Quandle cocycles from group cocycles

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(These slides are available.)

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Introduction

Dijkgraaf-Witten invariant

M : closed oriented 3-mfd, $\rho : \pi_1(M) \to G$: homom.

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(Though there is no reference as far as I know.)

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Aim

Construct a quandle cocycle from a group cocycle.
Relate the associated quandle cocycle inv to DW-inv.

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$\exists \mathrm{Vol} \in H^3(\mathrm{PSL}_2\mathbb{C}; \mathbb{R}) \cong H^3(K(\mathrm{PSL}_2\mathbb{C}, 1); \mathbb{R})$ s.t.

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Construct a quandle cocycle from the group cocycle Vol.

Apply it to give a diagrammatic computation of the volume (and Chern-Simons invariant) (w/ Ayumu Inoue).

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Construct a quandle cocycle from the group cocycle Vol.

Apply it to give a diagrammatic computation of the volume (and Chern-Simons invariant) (w/ Ayumu Inoue).

Quandle

Definition

A quandle X is a set with $* : X \times X \to X$ satisfying

(Q1) ${}^\forall x \in X$, $x * x = x$.

(Q2) ${}^\forall y \in X$, $*y : x \mapsto x * y$ is a bijection.

(Q3) $(x * y) * z = (x * z) * (y * z) \quad ({}^\forall x, y, z \in X)$

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Example (Conjugation quandle)

G : a group, $S \subset G$: a subset closed under conjugation.
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Define $x * y = y^{-1}xy \quad (x, y \in S)$. $(S, *)$ is a quandle.
(Q1) and (Q2) are clearly satisfied, and we have

$$\begin{aligned}(x * y) * z &= z^{-1}(y^{-1}xy)z = z^{-1}y^{-1}zz^{-1}xzz^{-1}yz \\ &= (y * z)^{-1}(x * z)(y * z) = (x * z) * (y * z).\end{aligned}$$

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Example (Dihedral quandle)

$X = \mathbb{Z}/p\mathbb{Z}$ ($p \geq 3$)
For $i, j \in X$, let $\quad i * j = 2j - i \mod p$.

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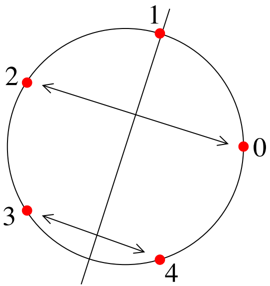
For $i, j \in X$, let $i * j = 2j - i \mod p$.
This is a conjugation quandle:
$G = D_{2p} = \langle r, x \mid r^2 = 1, x^p = 1, rx = x^{-1}r \rangle$
$S = \{x^i r x^{-i} \mid i = 0, \ldots, p-1\}$ 'reflection along i-axis'
$(x^j r x^{-j})^{-1}(x^i r x^{-i})(x^j r x^{-j}) = x^{2j-i} r x^{-(2j-i)}$



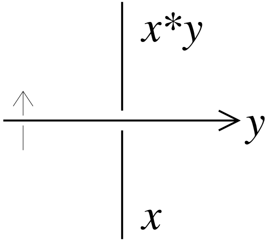
$(p = 5)$

$2 * 1 = 0,$
$3 * 1 = -1$

Quandle coloring

X : a quandle, L : an oriented link,
$D(L)$: a link projection

A quandle coloring is a map from 'arcs' of $D(L)$ to X satisfying the condition below at each crossing.



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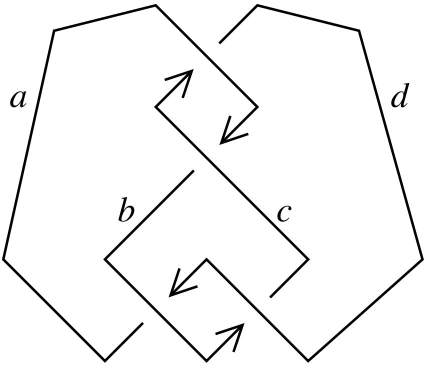

$$a, b, c, d \in X$$

$$c * a = d,$$

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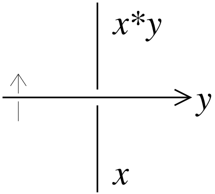

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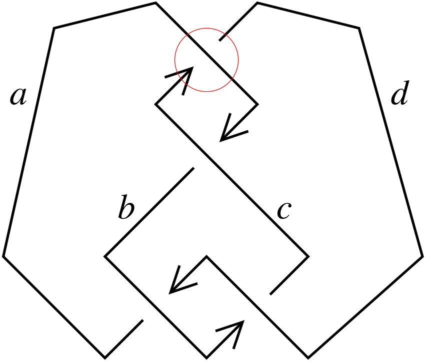

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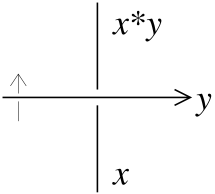

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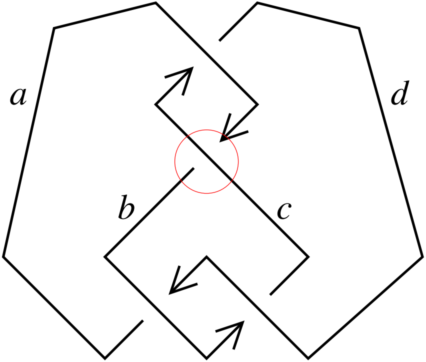

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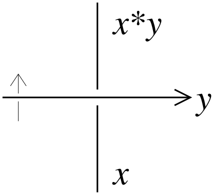

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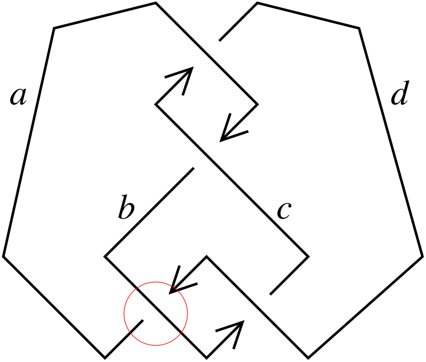

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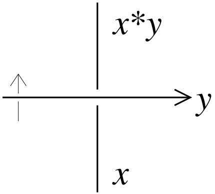

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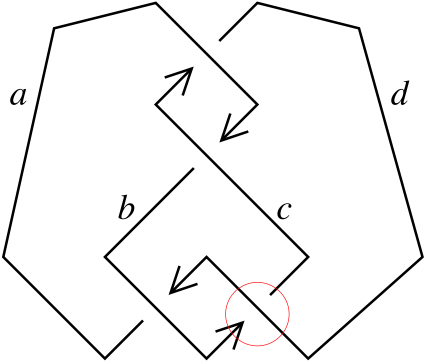

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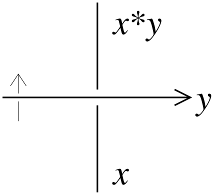

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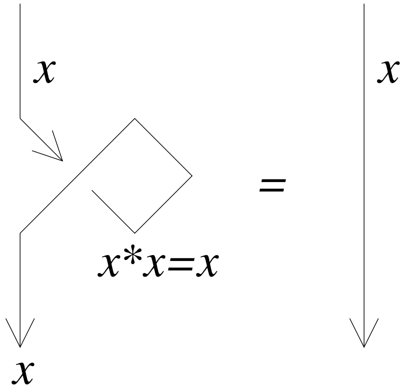
The coloring does not depend on the projection $D(L)$:
There are 1:1 corre. of colorings under R moves.



