Bending deformation of quasi-Fuchsian groups

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Outline

The shape of the set of discrete faithful representations in the character variety is very complicated.

In this talk, we study and visualize the shape using bending deformation in an explicit way.

Basics on 3-dim hyperbolic geometry

$\mathbb{H}^3 = \{(x, y, t) \mid t > 0\}$: the hyperbolic space

with metric $ds^2 = \frac{dx^2+dy^2+dt^2}{t^2}$. $K \equiv -1$ and complete.

As the one-point compactification of $\{t = 0\}$, $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ can be regarded as the ideal boundary of $\mathbb{H}^3$.

$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$ acting on $\mathbb{C}P^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

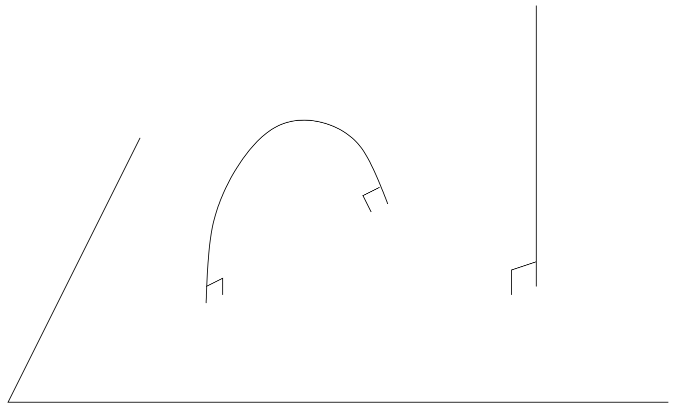
The action extends to the interior $\mathbb{H}^3$ as an isometry.

Moreover, $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$.

Basics on 3-dim hyperbolic geometry

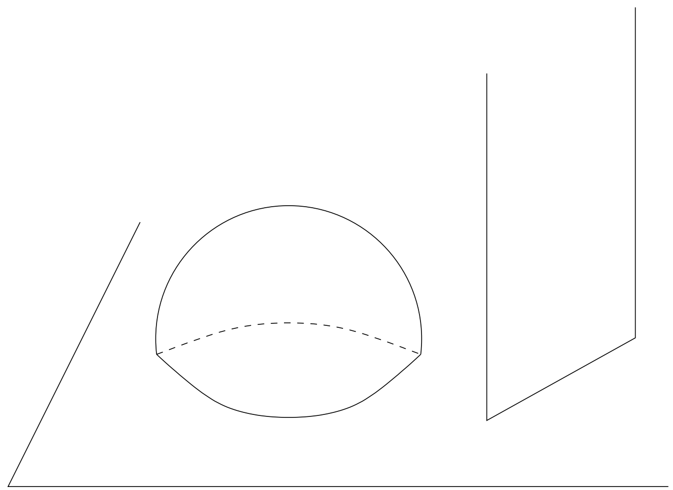
A geodesic in $\mathbb{H}^3$ is

- a line perpendicular to $\mathbb{C}$, or
- a (half) circle orthogonal to $\mathbb{C}$



A totally geodesic surface in $\mathbb{H}^3$ is

- a hemisphere orthogonal to $\mathbb{C}$, or
- a vertical plane orthogonal to $\mathbb{C}$



Isometries of $\mathbb{H}^3$

A non-trivial element in $A \in \mathrm{PSL}(2,\mathbb{C})$ is conjugate to

- $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ ($\Leftrightarrow$ A has exactly two fixed points on $\mathbb{C}P^1$), or
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ($\Leftrightarrow$ A has exactly one fixed point on $\mathbb{C}P^1$).

It acts on $\mathbb{C}P^1$ as $\quad z \mapsto \alpha^2 \cdot z$, respectively $\quad z \mapsto z + 1$.

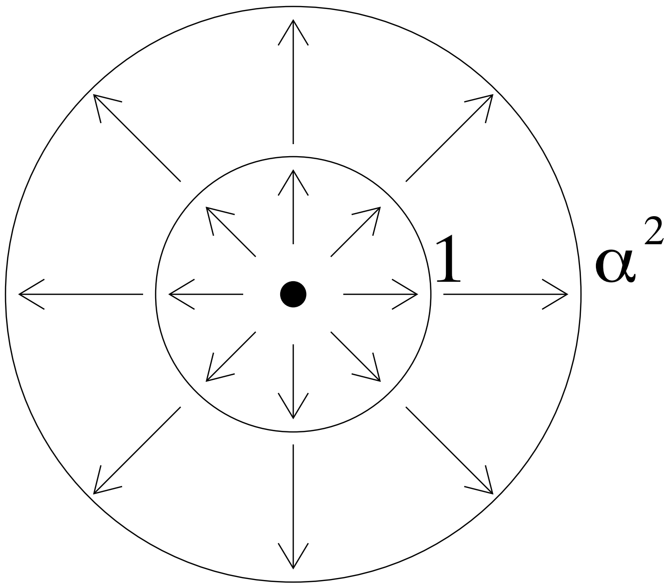
The latter case is called *parabolic*. The first case is further classified into

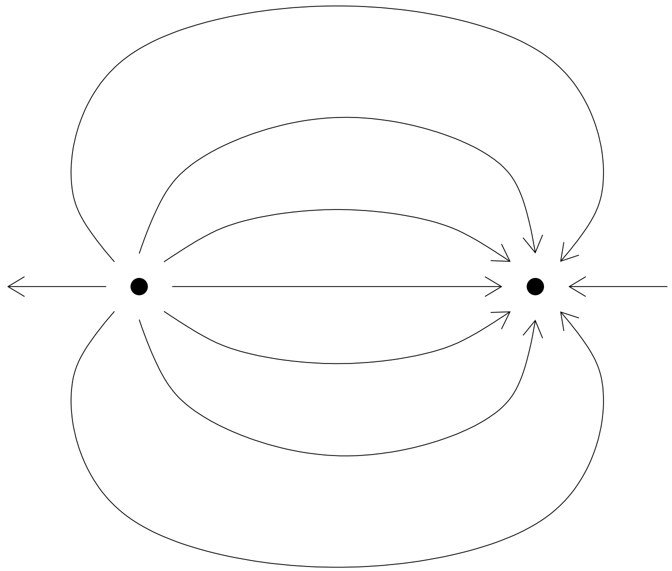
elliptic $\quad |\alpha| = 1$ (A has a fixed point in $\mathbb{H}^3$),

hyperbolic $\quad |\alpha| \neq 1$ and α is real,

loxodromic $\quad |\alpha| \neq 1$ and α is not real.

