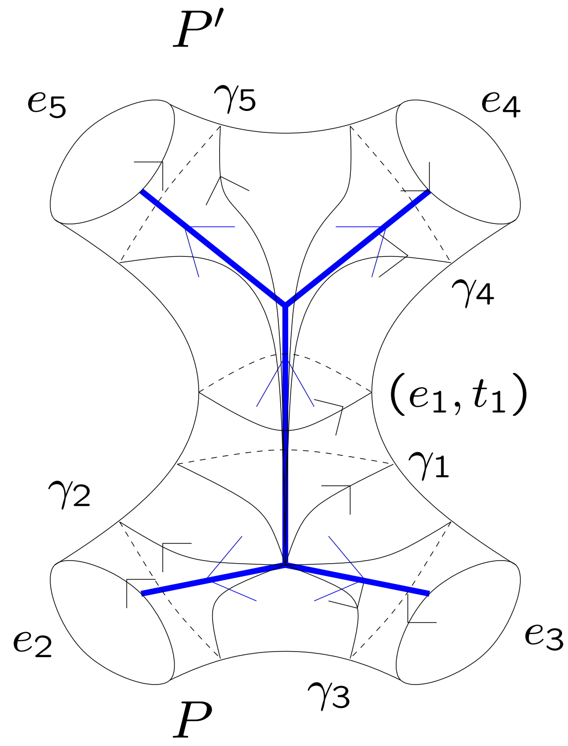
$$a_{12} = \frac{-e_1}{({e_1}^2 - 1)^2 e_2(e_1e_3 - e_2)e_4e_5t_1}$$
$$\times\left((e_1e_3e_5 + e_1e_2e_4)(t_1 + 1) - e_2e_5({e_1}^2 + t_1) - e_3e_4(1 + {e_1}^2 t_1)\right)$$
$$\times\left((e_1e_3e_4e_5 + e_1e_2)(t_1 + 1) - e_2e_4e_5({e_1}^2 + t_1) - e_3(1 + {e_1}^2 t_1)\right),$$

$$a_{21} = \frac{e_2(e_1e_5 - e_4)(e_1e_4e_5 - 1)}{e_1(e_1e_3 - e_2)e_4e_5t_1},$$

$$a_{22} = \frac{-(e_4 + {e_4}^{-1})}{({e_1}^2 - 1)} + \frac{e_1({e_5}^2 + 1)}{({e_1}^2 - 1)e_5} - \frac{(e_1e_4e_5 - 1)(e_1e_2 - e_3)(e_1e_5 - e_4)}{({e_1}^2 - 1)(e_1e_3 - e_2)e_4e_5t_1}.$$

For example, we have

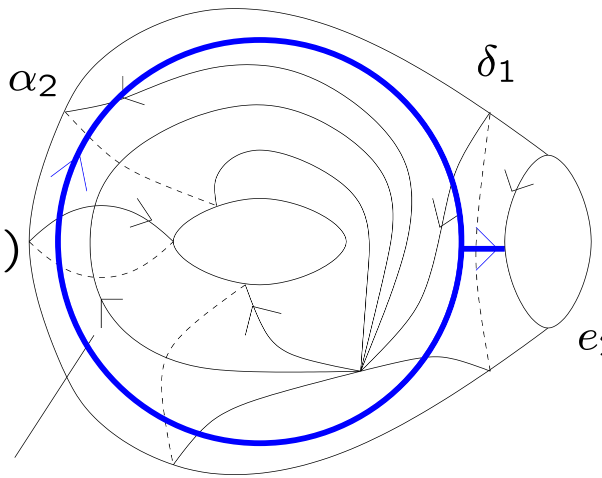
$$\mathrm{tr}(\rho(\gamma_3\gamma_4)) = \frac{a_1}{a_2},$$



$$\begin{aligned}
a_1 =& -\frac{(e_2e_3 - e_1)(e_1e_3 - e_2)}{e_1e_2e_3}\frac{(e_4e_5 - e_1)(e_1e_4 - e_5)}{e_1e_4e_5}t_1 \\
& - \frac{(1 - e_1e_2e_3)(e_1e_2 - e_3)}{e_1e_2e_3}\frac{(1 - e_1e_4e_5)(e_1e_5 - e_4)}{e_1e_4e_5}\frac{1}{t_1} \\
& + \chi_1(\chi_3\chi_5 + \chi_2\chi_4) - 2(\chi_2\chi_5 + \chi_3\chi_4), \\
a_2 =& (e_1 - {e_1}^{-1})^2.
\end{aligned}$$

where $\chi_i = e_i + {e_i}^{-1}$.

One holed torus

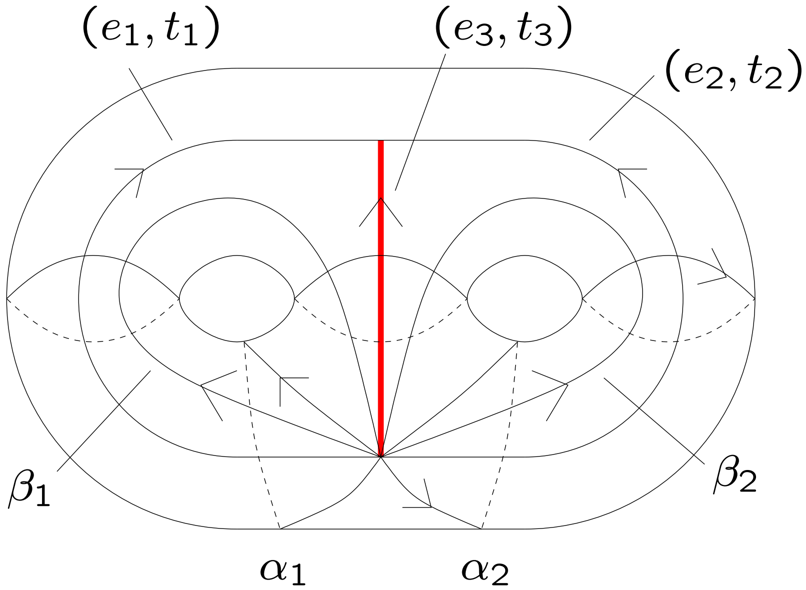


Define a pants decomposition, a dual graph and the parameters e_1, e_2, t_1 as in the right Figure. Then we have

$$\rho(\alpha_1) = \begin{pmatrix} e_1 & e_1^{-1} - e_1^{-1}e_2^{-1} \\ 0 & e_1^{-1} \end{pmatrix},$$

$$\rho(\beta_1) = \frac{1}{\sqrt{-e_2 t_1}(e_1^2 - 1)} \begin{pmatrix} (e_2 - e_1^2)t_1 + (e_2 - 1) & (t_1 + 1)(1 - e_2) \\ -e_2(e_1^2 - 1) & e_2(e_1^2 - 1) \end{pmatrix}$$

