曲面群の PGL(n,C) 表現の Fenchel-Nielsen 座標

蒲谷祐一
京都大学理学研究科
スライド：https://www.math.kyoto-u.ac.jp/~kabaya/

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S 上の標識付き双曲計量 $\overset{1:1}{\longleftrightarrow}$ $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$
injective, discrete (up to conj.)
($\rho \sim \rho'$: conjugate $\Leftrightarrow$ $\rho' = g\rho g^{-1}$ ($^\exists g \in \mathrm{PSL}(2, \mathbb{R})$))

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複素化

$$\{\text{injective, discrete}\} \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))/\sim_{\mathrm{conj.}}$$

は，$S \times (-1,1)$ 上の (marked) 双曲計量全体をパラメトライズ

($\mathbb{H}^3$：上半空間, $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$)

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Teichmüller space $\mathcal{T}(S)$ 上の $6g-6$ 個の関数で

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問題　$\mathrm{PSL}(2,\mathbb{C})$ を $\mathrm{PSL}(n,\mathbb{C})$ に一般化すると？

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問題　PSL$(2,\mathbb{C})$ を PSL$(n,\mathbb{C})$ に一般化すると？
(Xin Nie 氏との共同研究)

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そのほか様々な幾何構造を考えたい

$\mathrm{SO}^+(n, 1)$: hyperbolic, $\mathrm{PU}(n, 1)$: complex hyperbolic, $\cdots$

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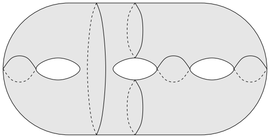

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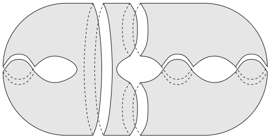

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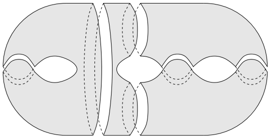

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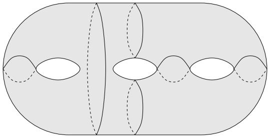

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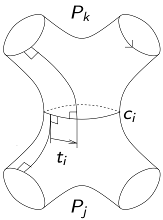

For $X \in \mathcal{T}(S)$, let

$l_i =$ length of c_i, $\quad t_i =$ twist along c_i

by the (marked) hyperbolic metric X.

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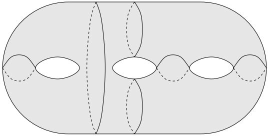

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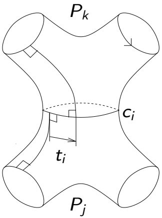

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Fenchel-Nielsen coordinates : $\mathcal{T}(S) \xrightarrow[\cong]{(l_i, t_i)} \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$

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$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ (metric $\frac{|dz|^2}{\mathrm{Im}(z)^2}$),

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$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ acts on $\mathbb{H}^2$ and $\mathbb{R}P^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

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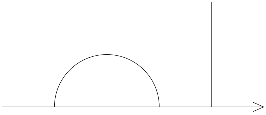
A geodesic in $\mathbb{H}^2$ is

- a line orthogonal to $\mathbb{R}$, or
- a (half) circle orthogonal to $\mathbb{R}$



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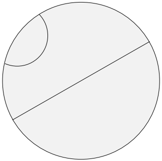
Sometimes, we draw $\mathbb{H}^2$ by the Poincaré disk model

$$\{z \in \mathbb{C} \mid |z| < 1\}$$



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Fact

$z_0, z_1, z_2 \in \mathbb{R}P^1$: distinct
$^{\exists}$ unique $g \in \mathrm{PGL}(2, \mathbb{R})$ so that $(gz_0, gz_1, gz_2) = (0, \infty, 1)$

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Definition (Cross ratio)

For $z_0, \ldots, z_3 \in \mathbb{R}P^1$ distinct, define

$$[z_0 : z_1 : z_2 : z_3] = \frac{(z_3 - z_0)(z_2 - z_1)}{(z_3 - z_1)(z_2 - z_0)}$$