Correspondence under Reidemeister I.

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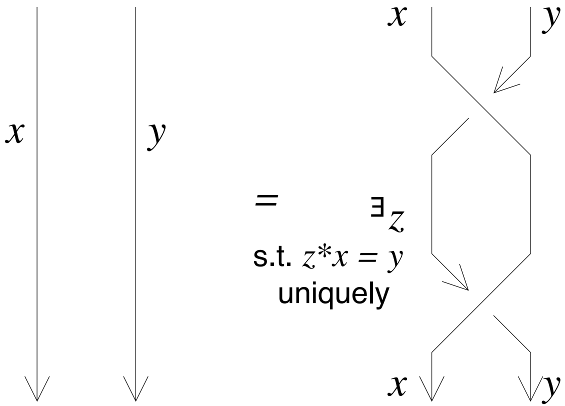

Correspondence under Reidemeister II.

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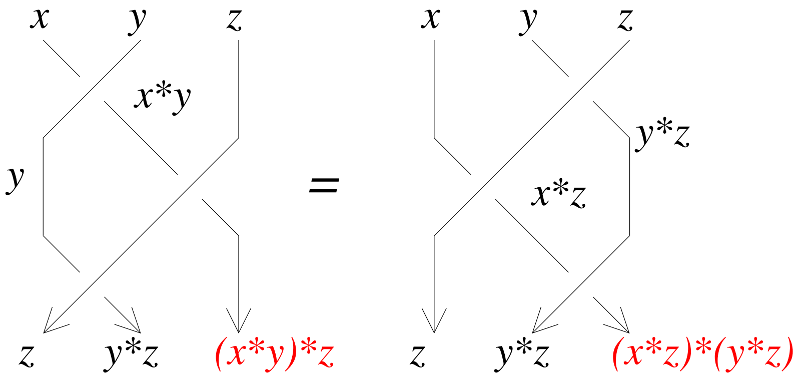
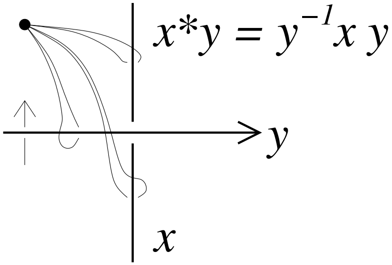


Correspondence under Reidemeister III.

Quandle coloring

If X is a conjugation quandle of a group G
(i.e. $X \subset G$ and $x * y = y^{-1}xy$ for $x, y \in X$),
then a coloring by X gives a rep $\pi_1(S^3 \setminus L) \to G$.



Recall the Wirtinger presentation of $\pi_1(S^3 \setminus L)$.

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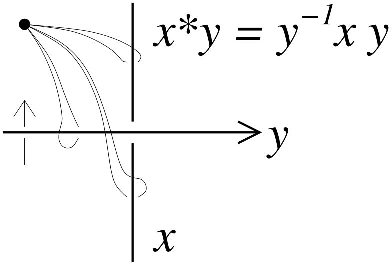


Recall the Wirtinger presentation of $\pi_1(S^3 \setminus L)$.

A quandle coloring by a conjugation quandle $X \subset G$.

$\overset{1:1}{\longleftrightarrow}$

A rep $\pi_1(S^3 \setminus L) \to G$ sending a meridian to an element of X.

Group homology

For a group G, let

$$C_n(G) = \operatorname{span}_{\mathbb{Z}[G]}\{[g_1|\cdots|g_n] \mid g_i \in G\}.$$

(free $\mathbb{Z}[G]$-module)

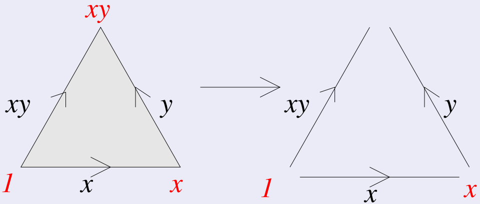
$\partial : C_n(G) \to C_{n-1}(G)$ is defined by

$$\partial[g_1|\cdots|g_n] = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

For example:

$$\partial[x|y] = x[y] - [xy] + [x],$$
$$\partial[x|y|z] = x[y|z] - [xy|z] + [x|yz] - [x|y].$$

Group homology



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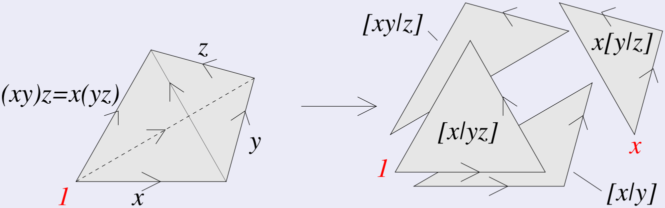

$$\partial[x|y|z] = x[y|z] - [xy|z] + [x|yz] - [x|y]$$

(It is easy to see that $\partial \circ \partial = 0$.)

Group homology

M : a right $\mathbb{Z}[G]$-module

$H_n(G; M) := H_n(M \otimes_{\mathbb{Z}[G]} C_*(G))$ group homology

Quandle homology

Definition (Adjoint group)

For a quandle X, define the adjoint group by

$$\mathrm{Ad}(X) = \langle x \in X \mid x * y = y^{-1} \cdot x \cdot y \rangle.$$

(also known as the associated group or enveloping group)

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Remark

For a Lie algebra L (a vector sp with $[,] : V \otimes V \to V$), the universal enveloping algebra is defined by

$$U(L) = \left(\bigoplus_{n \geq 0} L^{\otimes n} \right) \Big/ \{[v_1, v_2] = v_1 \otimes v_2 - v_2 \otimes v_1\}.$$

$\mathrm{Ad}(X)$ satisfies some universal properties as $U(L)$ does.

Quandle homology

Lie algebra (co)homology is defined as the (co)homology of the associative algebra $U(L)$.

But quandle (co)homology is NOT isomorphic to the (co)homology of the group $\mathrm{Ad}(X)$.

Quandle homology

For a quandle X, let

$$C_n^R(X) = \text{span}_{\mathbb{Z}[\text{Ad}(X)]}\{(x_1, \cdots, x_n) \mid x_i \in X\}.$$

Define the boundary operator $\partial : C_n^R(X) \to C_{n-1}^R(X)$ by

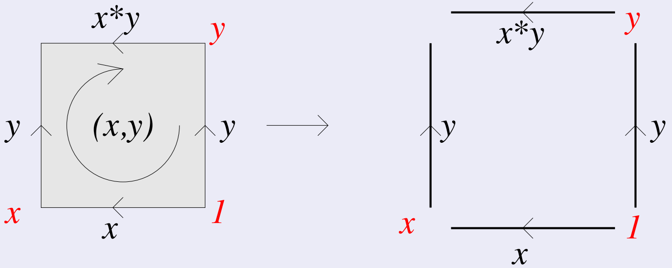
$$\begin{aligned}\partial(x_1, \cdots, x_n) = \sum_{i=1}^{n} (-1)^i \{ & (x_1, \cdots, \widehat{x_i}, \cdots, x_n) \\ & - x_i(x_1 * x_i, \cdots, x_{i-1} * x_i, x_{i+1}, \cdots, x_n)\}.\end{aligned}$$

For examples:

$$\begin{aligned}\partial(x, y) &= -((y) - x(y)) + ((x) - y(x * y)), \\ \partial(x, y, z) &= -((y, z) - x(y, z)) + ((x, z) - y(x * y, z)) \\ &\quad - ((x, y) - z(x * z, y * z)).\end{aligned}$$

Quandle homology

$\partial : C_2^R(X) \to C_1^R(X)$



$$\partial(x, y) = -(y) + x(y) + (x) - y(x * y)$$

(It is easy to check that $\partial \circ \partial = 0$.)