Closed surface of genus 2



$$\rho(\alpha_1)=\begin{pmatrix} e_1^{-1} & 0 \\ -e_1+e_2^{-1}e_3 & e_1 \end{pmatrix},$$

$$\rho(\alpha_2)=\begin{pmatrix} e_1e_3^{-1} & e_2-e_1e_3^{-1} \\ -e_2^{-1}+e_1e_3^{-1} & e_2+e_2^{-1}-e_1e_3^{-1} \end{pmatrix},$$

$$\rho(\beta_1)=\frac{1}{\sqrt{t_1t_3}}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \rho(\beta_2)=\frac{1}{(e_2{}^2-1)e_3\sqrt{t_2t_3}}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$a_{11}=1 \quad a_{12}=-\frac{(e_2e_3-e_1)(t_3+1)}{e_1(e_3{}^2-1)}, \quad a_{21}=\frac{e_1(t_1+1)(e_1e_2-e_3)}{(e_1{}^2-1)e_2},$$

$$a_{22}=\frac{(e_1e_2e_3-1)(e_1e_3-e_2)t_1t_3-(e_1e_2-e_3)(e_2e_3-e_1)(t_1+t_3+1)}{(e_1{}^2-1)e_2(e_3{}^2-1)},$$

$$b_{11}=(e_1e_2-e_3)t_2-e_2(e_2e_3-e_1),$$

$$b_{12}=-(e_2e_3-e_1)(e_3(e_1e_2e_3-1)t_2t_3+(e_3-e_1e_2)t_2 \\ +e_2e_3(e_2-e_1e_3)t_3-e_2(e_1-e_2e_3))/(e_1(e_3{}^2-1)),$$

$$b_{21}=(e_1e_2-e_3)(t_2+1),$$

$$b_{22}=-(e_3(e_1e_2e_3-1)(e_2e_3-e_1)t_2t_3-e_3(e_1e_2-e_3)(e_1e_3-e_2)t_3 \\ +(e_1e_2-e_3)(e_2e_3-e_1)(1+t_2))/(e_1(e_3{}^2-1)).$$

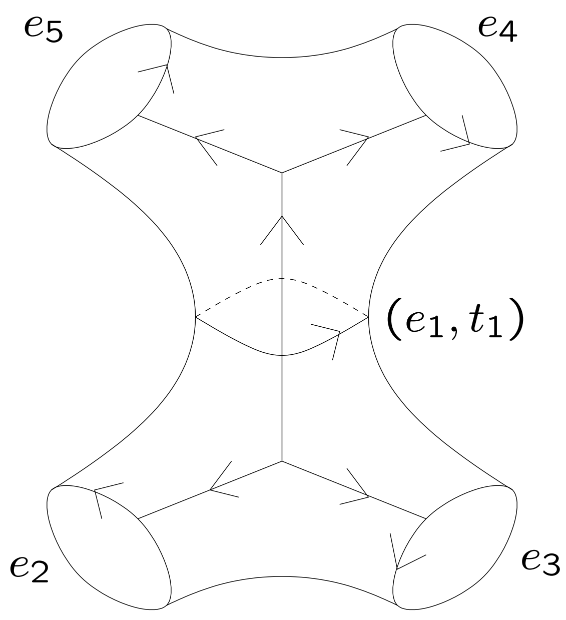
Action of $(\mathbb{Z}/2\mathbb{Z})^{3g-3}$

For each pair of pants, $(\mathbb{Z}_2)^3$ acts on the eigenvalues as

$$e_1 \mapsto {e_1}^{-1}, \quad e_2 \mapsto {e_2}^{-1}, \quad e_3 \mapsto {e_3}^{-1}.$$

In general the action affects on the twist parameters:

$$(e_1, t_1) \mapsto ({e_1}^{-1}, {t_1}^{-1}),$$
$$(e_2, t_1) \mapsto \left({e_2}^{-1}, \frac{e_2e_3 - e_1}{1 - e_1e_2e_3} \cdot \frac{e_1e_3 - e_2}{e_1e_2 - e_3} t_1\right),$$
$$e_3 \mapsto e_3^{-1},$$
$$e_4 \mapsto {e_4}^{-1},$$
$$(e_5, t_1) \mapsto \left({e_5}^{-1}, \frac{e_4e_5 - e_1}{1 - e_1e_4e_5} \cdot \frac{e_1e_4 - e_5}{e_1e_5 - e_4} t_1\right).$$



Globally $(\mathbb{Z}_2)^{3g-3}$ acts on the parameter space

$$E(S,C) \times T$$

where $T = (\mathbb{C} \setminus \{0\})^{3g-3}$ corresponds to the twist parameters. The map $E(S,C) \times T \to X_{PSL}(S)$ induces

$$(E(S,C) \times T)/(\mathbb{Z}_2)^{3g-3} \to X_{PSL}(S).$$

This is not injective since we can change the signs of the eigenvalue parameters as $(e_i, e_j, e_k) \mapsto (\varepsilon_i e_i, \varepsilon_j e_j, \varepsilon_k e_k)$ for $\varepsilon_i \varepsilon_j \varepsilon_k = 1$ if (e_i, e_j, e_k) belongs to a pair of pants. Globally the group is isomorphic to $H_1(G; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^g$.