For $g \in \mathrm{PGL}(2, \mathbb{R})$, $[gz_0 : gz_1 : gz_2 : gz_3] = [z_0 : z_1 : z_2 : z_3]$

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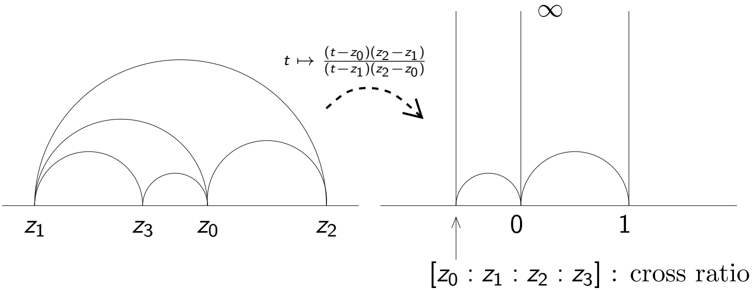

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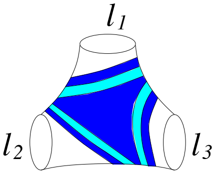
パンツ P_i (geodesic boundaries, lengths l_1, l_2, l_3) の 'ideal triangulation'



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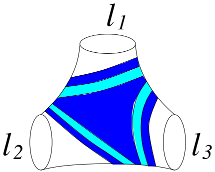

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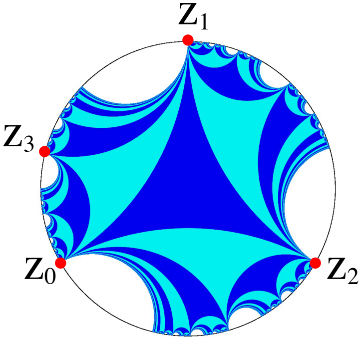

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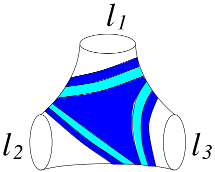

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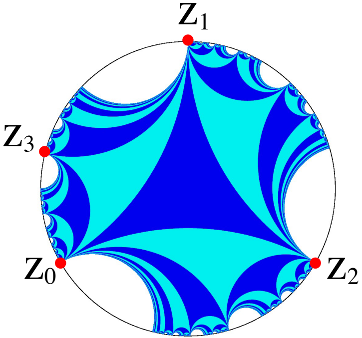

$$-[z_0 : z_1 : z_2 : z_3] = \exp\left(\frac{-l_1 - l_2 + l_3}{2}\right)$$

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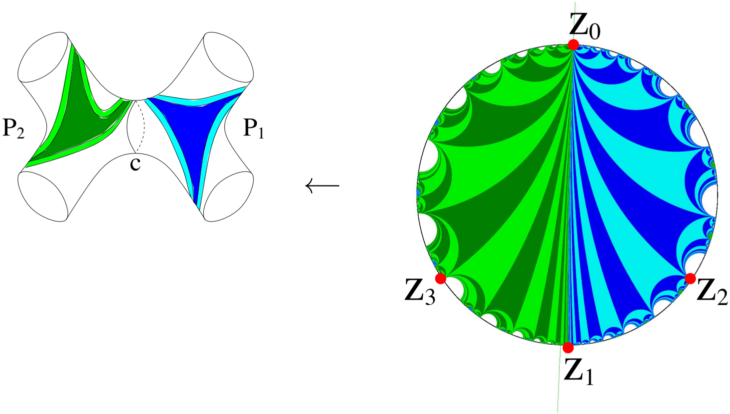
Twist parameter t_i is also described by cross ratios.