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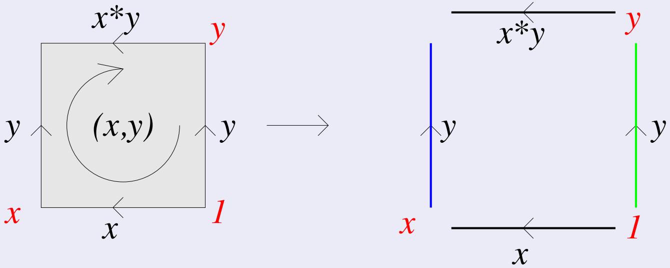

$$\partial(x, y) = {\color{green}-(y)}{\color{blue}+x(y)} + (x) - y(x * y)$$

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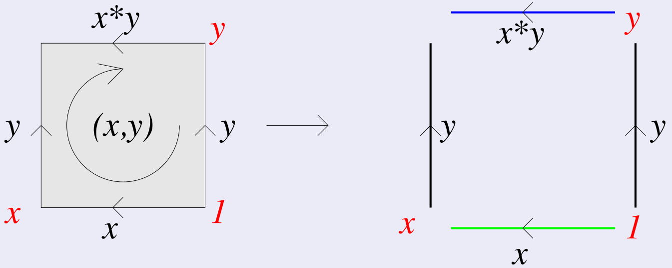

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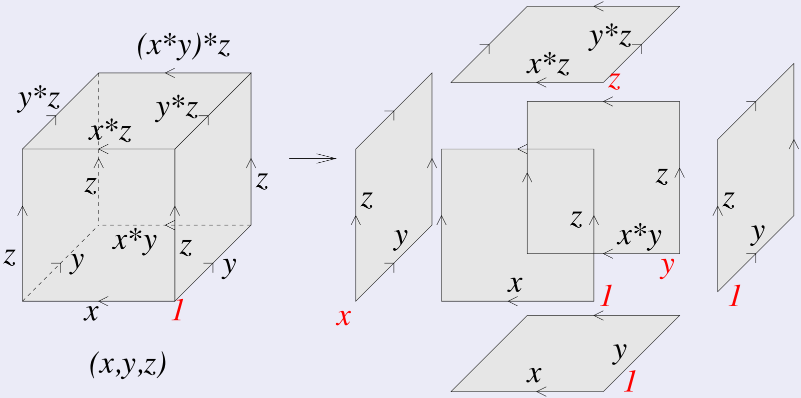

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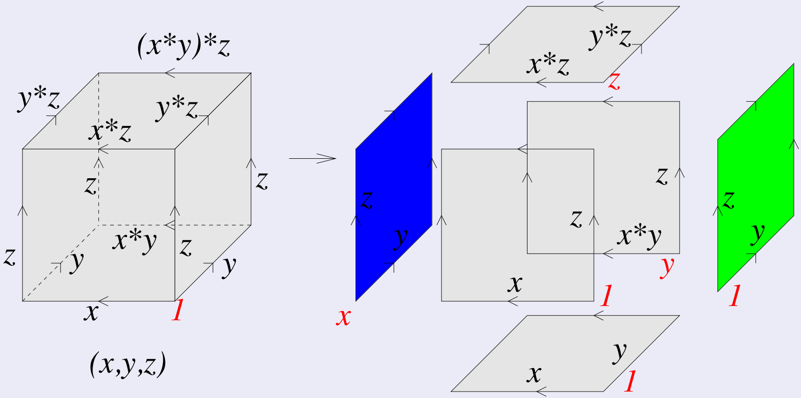

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$$-(x,y) + z(x*z, y*z)$$

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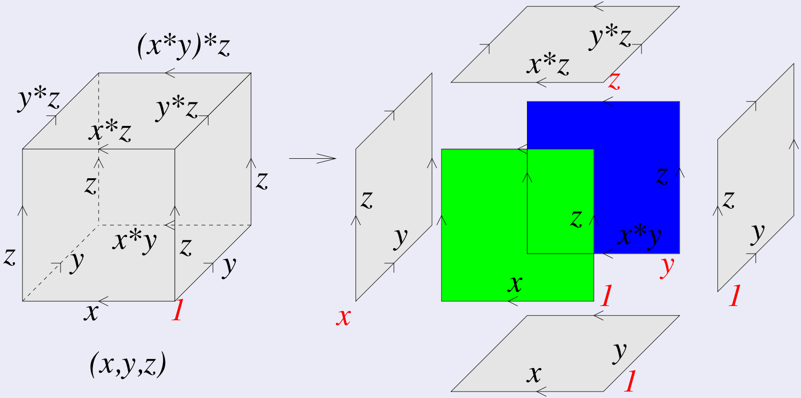

$$\partial(x, y, z) = -(y, z) + x(y, z) + (x, z) - y(x * y, z)$$
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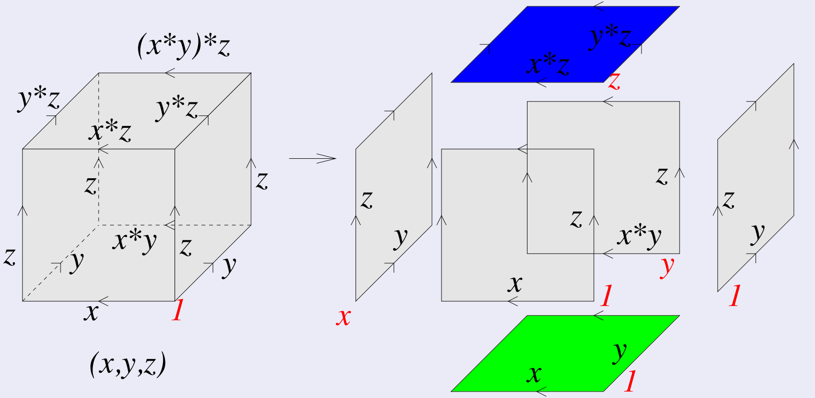

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Quandle homology

M : a right $\mathbb{Z}[\mathrm{Ad}(X)]$-module.

The homology group of

$$C_n^R(X;M) = M \otimes_{\mathbb{Z}[\mathrm{Ad}(X)]} C_n^R(X)$$

is called the rack homology $H_n^R(X;M)$.

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Then define a subcomplex $C_n^D(X) \subset C_n^R(X)$, and define the quandle homology by

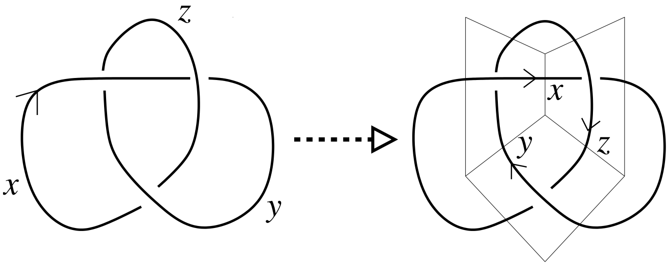
$$C_n^Q(X) := C_n^R(X)/C_n^D(X)$$
$$H_n^Q(X; M) := H_n(M \otimes_{\mathrm{Ad}(X)} C_*^Q(X))$$

Application to knot theory

$L \subset S^3$: an (oriented) link

Fix a diagram $D(L)$ (on S^2).