Theorem

$$((E(S,C) \times T)/(\mathbb{Z}_2)^{3g-3})/(\mathbb{Z}_2)^g \xrightarrow{\text{injective}} X_{PSL}(S).$$

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$$(E(S,C) \times T)/(\mathbb{Z}_2)^{3g-3} \to X_{PSL}(S).$$

This is not injective since we can change the signs of the eigen-value parameters as $(e_i, e_j, e_k) \mapsto (\varepsilon_i e_i, \varepsilon_j e_j, \varepsilon_k e_k)$ for $\varepsilon_i \varepsilon_j \varepsilon_k = 1$ if (e_i, e_j, e_k) belongs to a pair of pants. Globally the group is isomorphic to $H_1(G; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^g$.

Theorem

$$((E(S,C) \times T)/(\mathbb{Z}_2)^{3g-3})/(\mathbb{Z}_2)^g \xrightarrow{\text{injective}} X_{PSL}(S).$$

This map gives a parametrization of the representations satisfying (i) and (ii). In particular, it contains all quasi-Fuchsian representations.

As a summary, we have the following diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{(\mathbb{Z}_2)^{3g-3}} & X_{SL}(S) \\
\downarrow{\scriptstyle H^1(G;\mathbb{Z}_2)} & & \\
E(S,C)\times T & & \downarrow{\scriptstyle H^1(S;\mathbb{Z}_2)} \\
\downarrow{\scriptstyle H_1(G;\mathbb{Z}_2)} & & \\
(E(S,C)\times T/H_1(G;\mathbb{Z}_2)) & \xrightarrow{(\mathbb{Z}_2)^{3g-3}} & X_{PSL}(S)
\end{array}$$

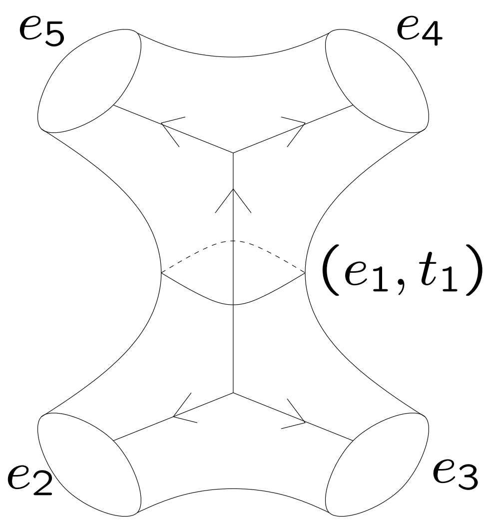
The horizontal maps induce injections after taking quotient.

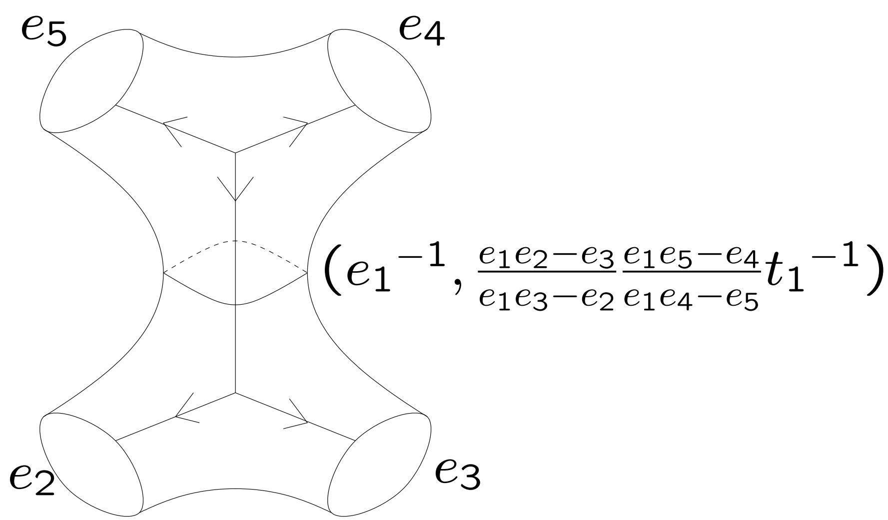
Coordinate change

Our coordinates depend on the choice of a pants decomposition with a dual oriented graph. We will give a formula for coordinate change.

Prop Any two pants decomposition with dual graphs are related by the following five types moves. Transformation formulas for such moves are given as follows.

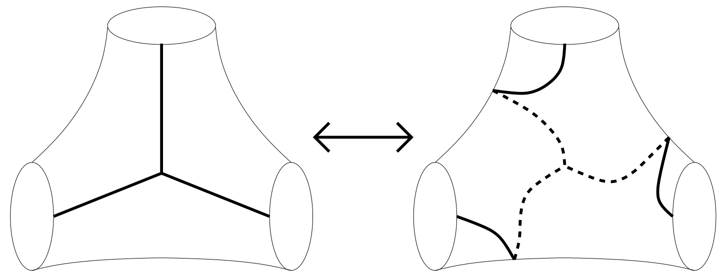
(I) Reverse orientation Reverse the orientation of an edge of the dual graph.