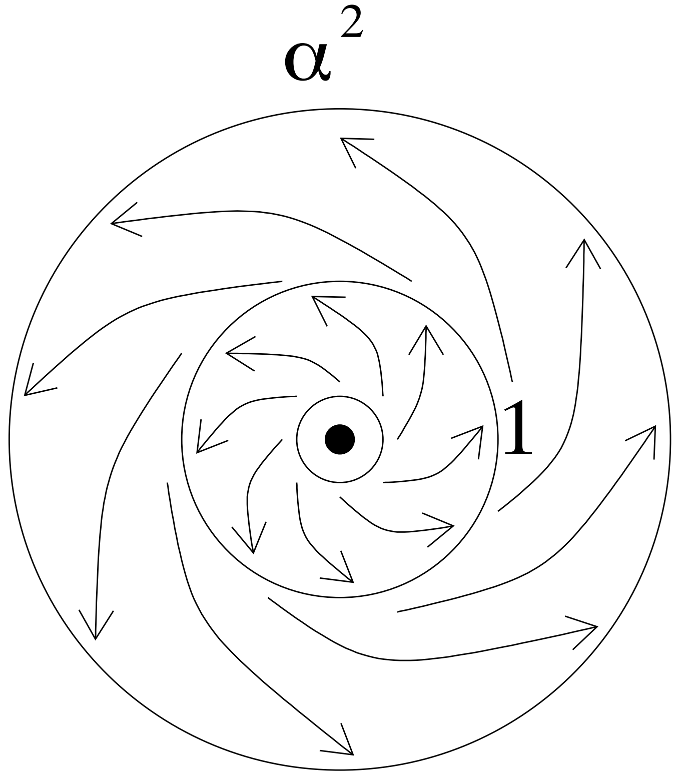
Hyperbolic transformation $\quad z \mapsto \alpha^2 \cdot z \quad (\alpha > 1)$

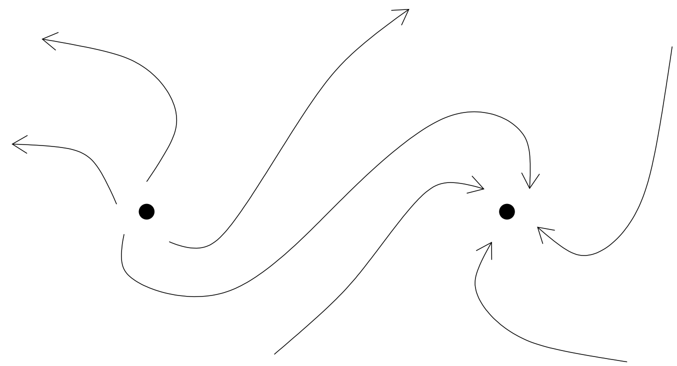




Dilation

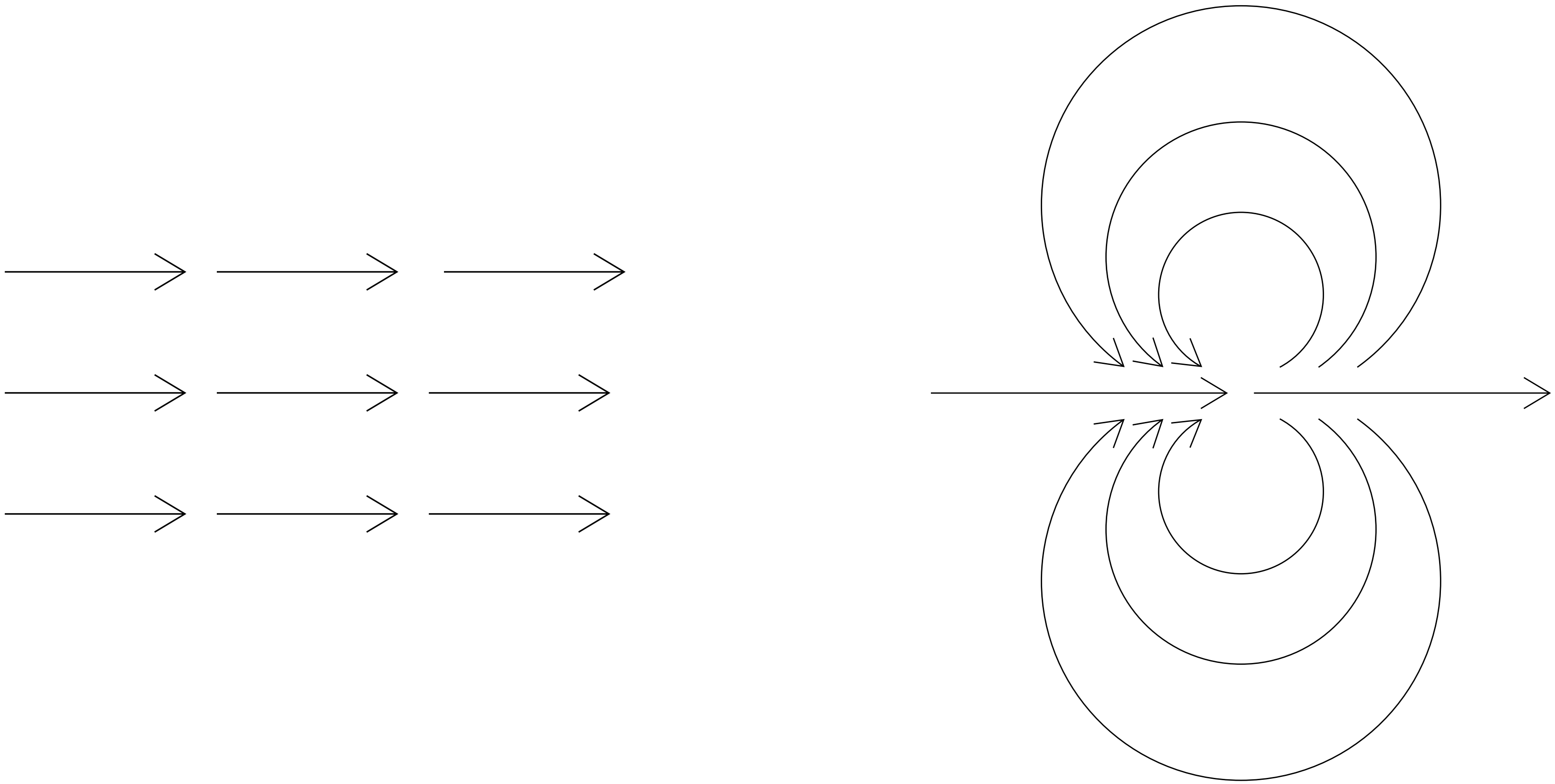
Loxodromic transformation $\quad z \mapsto \alpha^2 \cdot z \quad (|\alpha| > 1)$





Dilation and rotation

Parabolic transformation $z \mapsto z+1$



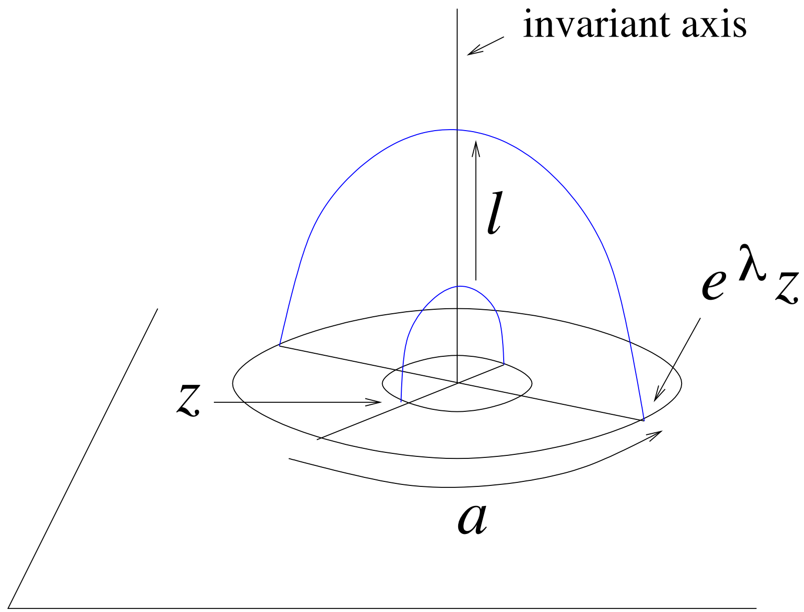
Translation

Complex length of an element of $\mathrm{PSL}(2,\mathbb{C})$

Let $\lambda = l + a\sqrt{-1}$ where $l \in \mathbb{R}_{\geq 0}$ and $a \in (-\pi, \pi]$. Then

$$A = \begin{pmatrix} \exp(\lambda/2) & 0 \\ 0 & \exp(-\lambda/2) \end{pmatrix}$$

(acts on $\mathbb{C}P^1$ as $z \mapsto \exp(\lambda) \cdot z$) preserves the axis $(0, \infty)$. The (hyperbolic) translation distance is l and the angle of rotation is a.