A coloring by a quandle X gives a cycle in $H_2^Q(X; \mathbb{Z})$:



The cycle $(x, z) + (z, y) + (y, x)$.

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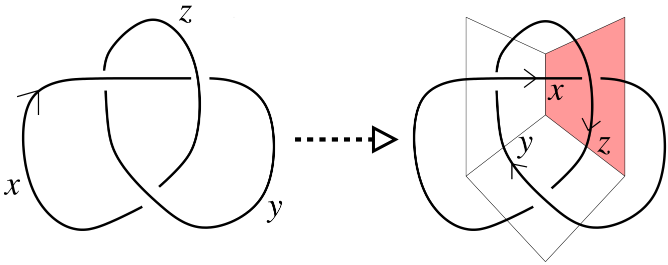

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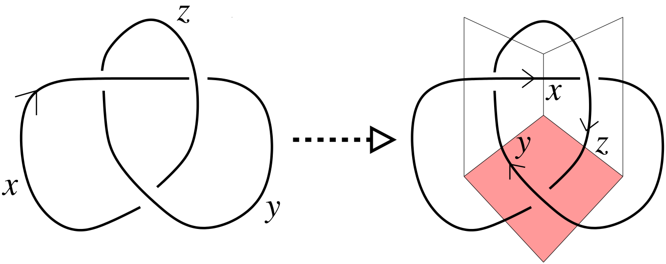

The cycle $(x,z) + {\color{red}(z,y)} + (y,x)$.

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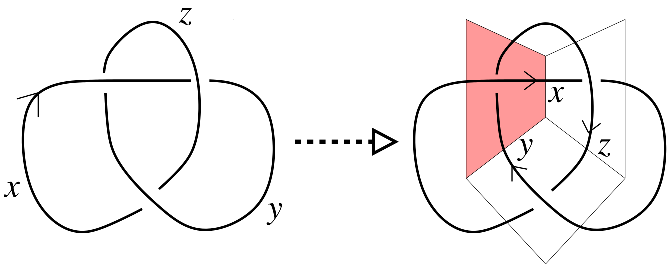
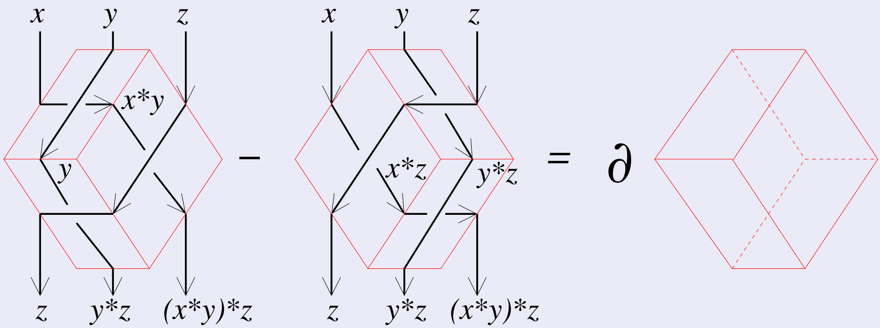


The cycle $(x, z) + (z, y) + (y, x)$.

This homology class in $H_2^Q(X;\mathbb{Z})$ does not depend on the choice of the diagram.

The invariance under Reidemeister III move



$$((x,y)+y(x*y,z)+(y,z))-((x,z)+x(y,z)+z(x*z,y*z))$$
$$=\partial(x,y,z)$$

(Untwisted) 2-cycle

Roughly, the 2-cycle associated to a coloring measures the longitudinal holonomy (Eisermann).

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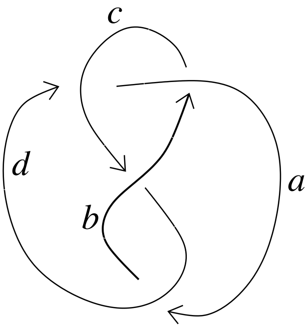

Given a knot with an arc coloring.

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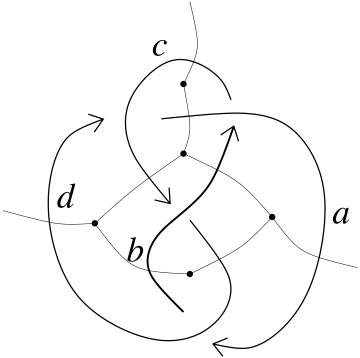

Subdivide S^2 into squares, they form a 2-cycle.

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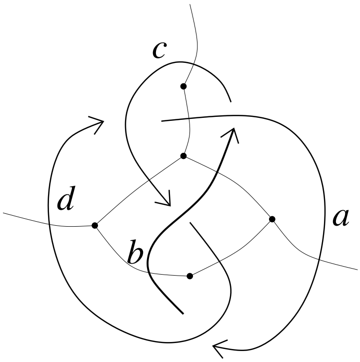

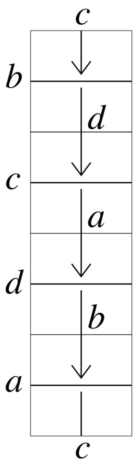

Rearrange the 2-cycle so that the squares form a torus.

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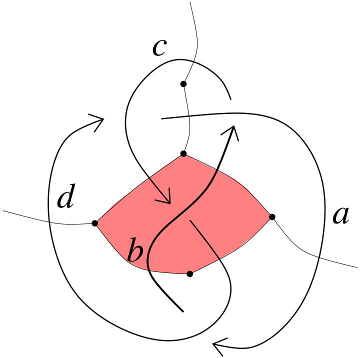

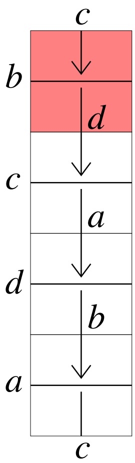

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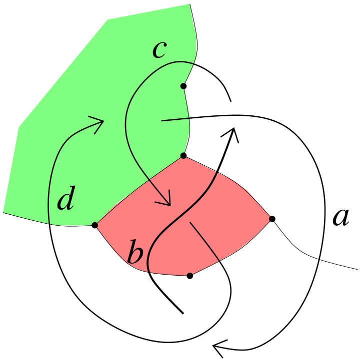

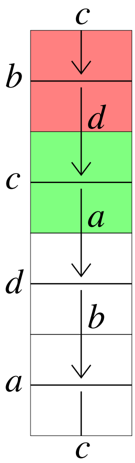

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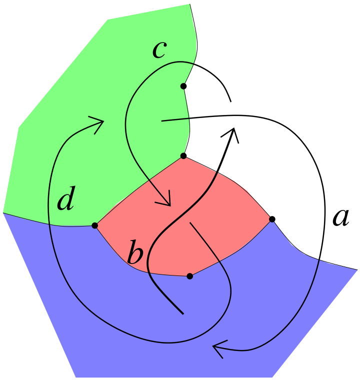