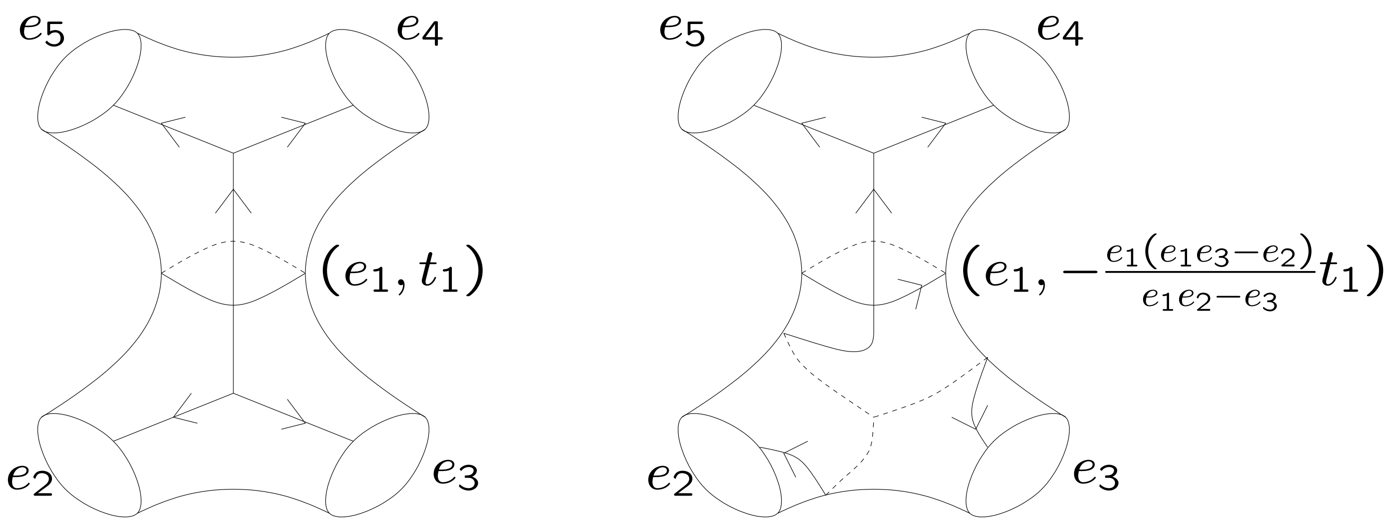


(II) Dehn twist Change the dual graph by a (left or right) Dehn twist along a pants curve.

(III) Vertex move For a vertex of the dual graph, change the edges adjacent to the vertex by their right half-twists as:

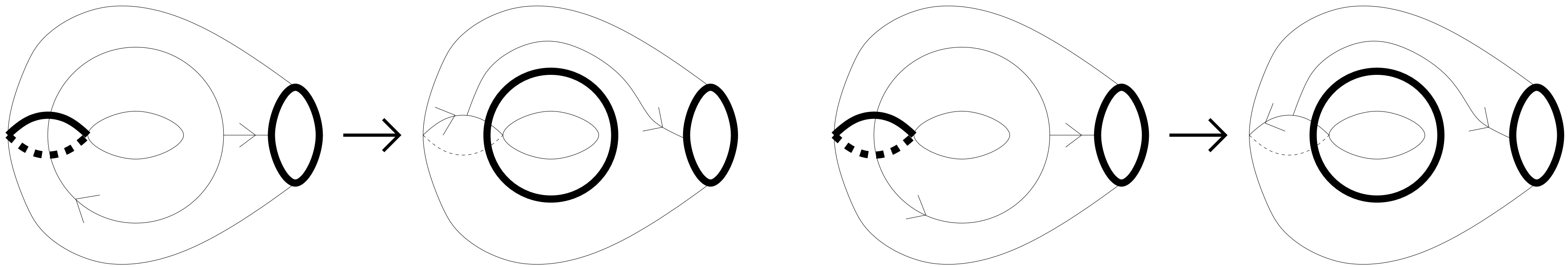


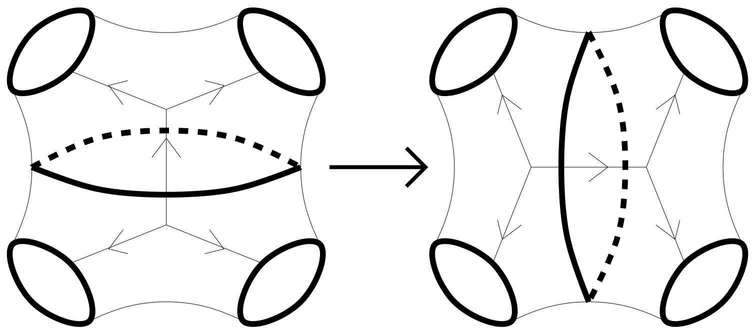
These moves are (locally) expressed by compositions of the following formula:



(IV) Graph automorphism Just change the variables by permutations.

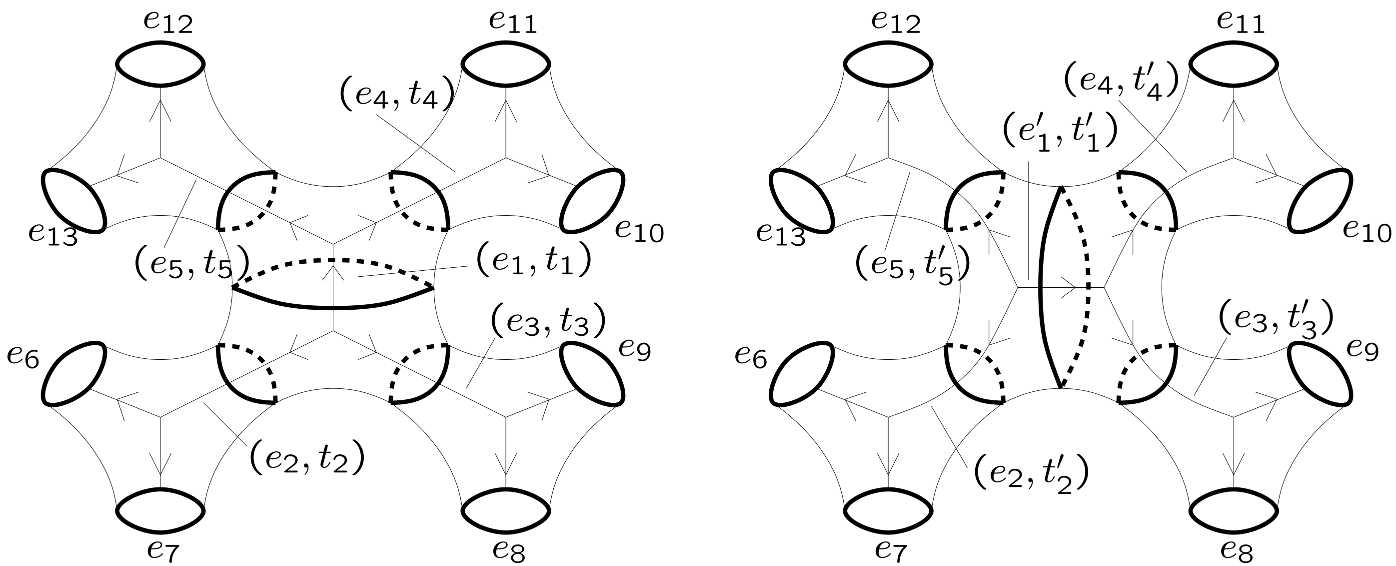
(V) Elementary move On a subsurface homeomorphic to a one-holed torus or a four-holed sphere, we define the moves by:





(a clockwise rotation of angle $\pi/2$)

The transformation formula for type (V) move is complicated...



$$t'_2 = \frac{(e_1e_2 - e_3)(e_1e_3 - e_2)(t_1 + 1)}{(e_2e_3 - e_1)(e_1e_3 - e_2)t_1 + (1 - e_1e_2e_3)(e_1e_2 - e_3)} \frac{e_2e'_1 - e_5}{e_2e_5 - e'_1} t_2,$$

$$t'_3 = \frac{(e_2e_3 - e_1)((e_1e_3 - e_2)(e_1e_4 - e_5)t_1 + (e_1e_2 - e_3)(e_1e_5 - e_4))}{(e_1e_3 - e_2)((e_2e_3 - e_1)(e_1e_4 - e_5)t_1 + (1 - e_1e_2e_3)(e_1e_5 - e_4))} t_3,$$

$$t'_4 = \frac{(e_1e_3 - e_2)(e_1e_4 - e_5)t_1 + (e_1e_2 - e_3)(e_1e_5 - e_4)}{(e_1e_3 - e_2)(e_4e_5 - e_1)t_1 + (e_1e_2 - e_3)(1 - e_1e_4e_5)} \frac{e_3e'_1 - e_4}{1 - e_3e_4e'_1} t_4,$$

$$t'_5 = \frac{(e_1e_5 - e_4)(e_1e_4e_5 - 1)(t_1 + 1)}{(e_1 - e_4e_5)(e_1e_4 - e_5)t_1 + (e_1e_4e_5 - 1)(e_1e_5 - e_4)} t_5,$$

where e_1' is one of the solution of

$$x^2 - \mathrm{tr}(\rho(\gamma_3\gamma_4))x + 1 = 0$$

where

$$\begin{aligned}\mathrm{tr}(\rho(\gamma_3\gamma_4)) = \frac{1}{(e_1 - {e_1}^{-1})^2}\Bigg(&-\frac{(e_2e_3 - e_1)(e_1e_3 - e_2)}{e_1e_2e_3}\frac{(e_4e_5 - e_1)(e_1e_4 - e_5)}{e_1e_4e_5}t_1 \\ &- \frac{(1 - e_1e_2e_3)(e_1e_2 - e_3)}{e_1e_2e_3}\frac{(1 - e_1e_4e_5)(e_1e_5 - e_4)}{e_1e_4e_5}\frac{1}{t_1} \\ &+ (e_1 + {e_1}^{-1})\big((e_3 + {e_3}^{-1})(e_5 + {e_5}^{-1}) + (e_2 + {e_2}^{-1})(e_4 + {e_4}^{-1})\big) \\ &- 2\big((e_2 + {e_2}^{-1})(e_5 + {e_5}^{-1}) + (e_3 + {e_3}^{-1})(e_4 + {e_4}^{-1})\big)\Bigg).\end{aligned}$$

And

$$\begin{aligned}t_1' = \frac{1}{e_1' + {e_1'}^{-1}}&\frac{e_1'e_5e_2}{(e_5e_2 - e_1')(e_1'e_2 - e_5)}\frac{e_1'e_3e_4}{(e_3e_4 - e_1')(e_1'e_3 - e_4)} \\ \Bigg(&(e_1' - {e_1'}^{-1})\big(e_1'\,\mathrm{tr}(\rho(\gamma_3\gamma_5) - {e_1'}^{-1}\,\mathrm{tr}(\rho(\gamma_2\gamma_4))\big) \\ &- (e_1' + {e_1'}^{-1})\big((e_2 + e_2^{-1})(e_3 + e_3^{-1}) + (e_4 + e_4^{-1})(e_5 + e_5^{-1})\big) \\ &+ 2\big((e_3 + e_3^{-1})(e_5 + e_5^{-1}) + (e_2 + e_2^{-1})(e_4 + e_4^{-1})\big)\Bigg)\end{aligned}$$

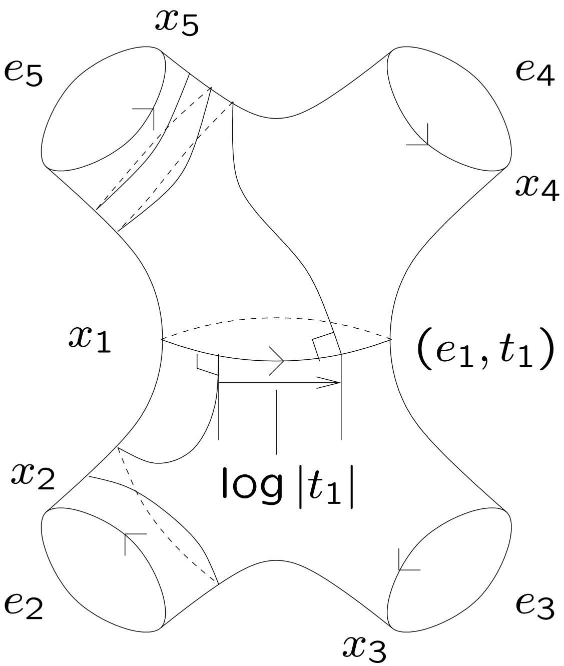
I omit the one-holed torus case.