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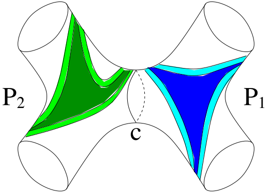

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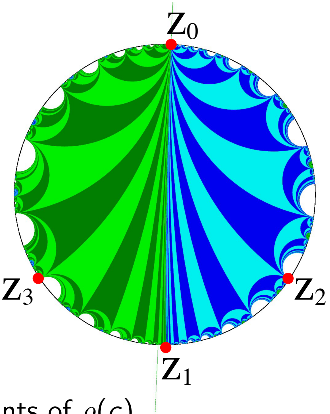

Note : $z_0, z_1 \in \mathbb{R}P^1$ are fixed points of $\rho(c)$.

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これまでの議論は，$\mathbb{R}P^1$ を $\mathbb{C}P^1$ に変えても同様にできる　$\mathrm{PSL}(2, \mathbb{C})$ 表現をパラメトライズできる

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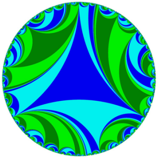

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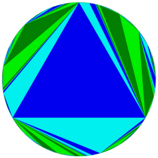

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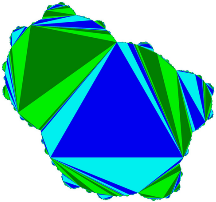

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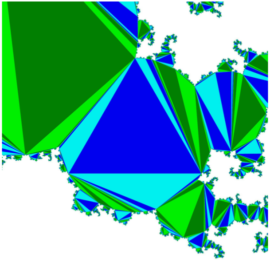

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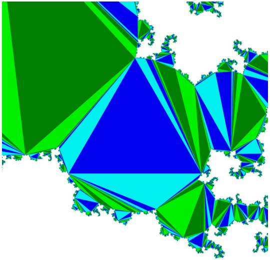

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PSL$(n, \mathbb{C})$-case　$\mathbb{C}P^1$ の代わりに flag manifold を利用する

3. Flags

Definition

A flag in $\mathbb{C}^n$ is a sequence of subspaces

$$\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \subsetneq \cdots \subsetneq V^n = \mathbb{C}^n.$$

$\mathcal{F}_n = \{\text{Flags in } \mathbb{C}^n\}$ (left action of $\mathrm{PGL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^*$)

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Fact

$\mathcal{F}_n \cong \mathrm{GL}(n,\mathbb{C})/B$ *where* $B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ o & & * \end{pmatrix} \right\}$

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Denote $X = (x^1 \dots x^n) \in \mathrm{GL}(n,\mathbb{C})$ ($x^1, \cdots, x^n$ column vectors)

$$\mathrm{GL}(n,\mathbb{C}) \to \mathcal{F}_n : X \mapsto V^i = \mathrm{span}_{\mathbb{C}}\{x^1, \dots, x^i\}$$

induces the bijection above.

3. Flags

$X_1, X_2, \cdots \in \mathcal{F}_n$ の代表元を $X_i = (x_i^1 \ldots x_i^n) \in \mathrm{GL}(n, \mathbb{C})$ の様に表す

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上の行列式は代表元の取り方によるが，単に $\det(X_1^{i_1} X_2^{i_2} \cdots X_k^{i_k})$ と書く

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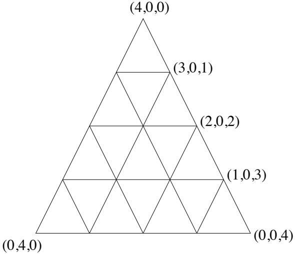

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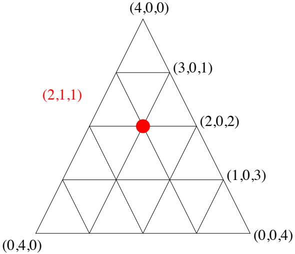

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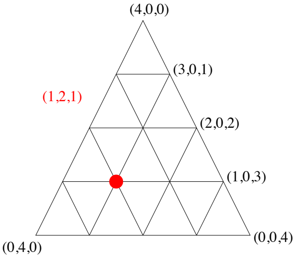