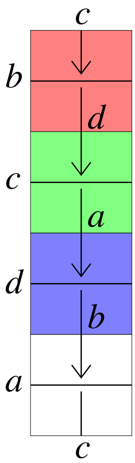

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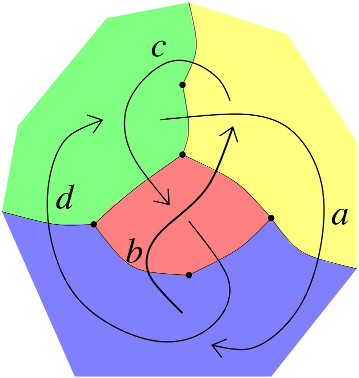

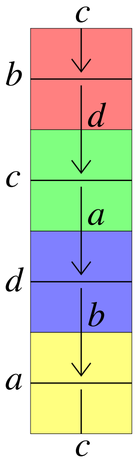

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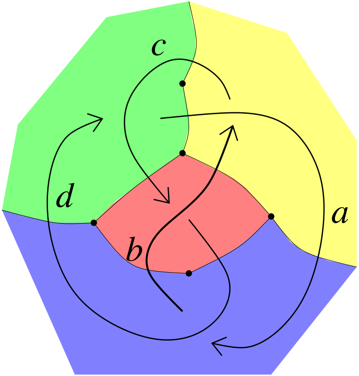

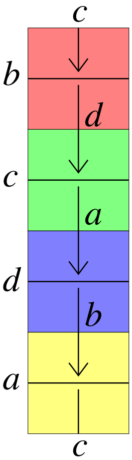

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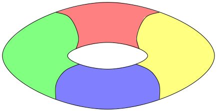

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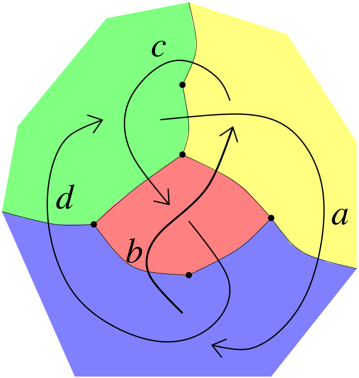

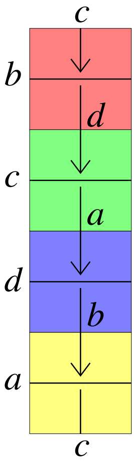

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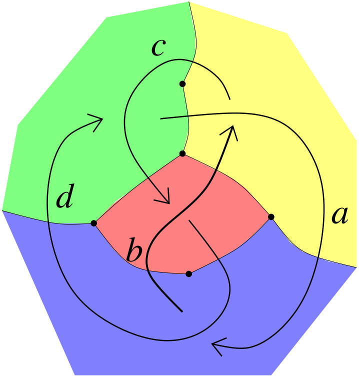

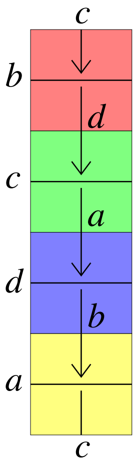

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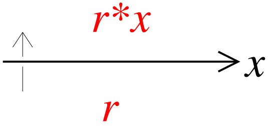
Refer to [Eisermann, Pacific J. 2007] for details.

Region coloring

$L \subset S^3$: oriented link, $D(L)$: a diagram as before

Fix a coloring of arcs by X.

A region coloring is a map from 'regions' of $D(L)$ to X satisfying the condition:



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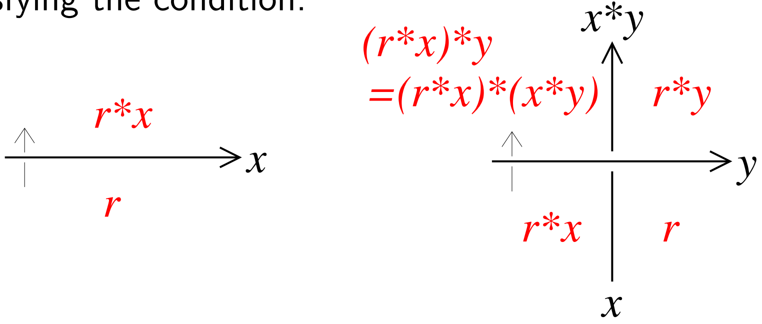
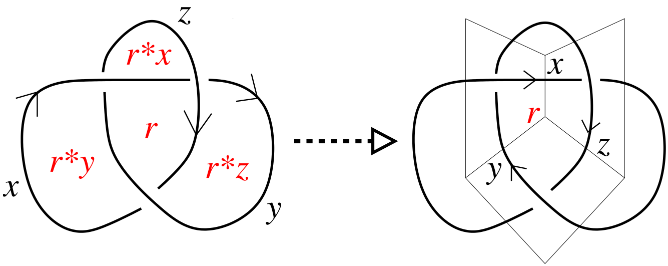


This is consistent with the arc coloring.

Region coloring

The pair, arc and region colorings gives a ‘twisted’ cycle :



$$r \otimes (x, z) + r \otimes (z, y) + r \otimes (y, x) \in \mathbb{Z}[X] \otimes_{\mathrm{Ad}(X)} C_2^Q(X).$$

Here, $\mathbb{Z}[X] = \mathrm{span}_{\mathbb{Z}} X$ is a right $\mathrm{Ad}(X)$-module by

$$r \cdot (x_1 x_2 \cdots x_n) = (\cdots (r * x_1) * x_2) \cdots) * x_n.$$

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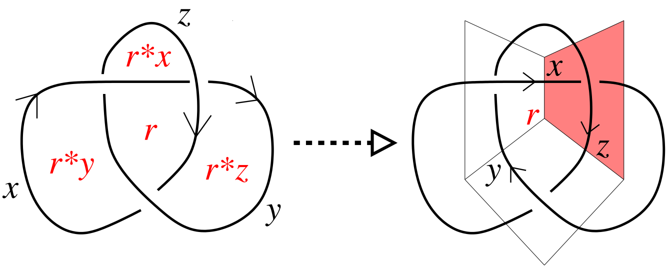

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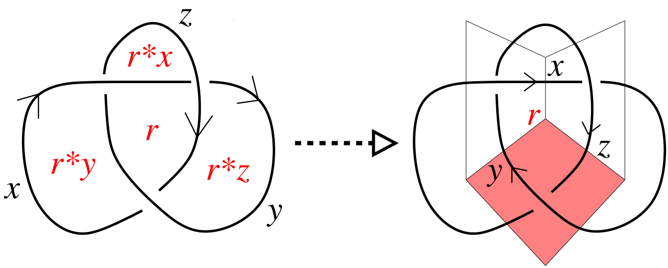

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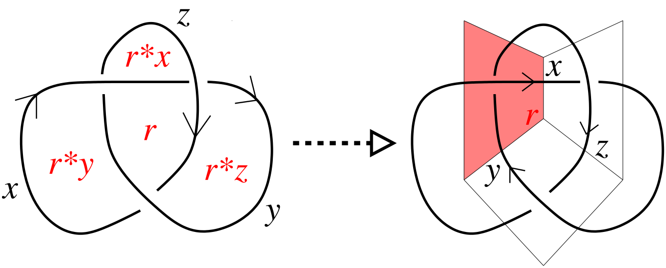

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Region coloring

Prop (see [A.Inoue-K.])

The (twisted) homology class in $H_2^Q(X;\mathbb{Z}[X])$ does not depend on the choice of the region coloring (undear a mild assumption on$X$).