$\lambda = l + a\sqrt{-1}$ is called the complex length of A.

We have $\mathrm{tr}(A) = \pm 2 \cdot \cosh(\lambda/2)$.

Basics on Kleinian groups

$\Gamma < \mathrm{PSL}(2, \mathbb{C})$: discrete subgroup (finitely generated)

If Γ is torsion-free, $\mathbb{H}^3/\Gamma$ is a hyperbolic manifold. Let

$$\Lambda(\Gamma) = \{\text{limit points of } \Gamma \cdot x \text{ in } \mathbb{C}P^1 \text{ for some } x \in \mathbb{H}^3\}.$$

In particular, $\Lambda(\Gamma) \supset \{\text{fixed points of non-elliptic } \gamma \in \Gamma\}$. Let

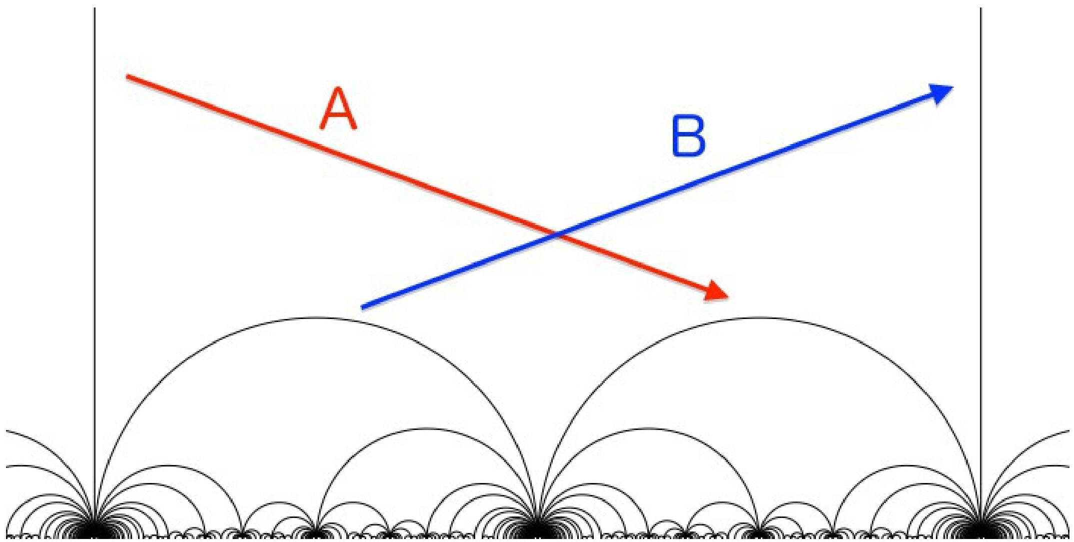
$$\Omega(\Gamma) = \mathbb{C}P^1 \setminus \Lambda(\Gamma).$$

Γ acts on $\Omega(\Gamma)$ properly discontinuously.

In this talk, we assume that Γ is isomorphic to a surface group $\pi_1(S)$ (or its quotient) for some hyperbolic surface S.

Example: Fuchsian groups

A 2-dim hyperbolic surface is represented as $\mathbb{H}^2/\Gamma$ by a discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$.



$\Gamma = \langle A, B \rangle$

punctured torus group

Since $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$, the action of Γ on $\mathbb{H}^2$ extends to $\mathbb{H}^3$.

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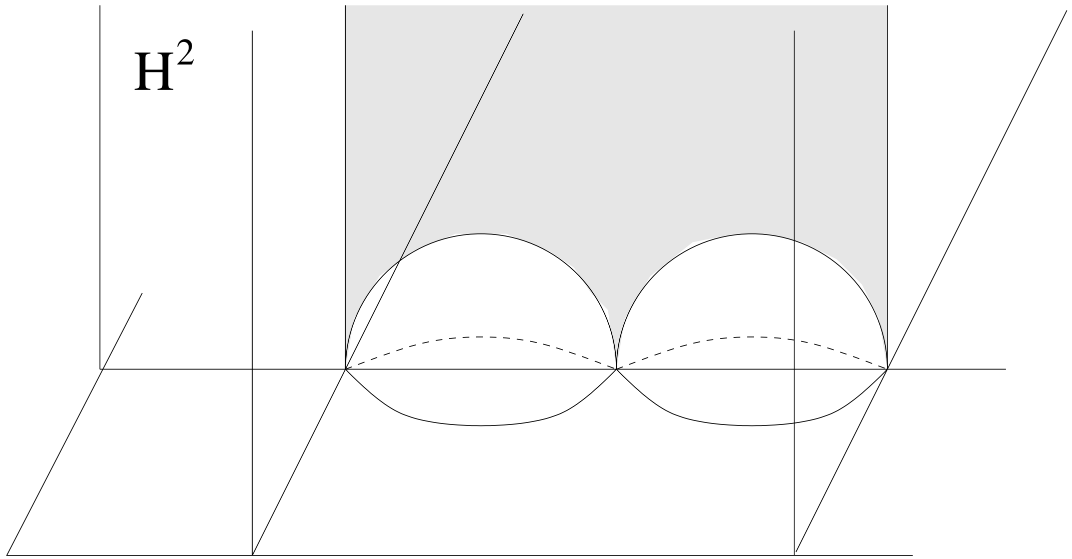
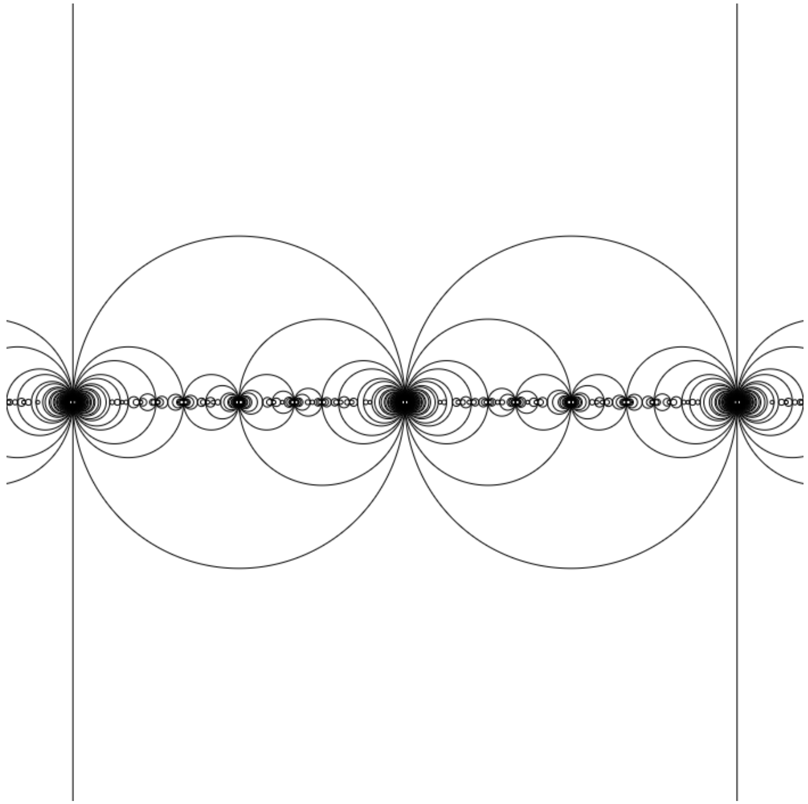




In this case, the limit set is $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ as above.