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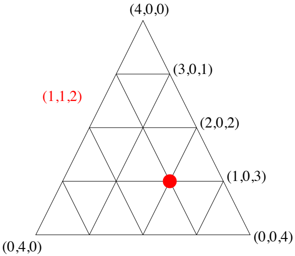

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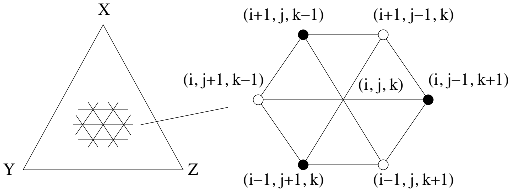

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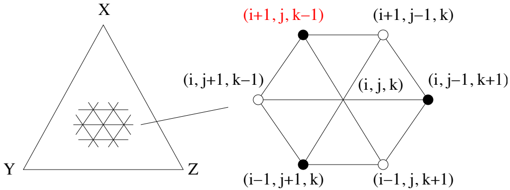

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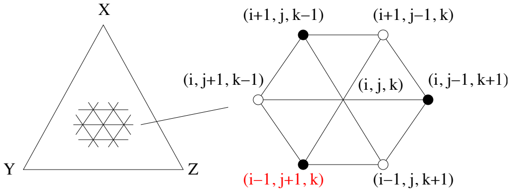

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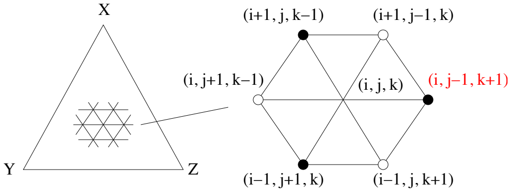

3. Flags

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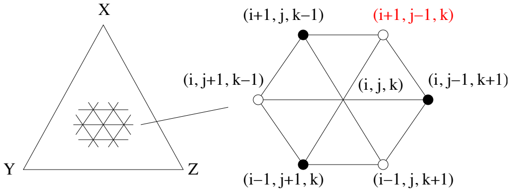

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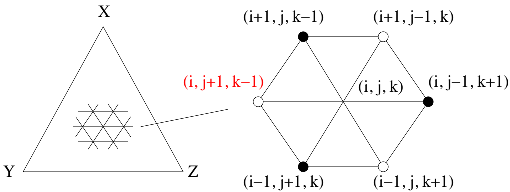

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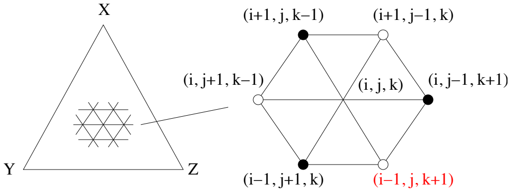

3. Flags

Let

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$$T_{i,j,k} : \mathrm{Conf}_3(\mathcal{F}_n) \to (\mathbb{C}^*)^{\frac{(n-1)(n-2)}{2}}$$

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3. Flags

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$X, Y \in \mathcal{F}_n$, $z \in \mathbb{C}P^{n-1}$ generic (i.e. $\det(X^i Y^j z^k) \neq 0$)
For any $c_{i,j,k} \in (\mathbb{C}^*)^{(n-1)(n-2)/2}$, $^\exists$ unique $Z \in \mathcal{F}_n$ s.t.
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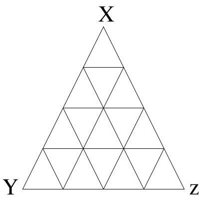

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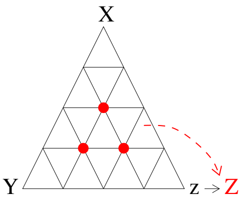

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$X', Y' \in \mathcal{F}_n$, $z' \in \mathbb{C}P^{n-1}$ generic
Then $^\exists$ unique $A \in \mathrm{PGL}(n, \mathbb{C})$ s.t.