We remark that

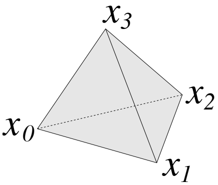
$$H^R_{k+1}(X;\mathbb{Z}) \cong H^R_k(X;\mathbb{Z}[X]).$$

via $(r, x_1, \cdots, x_k) \mapsto (-1)^k\, r \otimes (x_1, \cdots, x_k)$.

(There is a homom $H^Q_{k+1}(X;\mathbb{Z}) \to H^Q_k(X;\mathbb{Z}[X])$.)

$H_n^{\Delta}(X)$

For a quandle X, let
$C_n^{\Delta}(X) = \text{span}_{\mathbb{Z}}\{(x_0, \cdots, x_n) \mid x_i \in X\}$.
Define $\partial : C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ by



$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \cdots, \widehat{x_i}, \cdots, x_n).$$

Since $\text{Ad}(X)$ acts on X from the right, $C_n^{\Delta}(X)$ is a right $\mathbb{Z}[\text{Ad}(X)]$-module.

Def ('Simplicial quandle homology', A.Inoue-K.)

$$H_n^{\Delta}(X) := H_n(\, C_*^{\Delta}(X) \otimes_{\text{Ad}(X)} \mathbb{Z}\,)$$

$\varphi_* : H_n^R(X; \mathbb{Z}[X]) \to H_{n+1}^{\Delta}(X)$

We define a chain map

$$\varphi : \mathbb{Z}[X] \bigotimes_{\mathrm{Ad}(X)} C_n^R(X) \to C_{n+1}^{\Delta}(X) \bigotimes_{\mathrm{Ad}(X)} \mathbb{Z}$$

as follows.

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as follows.

Fix $p \in X$. $\quad I_n = \{\, \iota : \{1, 2, \cdots, n\} \to \{0, 1\}\,\}$.
For $\iota \in I_n$, $\;|\iota| = \#\{k \mid \iota(k) = 1,\ 1 \leq k \leq n\}$.

$$\begin{aligned}
&\varphi(r \otimes (x_1, x_2, \cdots, x_n)) \\
&\quad = \sum_{\iota \in I_n} (-1)^{|\iota|} (p, r(\iota), x(\iota, 1), x(\iota, 2), \cdots, x(\iota, n)).
\end{aligned}$$

where $r(\iota) = r * (x_1^{\iota(1)} x_2^{\iota(2)} \cdots x_n^{\iota(n)})$ and
$x(\iota, i) = x_i * (x_{i+1}^{\iota(i+1)} x_{i+2}^{\iota(i+2)} \cdots x_n^{\iota(n)})$.

$\varphi_* : H^R_n(X;\mathbb{Z}[X]) \to H^\Delta_{n+1}(X)$

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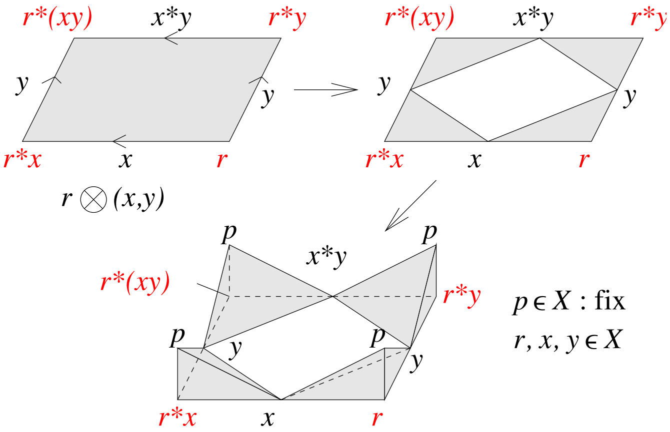
The pictorial definition is as follows:

$$\varphi : \mathbb{Z}[X] \otimes_{\mathrm{Ad}(X)} C_2^R(X) \to C_3^\Delta(X) \otimes_{\mathrm{Ad}(X)} \mathbb{Z}$$

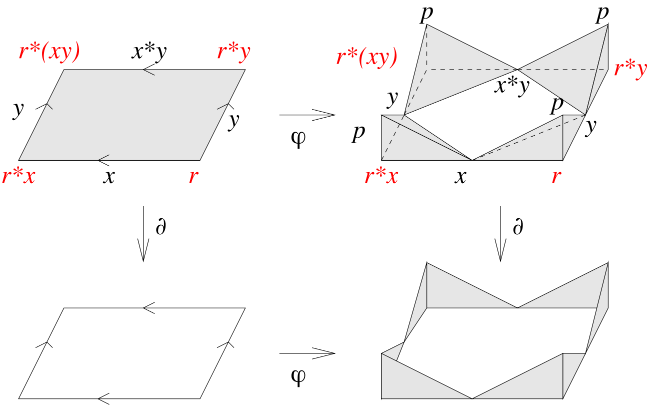


$p \in X$: fix

$r, x, y \in X$

$$\varphi(r \otimes (x,y)) = (p, r, x, y) - (p, r*x, x, y)$$
$$- (p, r*y, x*y, y) + (p, r*(xy), x*y, y)$$

φ is actually a chain map ($\partial \circ \varphi = \varphi \circ \partial$) [A.Inoue-K.].



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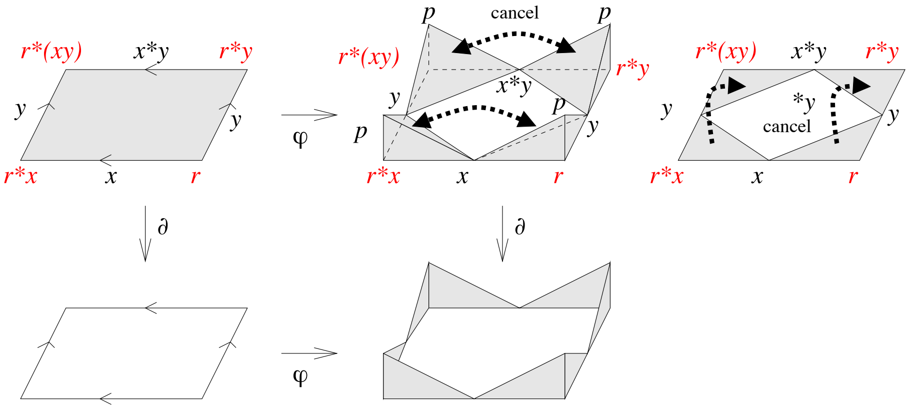
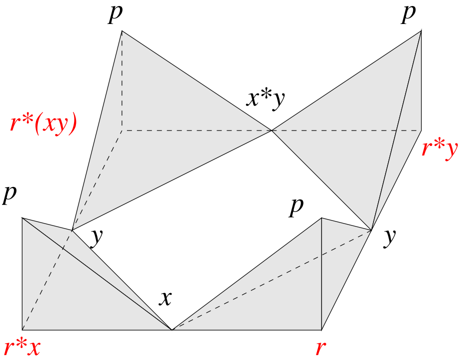
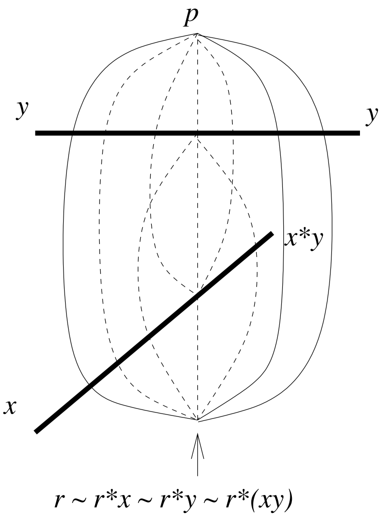


The definition of φ is motivated by a triangulation of $S^3 \setminus K$ near a crossing.