Quasi-Fuchsian groups

We can deform a Fuchsian group a little bit in $\mathrm{PSL}(2, \mathbb{C})$. The result dose not preserve $\mathbb{H}^2$ in general, and the limit set $\mathbb{R}P^1 \cong S^1$ is distorted.

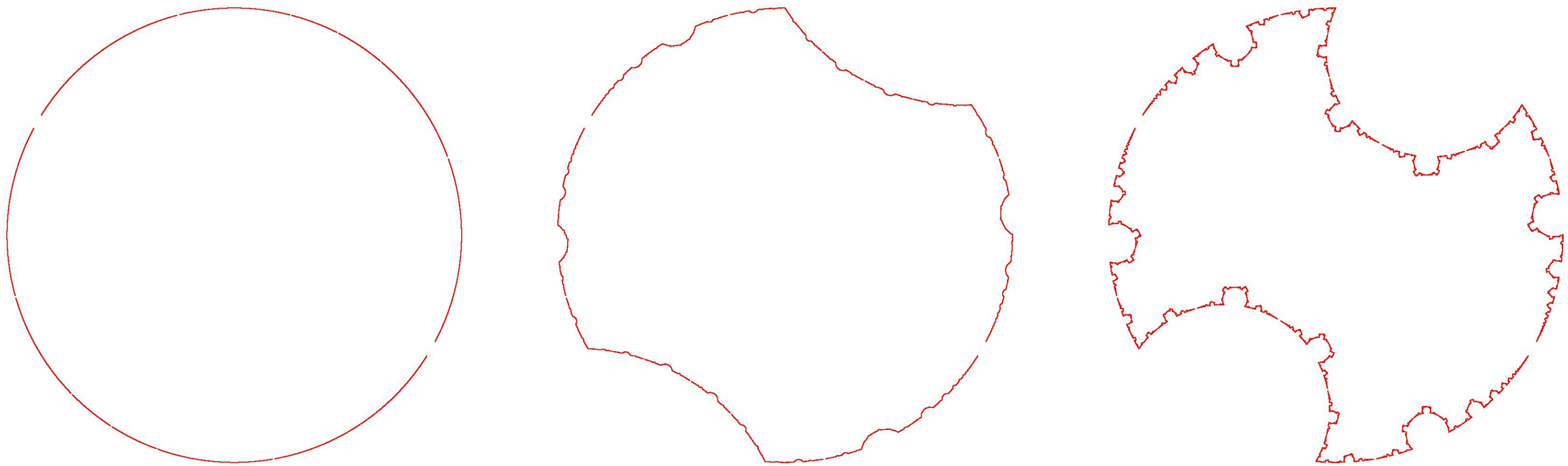
Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian group and ϕ a quasi-conformal homeomorphism of $\mathbb{C}P^1$. Then

$$\{\phi^{-1} \circ \gamma \circ \phi \mid \gamma \in \Gamma\}$$

is called a quasi-Fuchsian group.

Quasi-Fuchsian groups

The limit set of a quasi-Fuchsian group is homeomorphic to S^1 and Ω consists of two disks Ω_+ and Ω_-. By uniformization theorem, $\Omega_\pm$ are conformal to the unit disk and $\Omega_\pm/\Gamma$ give a pair of points in the Teichmüller space $\mathcal{T}(S)$.



Moreover, by Bers's simultaneous uniformization,

$$\mathcal{QF}(S) := \{\text{quasi-Fuchsian groups}\} \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S}).$$

Known facts

$X(S) = \mathsf{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))/\sim_{\text{conj.}}$: the character variety

$AH(S) = \{\text{discrete faithful reps}\}/\sim_{\text{conj.}} \subset X(S)$

$AH(S)$ is the set of (marked) hyp structures on $S \times (-1,1)$.

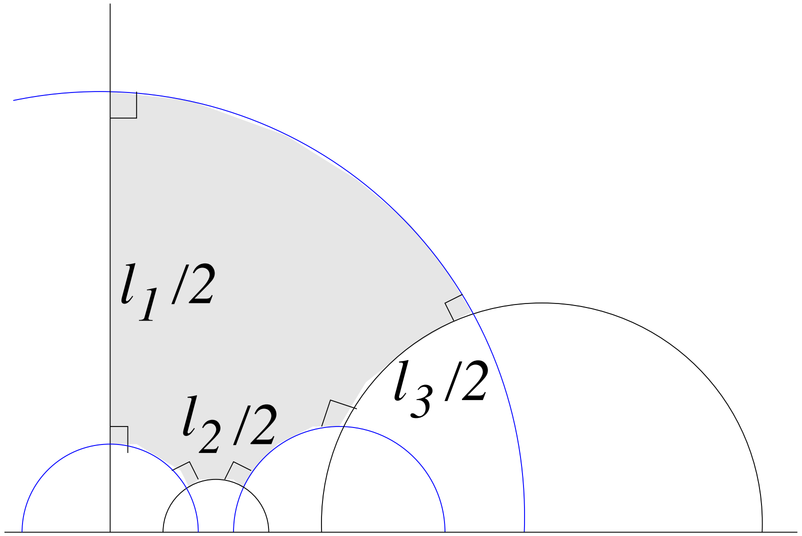
- $AH(S)$ is a closed subset in $X(S)$
- $\mathcal{QF}(S) \subset AH(S)$
- $\mathcal{QF}(S)$ is an open subset in $X(S)$
- $\overline{\mathcal{QF}(S)} = AH(S)$ (Density conjecture, now theorem)

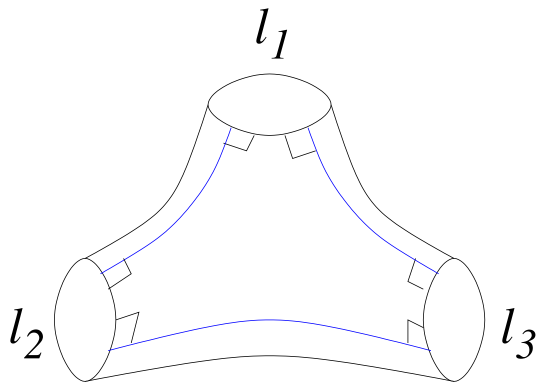
Although $\mathcal{QF}(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ is topologically a ball, the shape of $\mathcal{QF}(S)$ in $X(S)$ is very complicated.

Complex Fenchel-Nielsen coordinates

A pair of pants P is a 3-holed sphere.

Since a right angled hexagon is determined by three alternating side lengths, a hyp metric on P is determined by the lengths l_1, l_2, l_3 of the boundary curves.