$$AX = X', \quad AY = Y', \quad Az = z'.$$

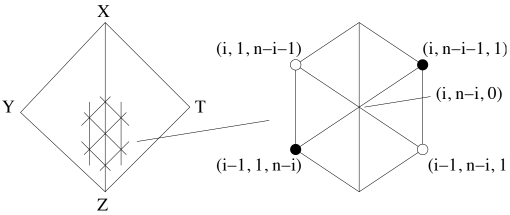
3. Flags

$(X, Y, Z, T) \in \mathcal{F}_n$ s.t. $(X, Y, Z), (X, Z, T) \in \mathcal{F}_n$ are generic

Definition (Edge function, quadruple ratio (Fock-Goncharov))

For $1 \leq i \leq n-1$,

$$\delta_i(X, Y, Z, T) = \frac{\det(X^{i-1} Y^1 Z^{n-i}) \det(X^i Z^{n-i-1} T^1)}{\det(X^i Y^1 Z^{n-i-1}) \det(X^{i-1} Z^{n-i} T^1)}$$



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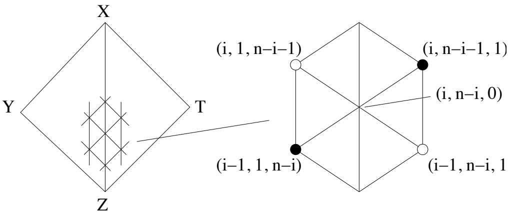

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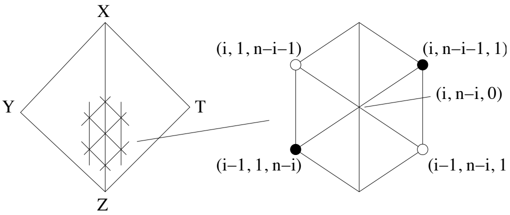

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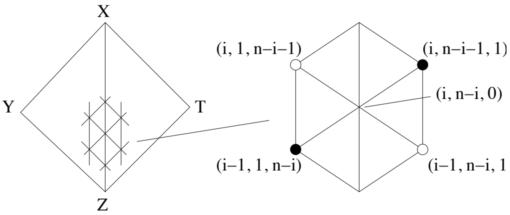

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- Defined for $X, Z \in \mathcal{F}_n$ and $Y^1, T^1 \in \mathbb{C}P^{n-1}$.

3. Flags

Example

$\underline{n=2}$ We can assume that $X = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$, $Y = \begin{pmatrix} 1 & * \\ 1 & * \end{pmatrix}$, $Z = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix}$ by the left action of $\mathrm{GL}(2, \mathbb{C})$. Let $T = \begin{pmatrix} t_1 & * \\ t_2 & * \end{pmatrix}$, then

$$\begin{aligned} \delta_1(X, Y, Z, T) &= \frac{\det(X^0 Y^1 Z^1) \det(X^1 Z^0 T^1)}{\det(X^1 Y^1 Z^0) \det(X^0 Z^1 T^1)} \\ &= \frac{1 \cdot t_2}{1 \cdot (-t_1)} \\ &= -t_2/t_1 = -[\infty : 0 : 1 : t_1/t_2] = -[X : Z : Y : T] \end{aligned}$$

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So $\delta_i(X, Y, Z, T)$ is a generalization of cross ratio.

3. Flags

Lemma

$X, Z \in \mathcal{F}_n$, $y \in \mathbb{C}P^{n-1}$ generic
For any $d_1, \ldots, d_{n-1} \in \mathbb{C}^*$, $^\exists$ unique $t \in \mathbb{C}P^{n-1}$ s.t.