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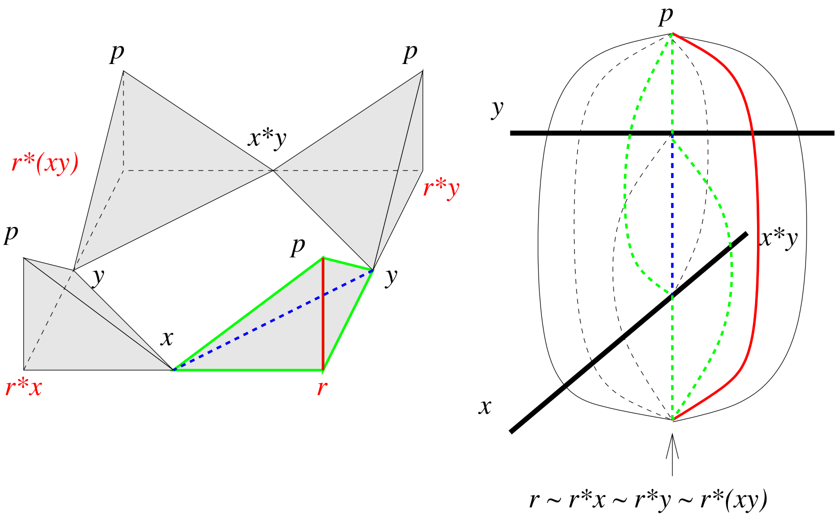

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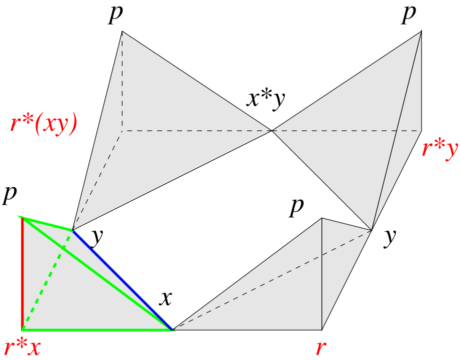

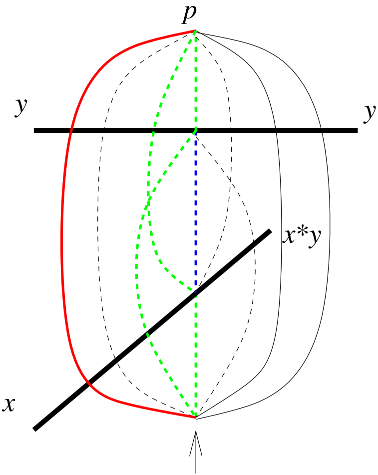

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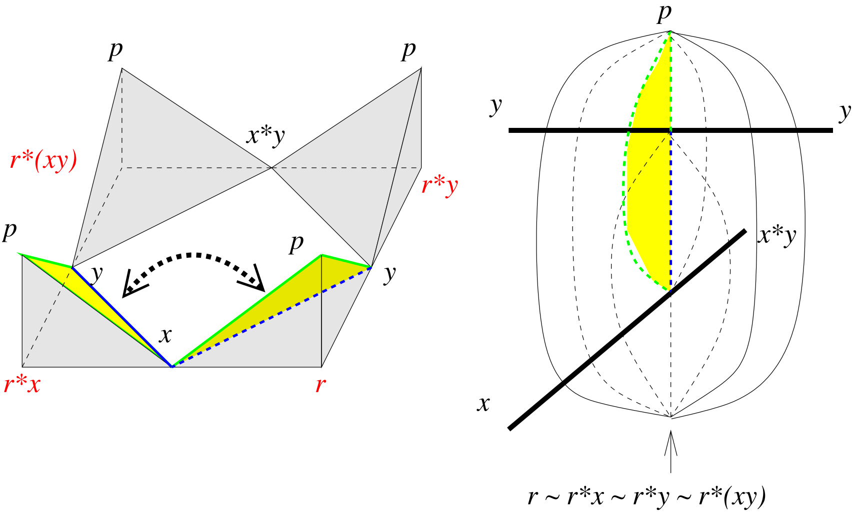

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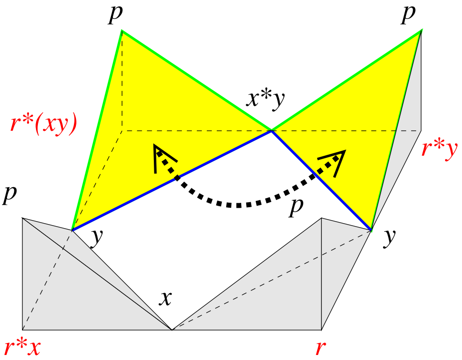

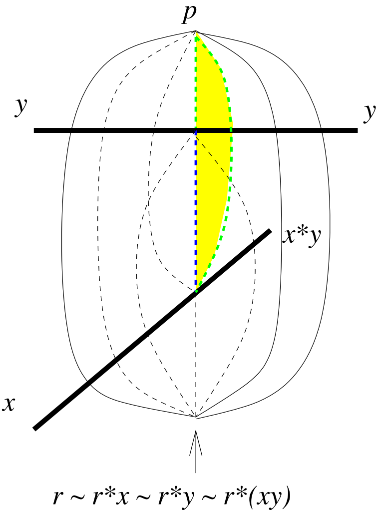

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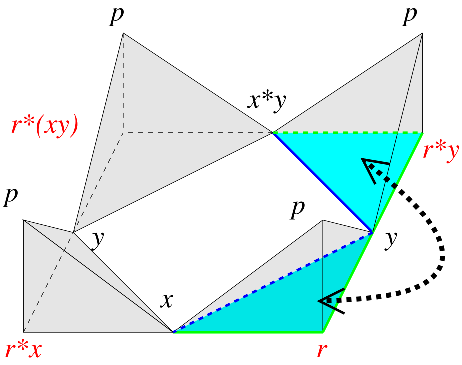

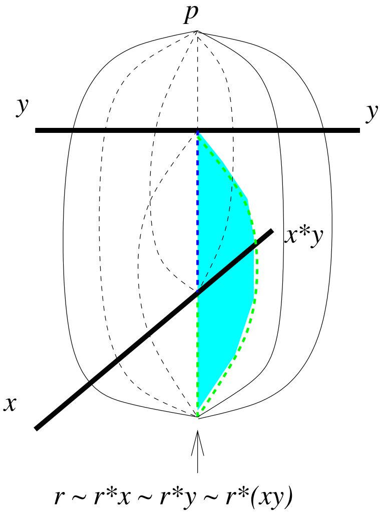

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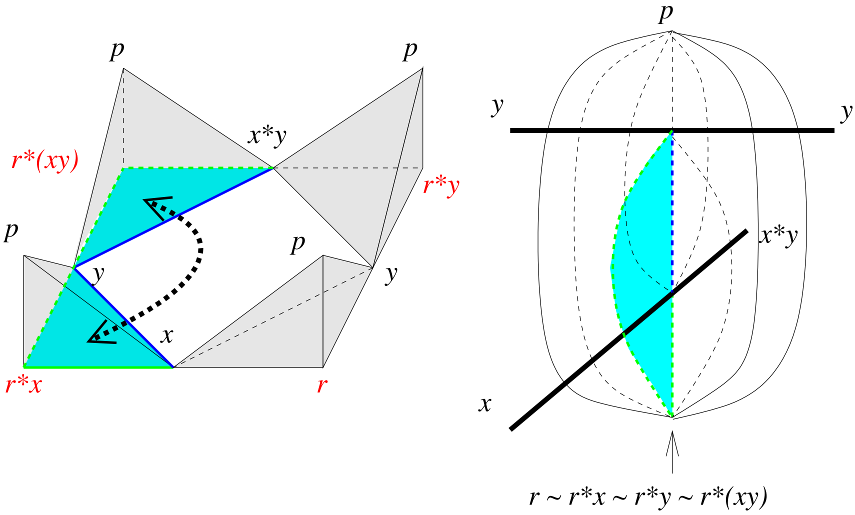