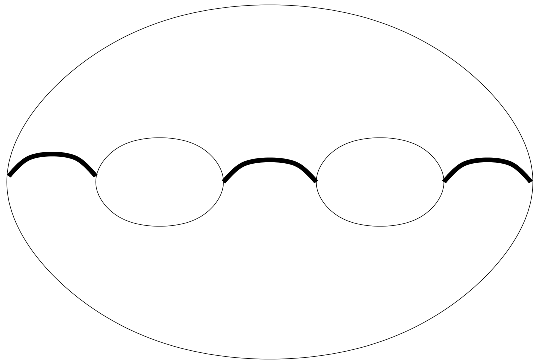


Similarly, a generic rep of $\pi_1(P) \to \mathrm{PSL}(2, \mathbb{C})$ is determined by three complex lengths $\lambda_i = l_i + a_i\sqrt{-1}$.

Complex Fenchel-Nielsen coordinates

Let S be a hyperbolic surface of genus g.

A pants decomposition is a set of simple closed curves on S s.t. the complement consists of pairs of pants.

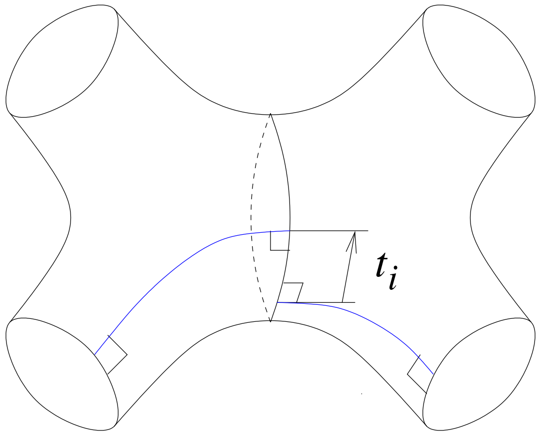


There are $3g-3$ such scc's. Denote the hyperbolic lengths of these curves by l_i's.

($\lambda_i = l_i + a_i\sqrt{-1}$ for PSL$(2,\mathbb{C})$-reps.)

Complex Fenchel-Nielsen coordinates

The hyperbolic structures on S is determined by l_i's and twist parameters t_i's :



Thus we have

$$\mathcal{T}(S) \cong (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} \ni (l_i, t_i).$$

l_i and t_i are called Fenchel-Nielsen coordinates.

Similarly, we can define the complex twist parameter $\tau_i = t_i + b_i\sqrt{-1}$ for a generic PSL$(2, \mathbb{C})$-rep of $\pi_1(S)$.

Complex Fenchel-Nielsen coordinates

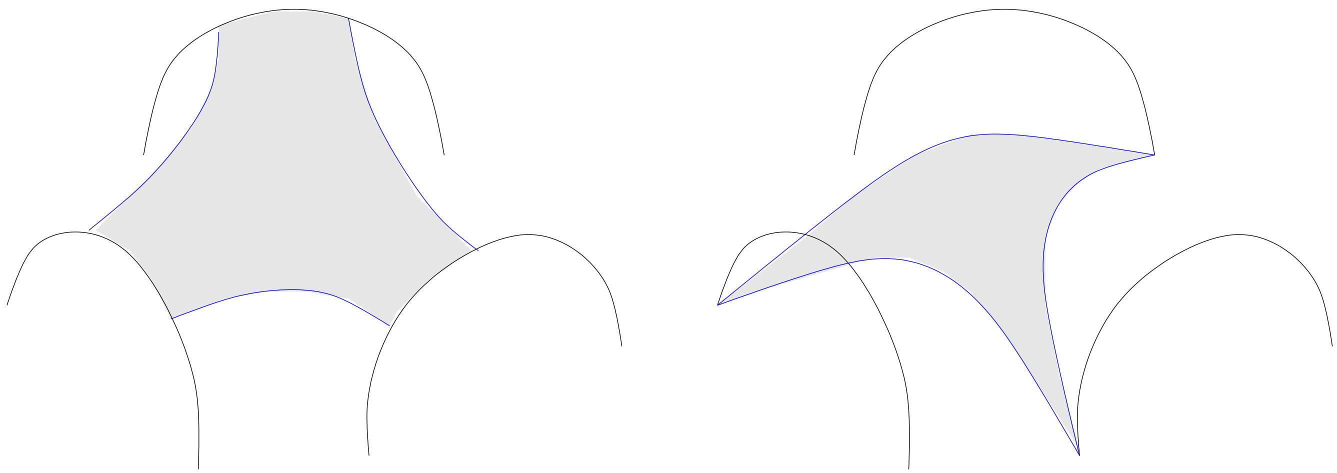
There are many subtleties about complex FN coordinates, at least we have a rational map

$$\mathbb{C}^{6g-6} \to X(S)$$

for a fixed pants decomposition, and we can parametrize the quasi-Fuchsian space $\mathcal{QF}(S)$.

Complex Fenchel-Nielsen coordinates

It is much easier to make the right angled hexagons to ideal triangles by spiraling the sides of the hexagons equivariantly, since the ideal vertices on $\mathbb{C}P^1$, a real 2-dim object.



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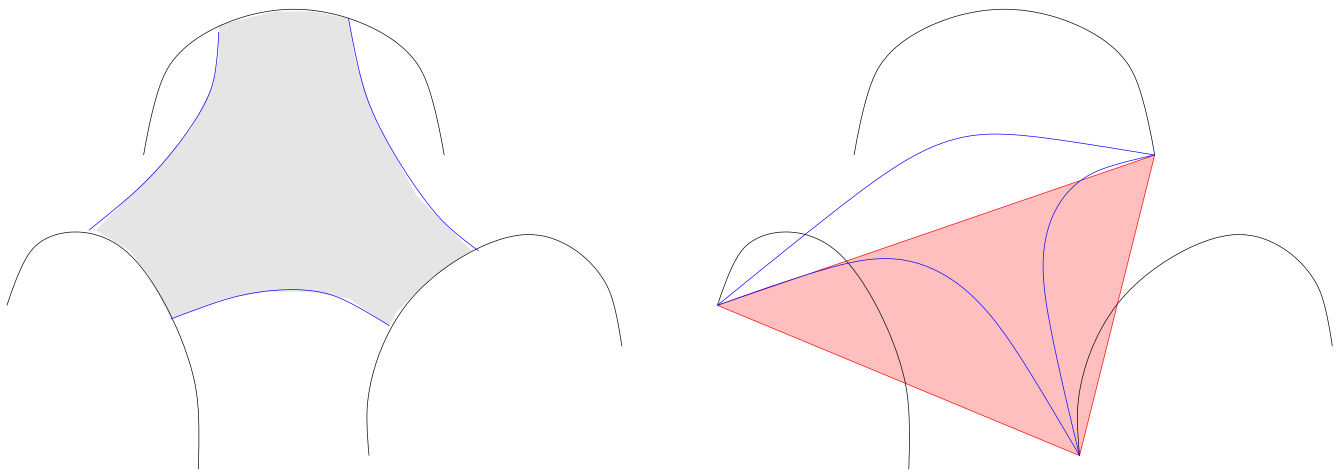
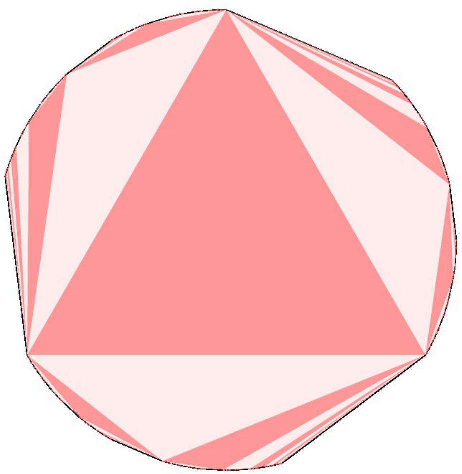
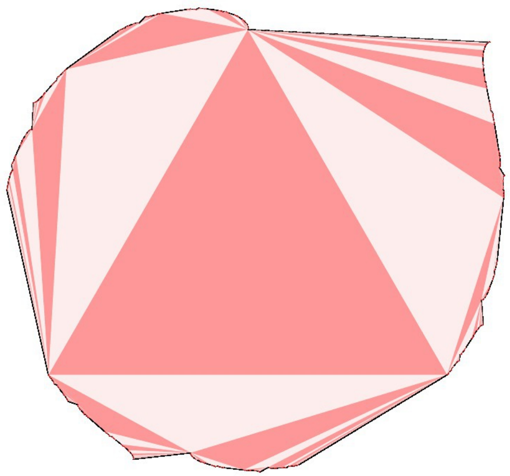
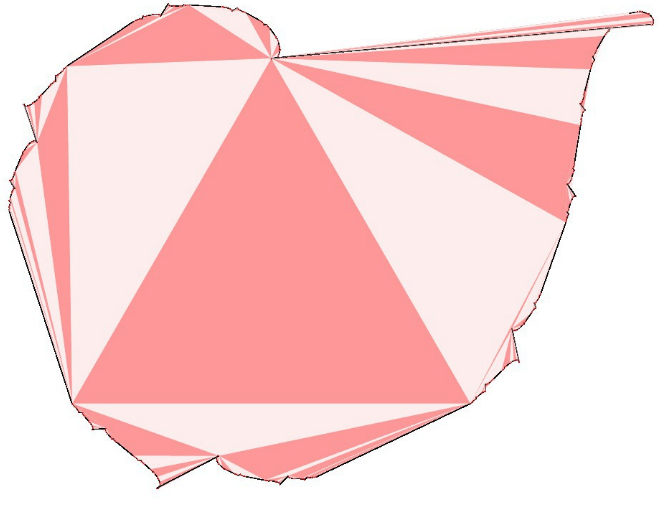