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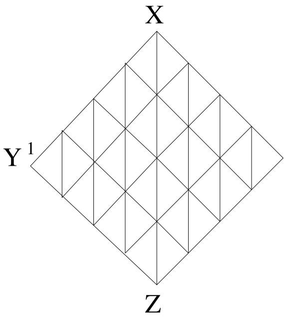

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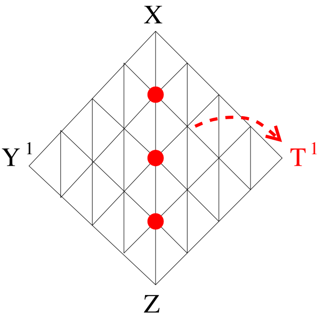

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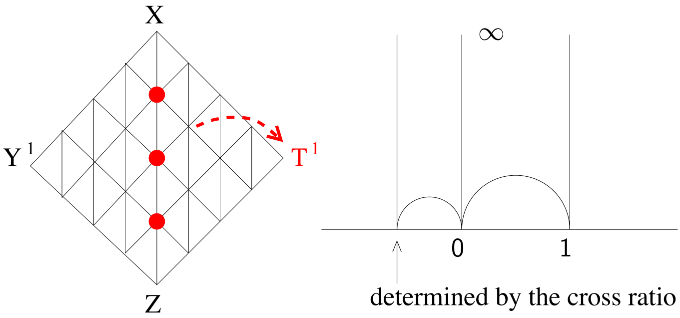

3. Flags

$(X, Y, Z, T) \in \mathrm{Conf}_4(\mathcal{F}_n)$ is completely determined by the following three types of functions :

- $T_{i,j,k}(X, Y, Z)$ $(n-1)(n-2)/2$
- $T_{i,j,k}(X, Z, T)$ $(n-1)(n-2)/2$
- $\delta_i(X, Y, Z, T)$ $n-1$

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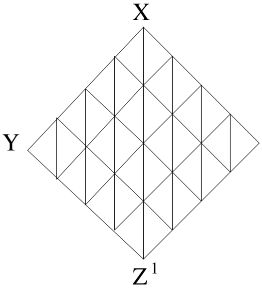

Fix $X, Y \in \mathcal{F}_n$ and $Z^1 \in \mathbb{C}P^{n-1}$ arbitrary.

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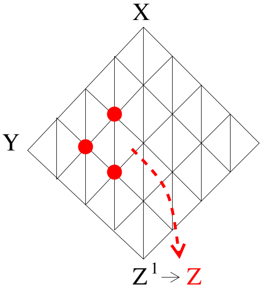

$T_{i,j,k}(X, Y, Z)$ determine $Z \in \mathcal{F}_n$ uniquely.

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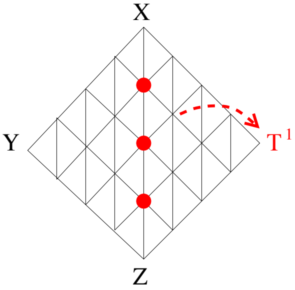

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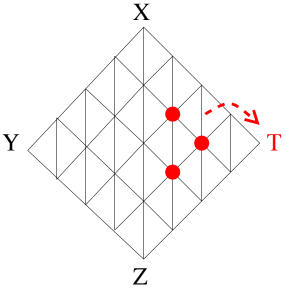

$T_{i,j,k}(X, Z, T)$ determine $T \in \mathcal{F}_n$ uniquely.
Thus (X, Y, Z, T) is completely determined.

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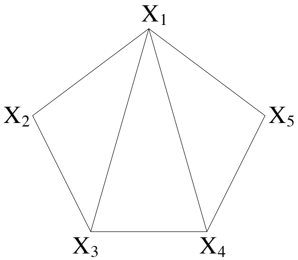
Thus we have

$$\mathrm{Conf}_4(\mathcal{F}_n) \xrightarrow{\text{injective}} (\mathbb{C}^*)^{n^2-1}$$

3. Flags

By 'subdividing' a k-tuple of generic flags into $(k-2)$-triangles, $\mathrm{Conf}_k(\mathcal{F}_n)$ is completely determined by