Geometric meaning of the twist parameters

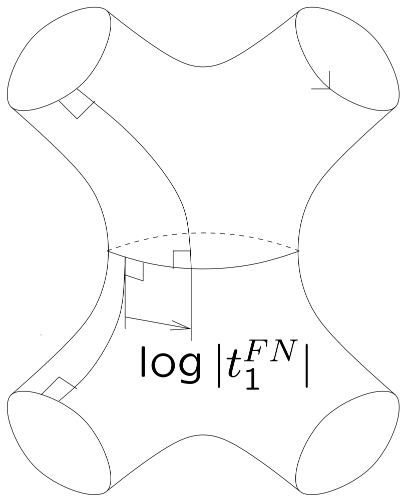
It is easy to see that

$$\{(e_i, t_i) \in \mathbb{R}^{2(3g-3)} | e_i < -1, \quad t_i > 0\}$$

corresponds to the Fuchsian representations. Restrict to this subset, we can interpret our twist parameter as:



On the other hand, usual F-N twist parameters are defined as:



Prop Let t_1^{FN} be the exponential of the usual F-N twist parameter. Then we have

$$t_1^{FN} = \sqrt{\frac{(e_1e_3 - e_2)(e_2e_3 - e_1)(e_1e_4 - e_5)(e_4e_5 - e_1)}{(e_1e_2 - e_3)(e_1e_2e_3 - 1)(e_1e_5 - e_4)(e_1e_4e_5 - 1)}}t_1.$$

Remark

t_i^{FN} is invariant under the action of $(\mathbb{Z}/2\mathbb{Z})^{3g-3}$ up to sign. It might be a better parametrization although I do not know any appropriate choice of signs.

The traces of the four holed sphere seem to be much simpler.

$$\mathrm{tr}(\rho(\gamma_3\gamma_4)) = \frac{1}{(e_i - {e_i}^{-1})^2}\Big(-\sqrt{(\chi_1^2+\chi_2^2+\chi_3^2-\chi_1\chi_2\chi_3-4)}\times \sqrt{(\chi_1^2+\chi_4^2+\chi_5^2-\chi_1\chi_4\chi_5-4)}(t_1^{FN}+(t_1^{FN})^{-1}) + \chi_1(\chi_3\chi_5+\chi_2\chi_4)-2(\chi_2\chi_5+\chi_3\chi_4)\Big)$$

where $\chi_i = e_i + {e_i}^{-1}$.

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where $\chi_i = e_i + {e_i}^{-1}$.

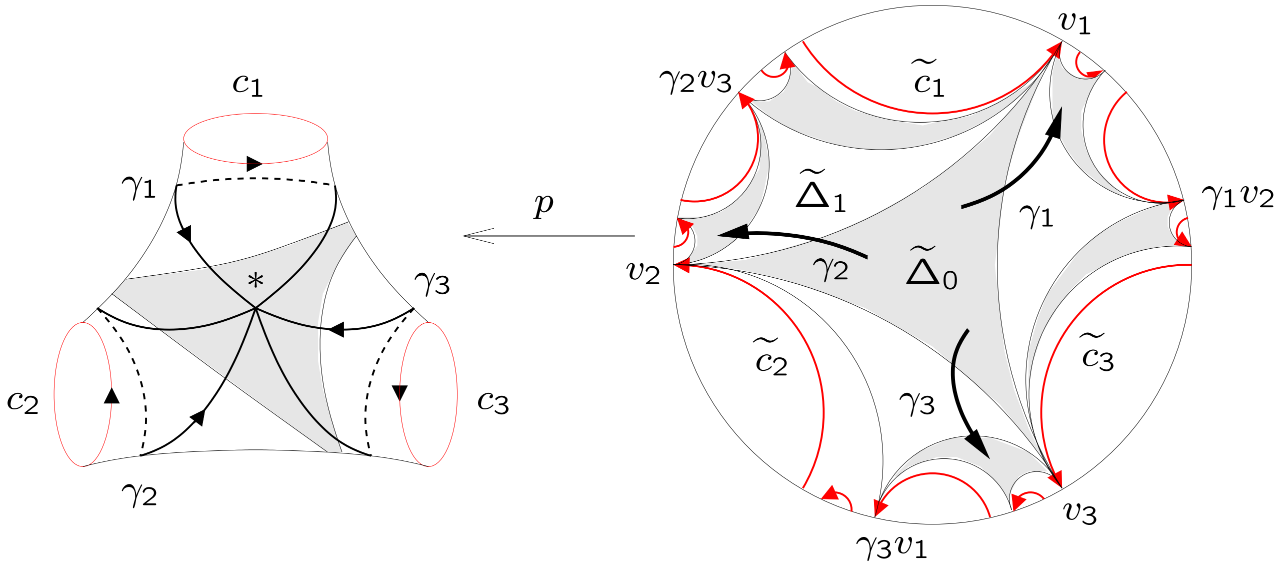
Developing map

S : a bordered surface (mainly interested in a pair of pants)

T : ideal triangulation of S

The universal cover $\tilde{S}$ of S has an ideal triangulation $\widetilde{T}$ lifted from T.

An equivariant map $f : \partial_\infty \widetilde{T} \to \mathbb{C}P^1$ is called a developing map.

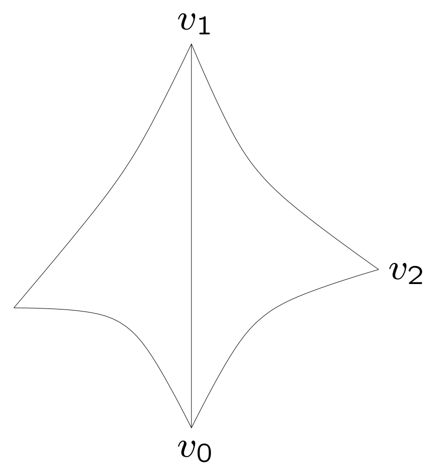


$v_i \in \partial_\infty \widetilde{T}$,

$f(v_i) \in \mathbb{C}P^1$.

By **Lem A**, a developing map determines a representation $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$.

For a developing map $f : \partial_\infty \widetilde{T} \to \mathbb{C}P^1$, we assign to each edge of T a complex number defined by the cross ratio $[f(v_0), f(v_1), f(v_2), f(v_3)]$ where (v_0, v_1, v_2) and (v_0, v_1, v_3) are ideal triangles of $\widetilde{T}$ as in the right figure.