$a_0 = 0.0$



$a_0 = 0.5$

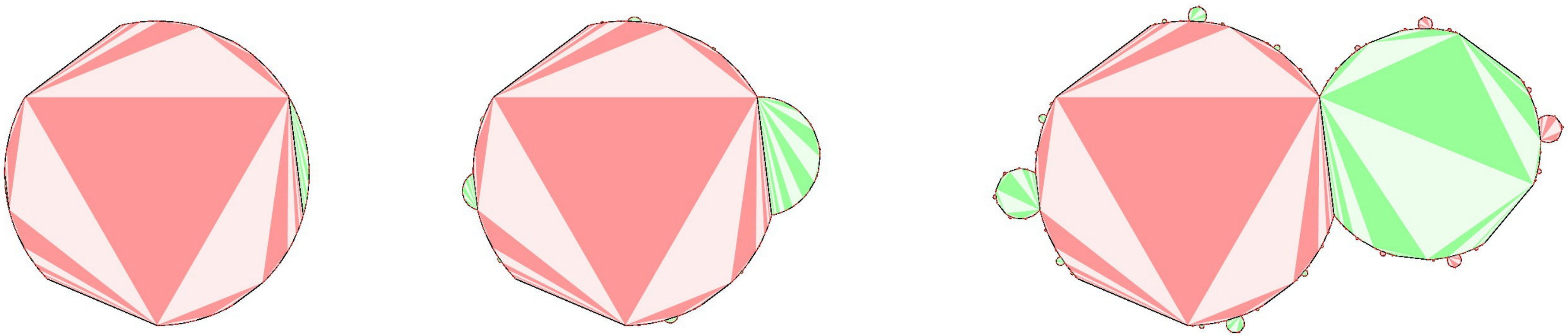


$a_0 = 1.0$

$$l_1 = l_2 = l_3 = 1.0, \quad \text{and} \quad a_2 = a_3 = 0.0.$$

(Recall $\lambda_i = l_i + a_i\sqrt{-1}$ is the complex length.)

A deformation by the twist parameter can be seen as :



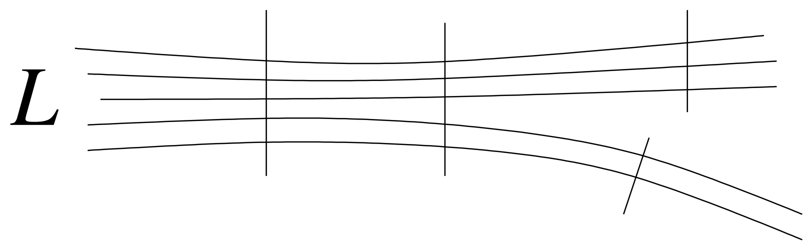
If $a_i = 0$ for all i, this is a 'bending deformation'.

Measured laminations

Fix a hyperbolic metric on S.

A closed subset L on S foliated by geodesics is called a geodesic lamination.

A lamination with a homotopy invariant transverse measure μ is called a measured lamination.



An easy example is a simple closed geodesic with a Dirac measure.

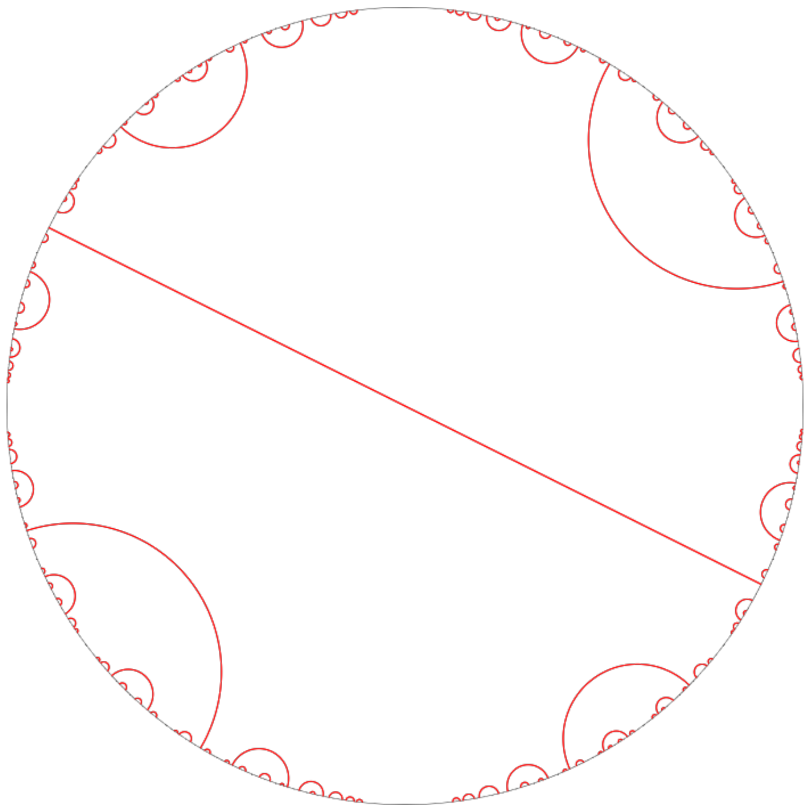
It is known that every measured lamination is obtained as a limit of such simple closed geodesics with Dirac measures.