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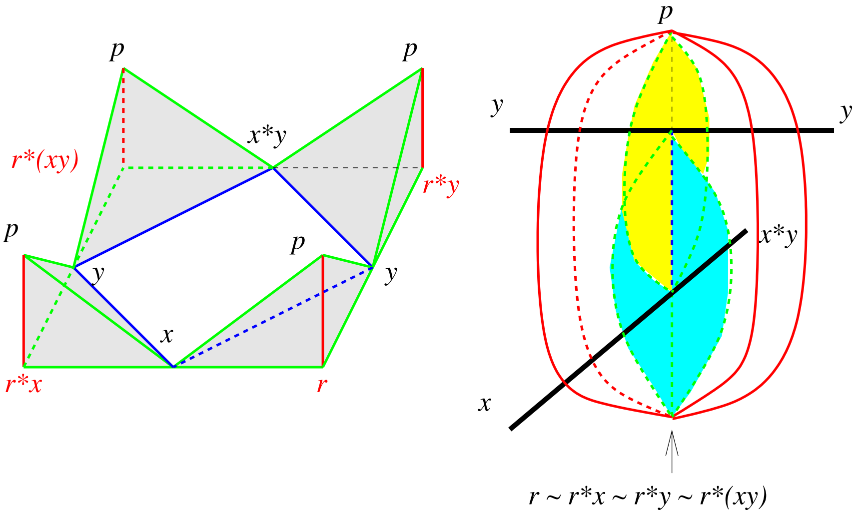

The map φ induces a homomorphism

$$H^R_{n+1}(X;\mathbb{Z}) \cong H^R_n(X;\mathbb{Z}[X]) \to H^\Delta_{n+1}(X).$$

We can construct a quandle cocycle from a cocycle of $H^\Delta_k(X)$, that is $f : X^{k+1} \to A$ (abelian gp) satisfying

(1) $\sum_i (-1)^i f(x_0, \cdots, \widehat{x_i}, \cdots, x_{k+1}) = 0$,
(2) $f(x_0 * y, \cdots, x_k * y) = f(x_0, \cdots, x_k)$,
(3) $f(x_0, \cdots, x_k) = 0$ if $x_i = x_{i+1}$ for some i.

The pull-back of f is a cocycle of $H^Q_k(X;\mathbb{Z})$.

Dihedral quandle $X = \mathbb{Z}/p\mathbb{Z}$

Define $d : X^2 \to \mathbb{Z}/p\mathbb{Z}$ by

$$d(y,z) = \begin{cases} 1 & \text{if } \bar{y} + \bar{z} > p \\ -1 & \text{if } \bar{y} + \bar{z} < p \text{ and } yz \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{x}$ is an integer $0 \leq \bar{x} < p$ with $x = \bar{x} \mod p$.

Then $[x|y|z] \mapsto x \cdot d(y,z)$ is a group 3-cocycle of $\mathbb{Z}/p\mathbb{Z}$.

In the 'homogeneous notation',

$$(0,\, x,\, x+y,\, x+y+z) \mapsto x \cdot d(y,z)$$

This satisfies (1) and (3). To satisfy (2) we consider

$$(0,\, x,\, x+y,\, x+y+z) \mapsto x \cdot d(y,z) - x \cdot d(-y,-z)$$

Dihedral quandle $X = \mathbb{Z}/p\mathbb{Z}$

Apply φ, we obtain

$$
\begin{aligned}
(x, y, z) \overset{\varphi}{\mapsto} & (0, x, y, z) - (0, x * y, y, z) \\
& - (0, x * z, y * z, z) + (0, (x * y) * z, y * z, z) \\
\mapsto & x \cdot d(y - x, z - y) - x * y \cdot d(y - x * y, z - y) \\
& - x * z \cdot d(y * z - x * z, z - y * z) \\
& + (x * y) * z \cdot d(y * z - (x * y) * z, z - y * z) \\
= & 2z(d(y - x, z - y) + d(y - x, y - z)).
\end{aligned}
$$

(Recall $x * y = 2y - x$.)

Theorem (K.)

This is a non-trivial 3-cocycle.

Complex volume

$\mathbb{H}^3$: hyperbolic 3-space.

$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}/\{\pm 1\}$

$\mathcal{P} = \{g^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g \mid g \in \mathrm{PSL}_2\mathbb{C}\}$: parabolic elements

$\mathcal{P}$ is closed under conjugation, thus has a quandle structure defined by $x * y = y^{-1}xy$.

$P = \{\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} \mid z \in \mathbb{C}^*\}$: a parabolic subgroup

We have $\mathcal{P} \cong P\backslash\mathrm{PSL}_2\mathbb{C}$ by $g^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g \mapsto Pg$.

Complex volume

Since $\mathcal{P} \cong P\backslash \mathrm{PSL}_2\mathbb{C}$, we can regard $H_k^\Delta(\mathcal{P})$ as the Hochschild relative homology $H_k([G:P];\mathbb{Z})$.

Thm (Arciniega-Nevárez and Cisneros-Molina, arXiv:1303.2986)

$H_k([G:P];\mathbb{Z})$ is isomorphic to the Takasu relative homology $H_k(G,P;\mathbb{Z})$ (usual relative group homology).

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Zickert constructed

$$\widehat{L} \circ \Psi : H_3(G,P;\mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C}) \to \mathbb{C}/\pi^2\mathbb{Z}$$

which gives $i(\mathrm{Vol} + i\mathrm{CS})$. Thus we obtain a map

$$H_2^Q(\mathcal{P};\mathbb{Z}[\mathcal{P}]) \to H_3^{\Delta}(\mathcal{P}) \cong H_3(G,P;\mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}.$$

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For a hyperbolic knot K, we have a coloring by $\mathcal{P}$ corresponding to the discrete faithful representation.

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This gives a (twisted) 2-cycle in $H_2^Q(\mathcal{P};\mathbb{Z}[\mathcal{P}])$.

By construction, the image of this 2-cycle by

$$H_2^Q(\mathcal{P};\mathbb{Z}[\mathcal{P}]) \to H_3^\Delta(\mathcal{P}) \cong H_3(G,P;\mathbb{Z}) \to \mathbb{C}/\pi^2\mathbb{Z}.$$

computes $i(\mathrm{Vol}(S^3 \setminus K) + i\mathrm{CS}(S^3 \setminus K))$.

In [A.Inoue-K.], we gave a more explicit description.