Conversely, a developing map is completely determined by these complex parameters:

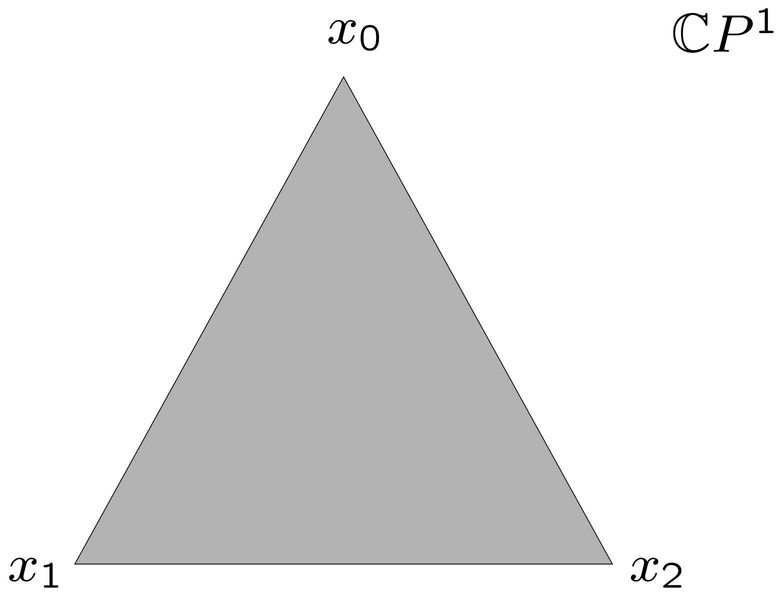
Lem C

x_0, x_1, x_2 : distinct points of $\mathbb{C}P^1$, $\quad z \in \mathbb{C} \setminus \{0\}$

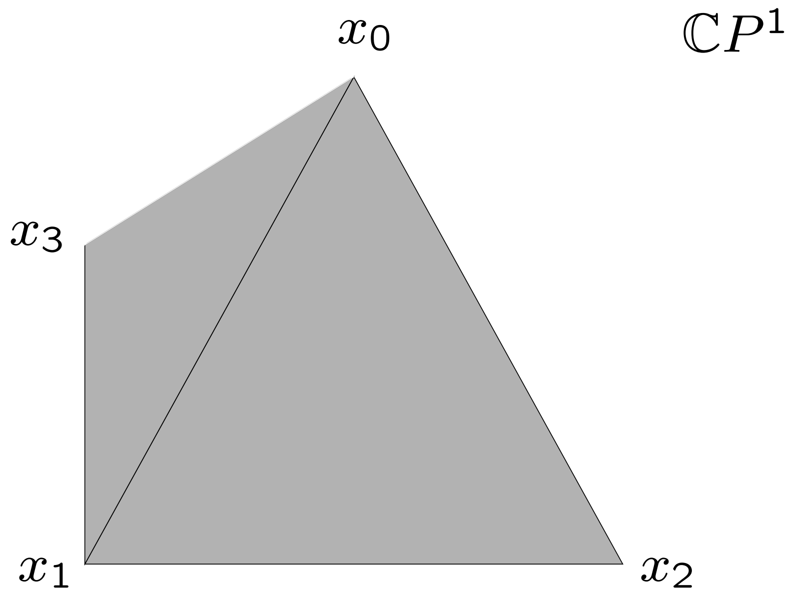
Then there exists a unique $x_3 \in \mathbb{C}P^1$ s.t. $[x_0, x_1, x_2, x_3] = z$.

(Set $x_3 = \frac{x_0(x_2-x_1)-zx_1(x_2-x_0)}{(x_2-x_1)-z(x_2-x_0))}$.)

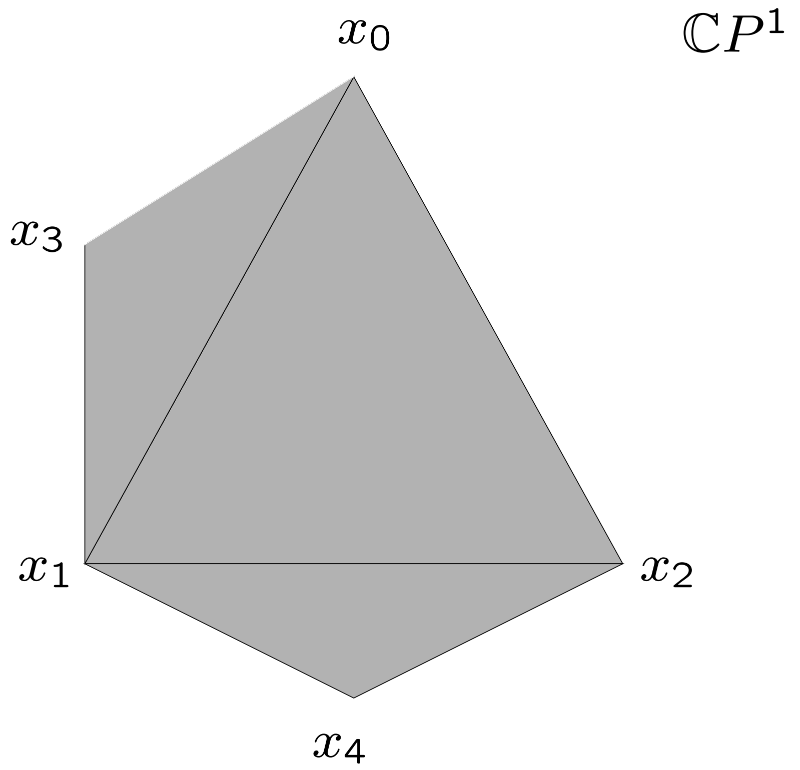
We can develop $\widetilde{T}$ to $\mathbb{C}P^1$ by **Lem C** inductively:



We can develop $\widetilde{T}$ to $\mathbb{C}P^1$ by **Lem C** inductively:



We can develop $\widetilde{T}$ to $\mathbb{C}P^1$ by **Lem C** inductively:



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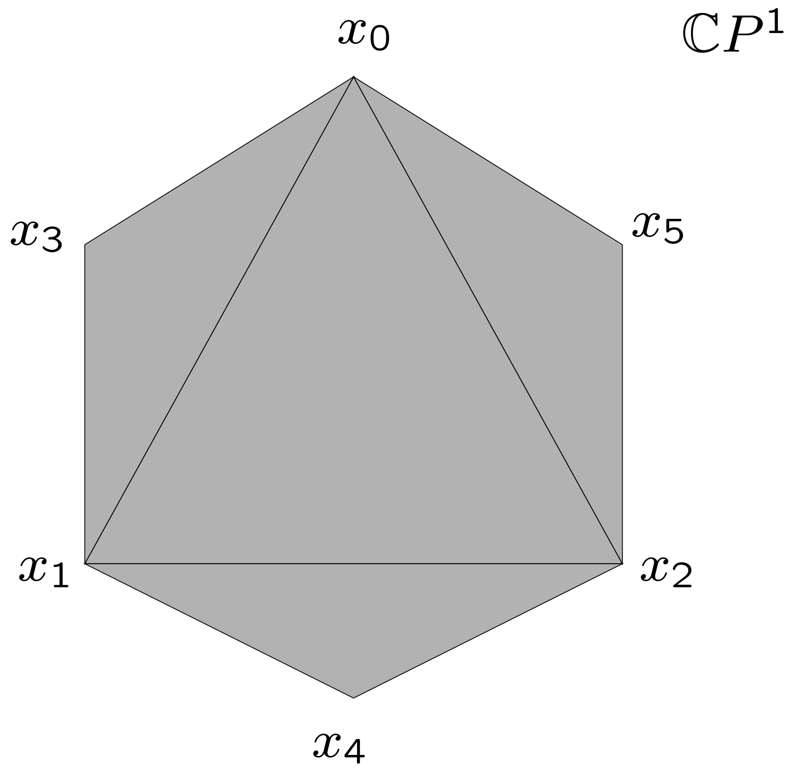

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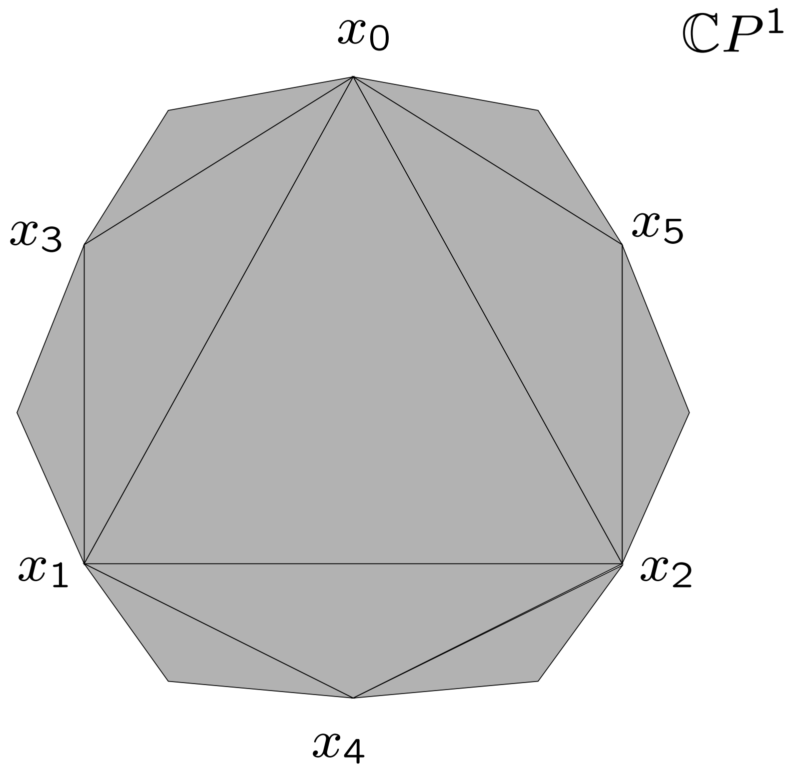
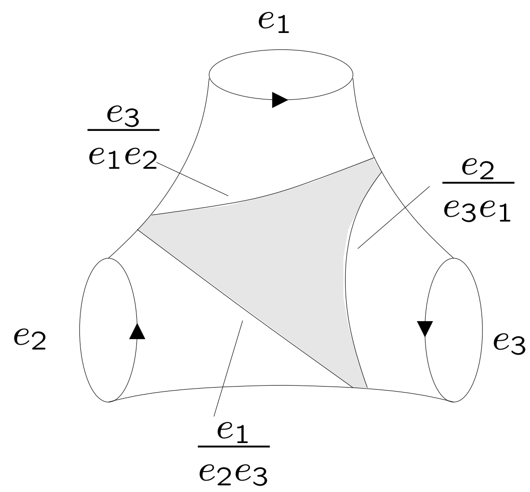


Counterparts of **Prop 1** and **Prop 2** in this context are expressed as follows:

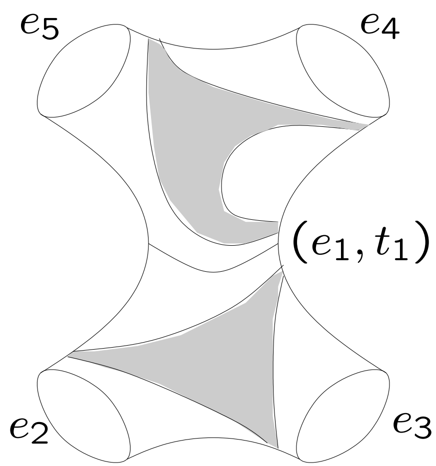
Prop 1'

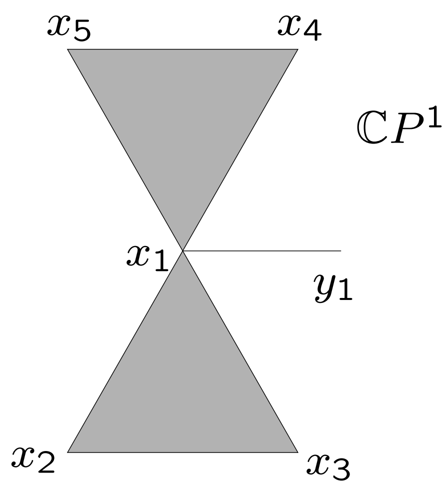
A pair of pants whose eigenvalue parameters e_1, e_2, e_3 has complex parameters as in the left figure.



Prop 2'

The twist parameter t_1 describes the relative position of the two developed triangles.





Continue to the second part