Bending deformation

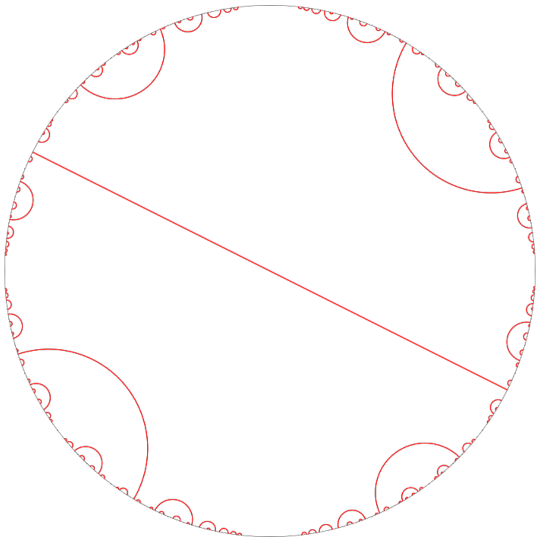
We can identify the universal cover $\tilde{S}$ of S with a totally geodesic surface in $\mathbb{H}^3$.

For a geodesic lamination (L, μ), we can 'bend' $\tilde{S}$ in $\mathbb{H}^3$.

Assume L is a simple closed curve. Let $\tilde{L}$ be the preimage of L in the universal cover $\tilde{S}$.



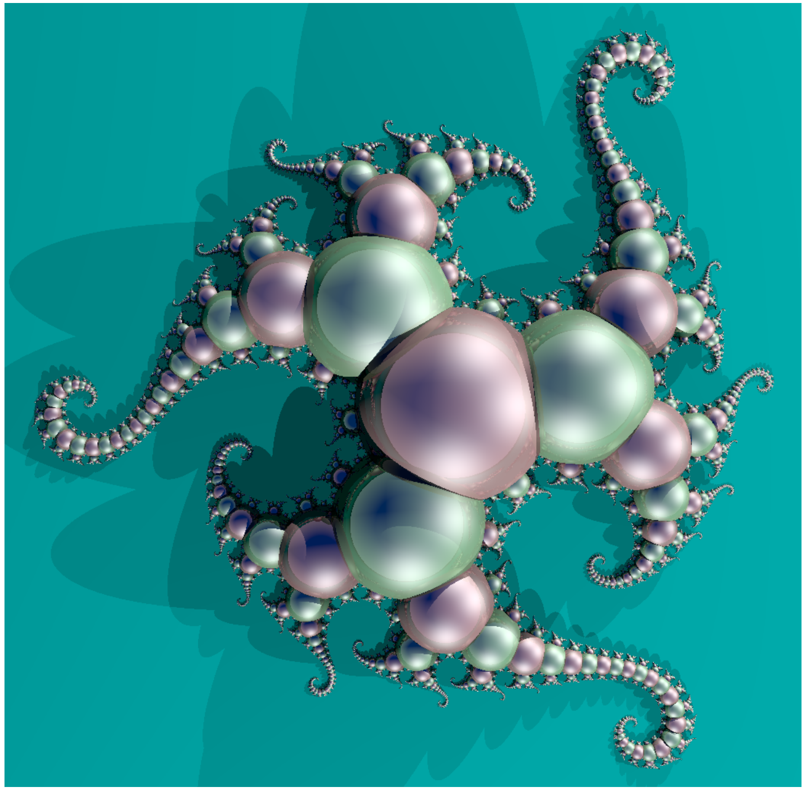
Bending deformation



Bend the disk

by the angle $\tilde{\mu}$

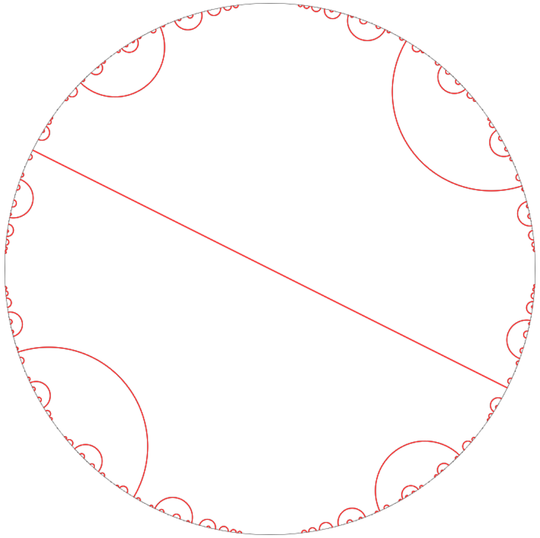
$\longrightarrow$



We call this procedure a bending deformation, and call the map $\tilde{S} \to \mathbb{H}^3$ the developing map.

Since the construction is $\pi_1(S)$-equivariant, we obtain a representation $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ from the developing map.

Bending deformation



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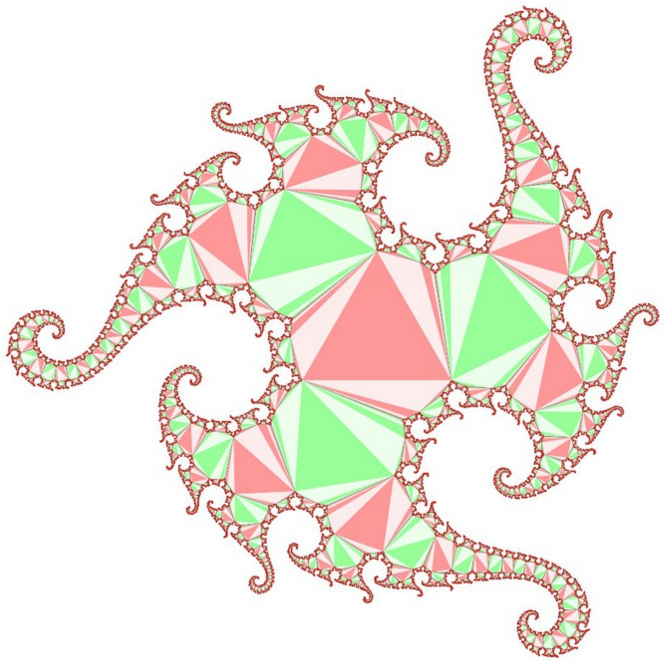

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Since the construction is $\pi_1(S)$-equivariant, we obtain a representation $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ from the developing map.

From the bending construction, we obtain a PSL$(2, \mathbb{C})$-rep from a hyp metric on S and a bending measure μ.

Since $\mathcal{QF}(S)$ is an open set containing all Fuchsian groups, a small bending gives a quasi-Fuchsian group.

Question How much can we bend the surface within $\mathcal{QF}$?

In the complex FN coordinates

$$\lambda_i = l_i + a_i\sqrt{-1}, \quad \tau_i = t_i + b_i\sqrt{-1},$$

if $a_i = 0$, the rep is obtained by the hyp surface determined by (l_i, t_i) bending along the pants curves by angles b_i.