- $T_{i,j,k}(\cdot,\cdot,\cdot)$: $(k-2)(n-1)(n-2)/2$ triple ratios
- $\delta_i(\cdot,\cdot,\cdot,\dot{)}$: $(k-3)(n-1)$ edge functions



$$\mathrm{Conf}_k(\mathcal{F}_n) \xrightarrow{\text{injective}} (\mathbb{C}^*)^{(k-2)(n-1)(n-2)/2+(k-3)(n-1)}$$

4. 穴あき曲面の場合

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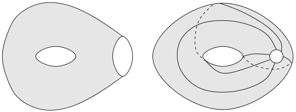

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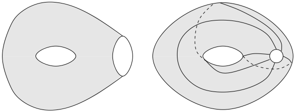

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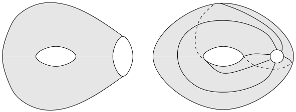

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gives a rep $\rho : \pi_1(S) \to \mathrm{PGL}(n, \mathbb{C})$ up to conjugation (Fock-Goncharov coordinates).

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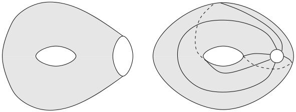

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P : a pair of pants, $\mathrm{Hom}(\pi_1(P), PGL(n, \mathbb{C}))/\sim$ is parametrized by

$$(n-1)(n-2) + 3(n-1) = (n^2 - 1)$$

parameters.

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$\rho : \pi_1(P) \to \mathrm{PGL}(n, \mathbb{C})$, $\rho' : \pi_1(P') \to \mathrm{PGL}(n, \mathbb{C})$: representations

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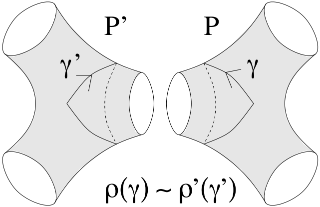

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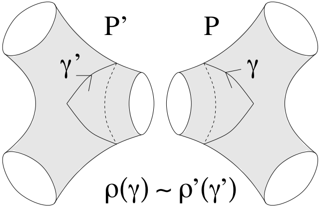

So we need to compute the eigenvalues in terms of Fock-Goncharov coordinates.

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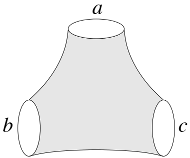

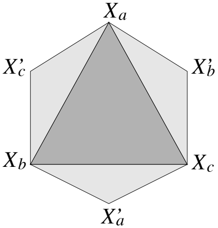

$X_a, X'_a, \cdots \in \mathcal{F}_n$: as above

$$T^{a,b,c}_{i,j,k} := T_{i,j,k}(X_a, X_b, X_c), \quad U^{a,c,b}_{i,j,k} := T_{i,j,k}(X_a, X'_c, X_b).$$

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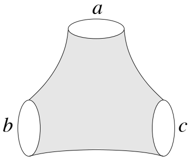

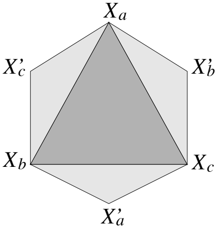

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$$\frac{e_{a,i+1}}{e_{a,i}} = \delta^{a,b}_i \delta^{a,c}_i \prod_{l=1}^{n-1-i} T^{a,b,c}_{i,l,n-i-l} U^{a,c,b}_{i,l,n-i-l}, \quad \text{for } i = 1, \dots, n-1.$$

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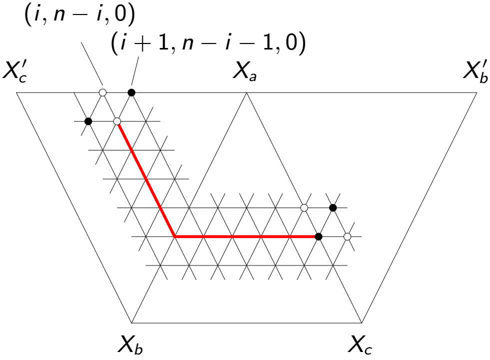