Convex core

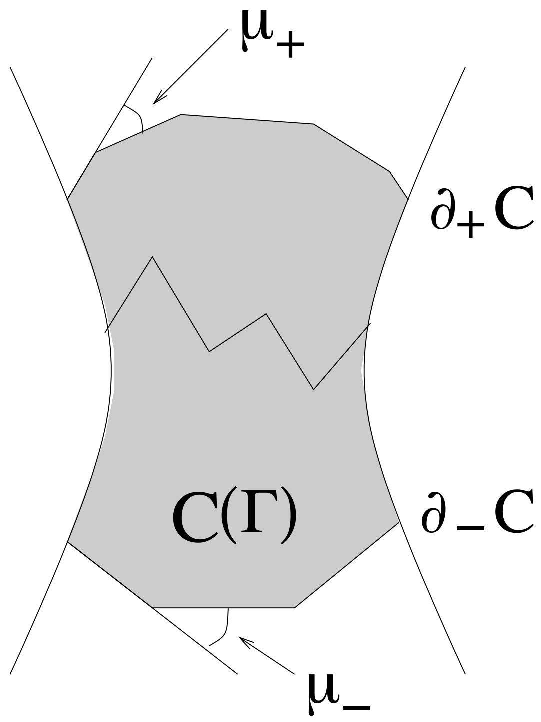
$\pi_1(S) \cong \Gamma \subset \mathrm{PSL}(2,\mathbb{C})$: quasi-Fuchsian group

$\Lambda(\Gamma)$: limit set $(\subset \mathbb{C}P^1)$

$CH(\Gamma)$: convex hull of $\Lambda(\Gamma)$ in $\mathbb{H}^3$

The boundary of $CH(\Gamma)$ consists of two totally geodesic surfaces $\partial_\pm CH(\Gamma)$ bent along measured laminations $\tilde{\mu}_\pm$. Denote $\partial_\pm C(\Gamma) = \partial_\pm CH(\Gamma)/\Gamma$ and the induced measured laminations by $\mu_\pm$.

Γ is uniquely determined by one of the pairs $(\partial_\pm C, \mu_\pm)$.



Question Can we describe $(\partial_- C, \mu_-)$ in terms of $(\partial_+ C, \mu_+)$?

Since the dim of $X(S)$ is greater than 1, it is natural to study the slice fixing some of λ_i and τ_i.

We study the deformation space of τ_1 fixing the other parameters.

Facts

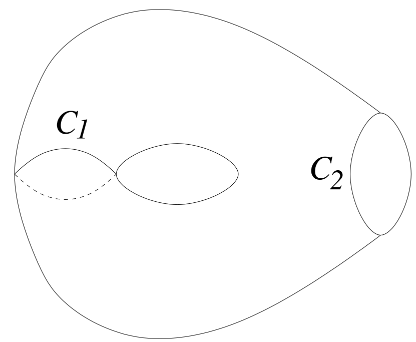
When we change a PSL$(2,\mathbb{C})$-rep by the Dehn twist along i-th pants curve, then

$$(\lambda_i, \tau_i) \mapsto (\lambda_i, \tau_i + \lambda_i)$$

So it is sufficient to study τ_1 in the range $0 \leq \mathrm{Re}(\tau_1) \leq l_1$ and $-\pi \leq \mathrm{Im}(\tau_1) \leq \pi$.

Zero and half twist

For simplicity, we assume that C_1 is on the torus with boundary C_2. (C_i means the i-th pants curve.) We assume λ_1 is real and λ_i, τ_i are real for $i \neq 1$.



Prop • When $\tau_1 = 0 + b_1\sqrt{-1}$, if

$$|b_1| < 2\,\mathrm{arccos}\left(\frac{2\sinh(l_1/2)}{\sqrt{2\cosh(l_1) + 2\cosh(l_2/2)}}\right),$$

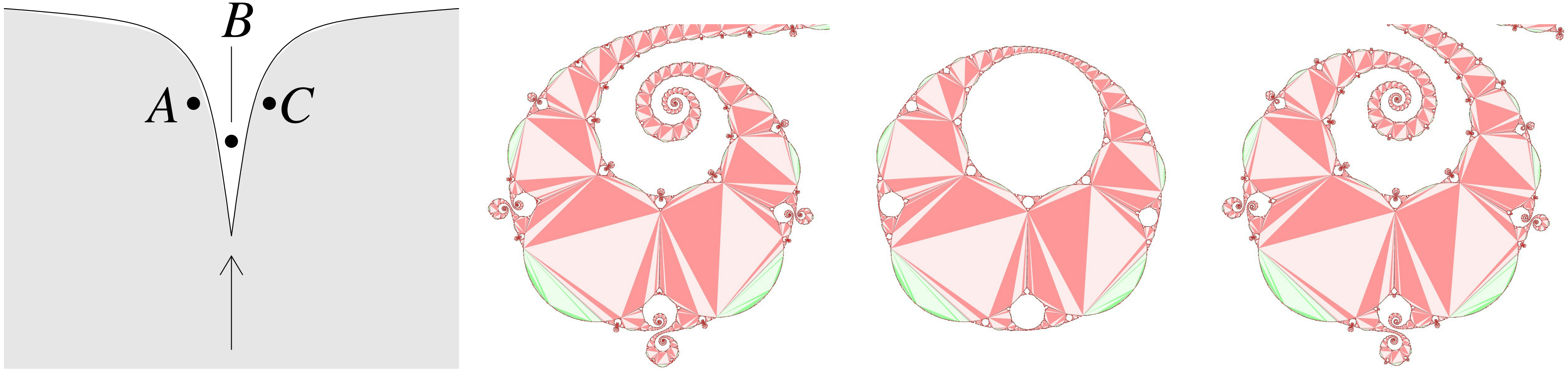
then the rep is quasi-Fuchsian. The bending loci are $C_1(= 1/0)$ and $0/1$.

• When $\tau_1 = l_1/2 + b_1\sqrt{-1}$, there is such a bound and the bending loci are C_1 and $1/2$.

Remark The first statement was already proved by Parker-Series for the once punctured tours case.

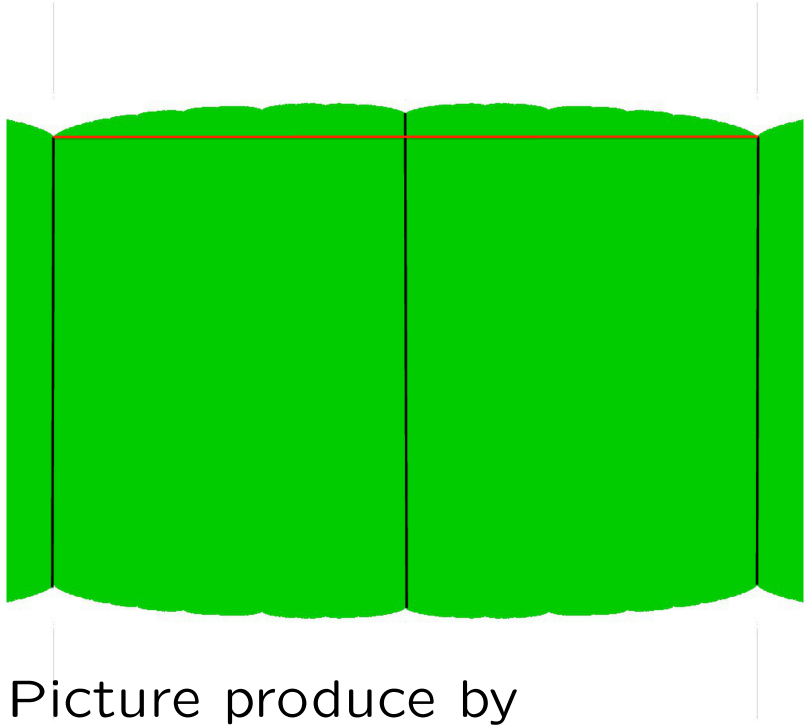
Behavior near a cusp group

On the slice, a 'cusp group' has a cusp-like neighborhood (Miyachi). This phenomenon can be observed from the developing maps:



Behavior near a cusp group

If we consider the once-punctured torus group, the shape of $\mathcal{QF}$ in τ_1-plane has the shape as:



Picture produce by

Yamashita's program

Parker-Parkkonen showed that we can take the red line in the above picture within $\mathcal{QF}$ for once-punctured group case. A similar statement holds in general case.