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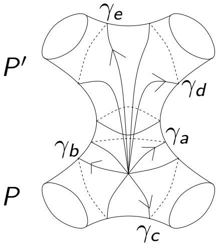
P, P' : pairs of pants, $\quad S = P \cup P'$ four-holed sphere



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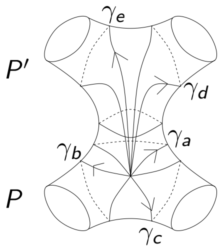

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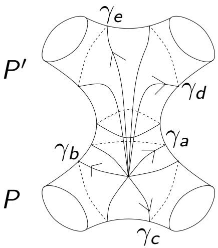

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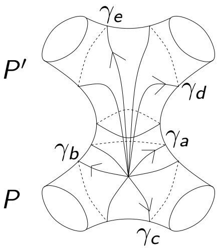

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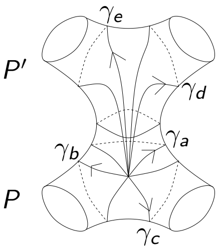

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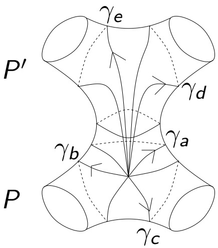

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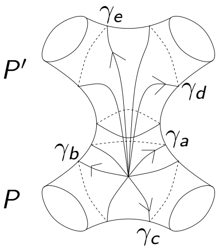

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By definition

$$\rho(\gamma_a) X_a = X_a, \quad \rho(\gamma_a) Y_a = Y_a$$

and (X_a, Y_a) is a generic pair. (X_a, Y_a are fixed pts of $\rho(\gamma_a)$ on $\mathcal{F}_n$.)

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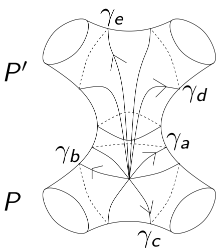
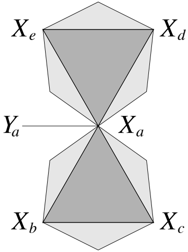
Define $X_b, X_c, \ldots$ etc.





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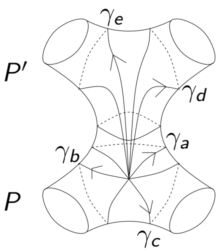

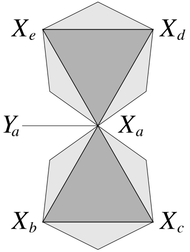

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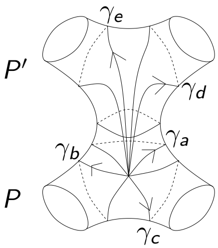

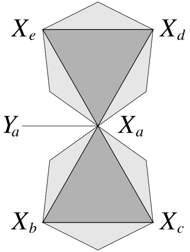

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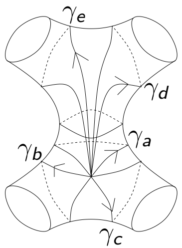

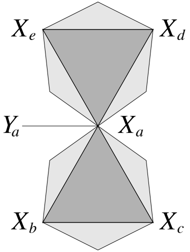

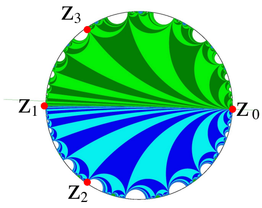

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($[z_0 : z_1 : z_2 : z_3] = -\delta_1(z_0, z_2, z_1, z_3)$)

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$\mathrm{Hom}(\pi_1(S), \mathrm{PGL}(n,\mathbb{C}))/\sim$ is parametrized by a variety of dimension

$$\begin{aligned}(2g-2)(n^2-1)+&(3g-3)(n-1)-(3g-3)(n-1)\\ &= (2g-2)(n^2-1) = |\chi(S)| \cdot \dim_{\mathbb{C}} \mathrm{PGL}(n,\mathbb{C}).\end{aligned}$$