Exotic components in linear slices of quasi-Fuchsian groups

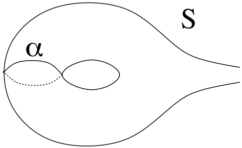
Yuichi Kabaya
(Kyoto University)
https://www.math.kyoto-u.ac.jp/~kabaya/

(These slides are available.)

Osaka, February 14 2015

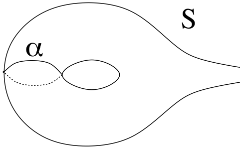
Outline

S : once punctured torus
$\alpha \subset S$: essential simple closed curve



Outline

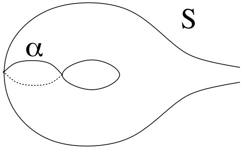
S : once punctured torus
$\alpha \subset S$: essential simple closed curve



$$QF(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{injective, } \rho(\pi_1(S)) \text{ quasi-Fuchsian}\} / \sim_{conj.}$$

Outline

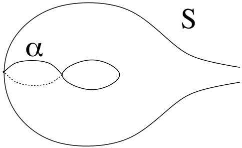
S : once punctured torus
$\alpha \subset S$: essential simple closed curve



$$QF(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{injective, } \rho(\pi_1(S)) \text{ quasi-Fuchsian}\}/\sim_{conj.}$$

$\lambda_\alpha : QF(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$: the (complex) length of α

Outline



S : once punctured torus
$\alpha \subset S$: essential simple closed curve

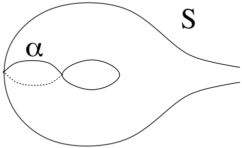
$$QF(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{injective, } \rho(\pi_1(S)) \text{ quasi-Fuchsian}\}/\sim_{conj.}$$

$\lambda_\alpha : QF(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$: the (complex) length of α

For $\ell > 0$, consider a slice of $QF(S)$

$$QF(\ell) = \{\rho \in QF(S) \mid \lambda_\alpha(\rho) = \ell\}.$$

Outline



S : once punctured torus
$\alpha \subset S$: essential simple closed curve

$$QF(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{injective, } \rho(\pi_1(S)) \text{ quasi-Fuchsian}\} / \sim_{conj.}$$

$\lambda_\alpha : QF(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$: the (complex) length of α

For $\ell > 0$, consider a slice of $QF(S)$

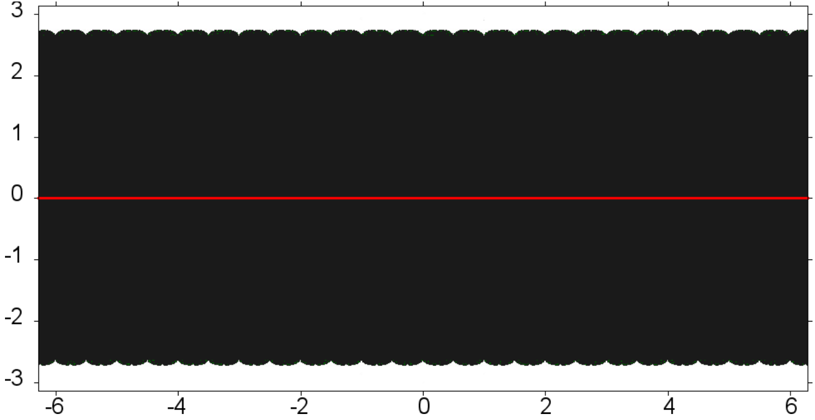
$$QF(\ell) = \{\rho \in QF(S) \mid \lambda_\alpha(\rho) = \ell\}.$$

This can be regarded as a subset of

$$\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \leq \pi\}.$$

Outline

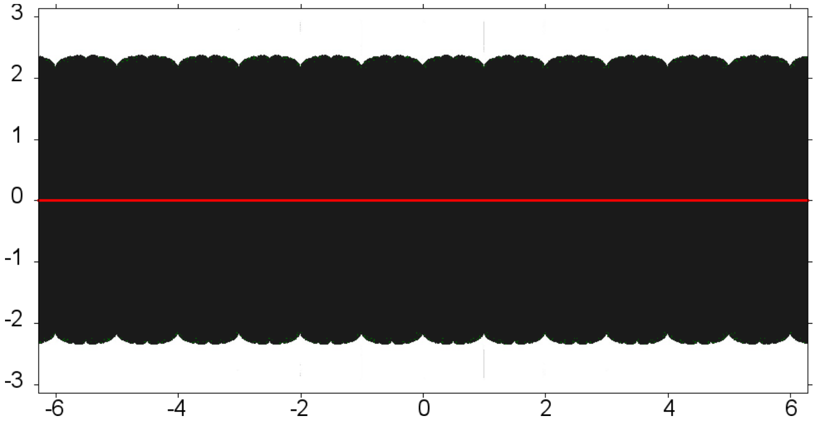
$QF(\ell)$ looks like this:



$\ell = 0.5$

Outline

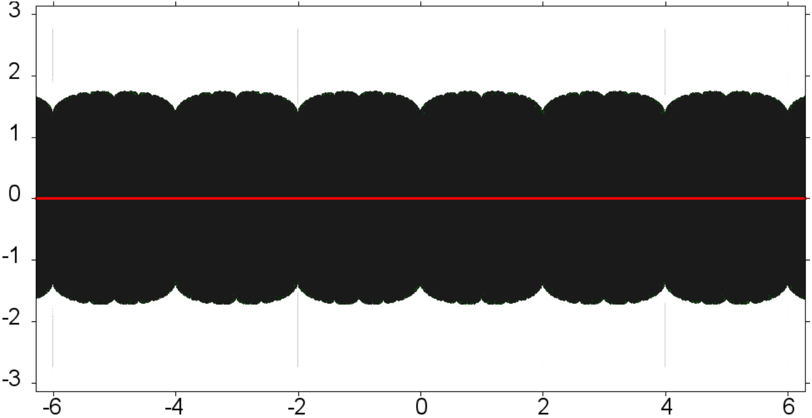
$QF(\ell)$ looks like this:



$\ell = 1.0$

Outline

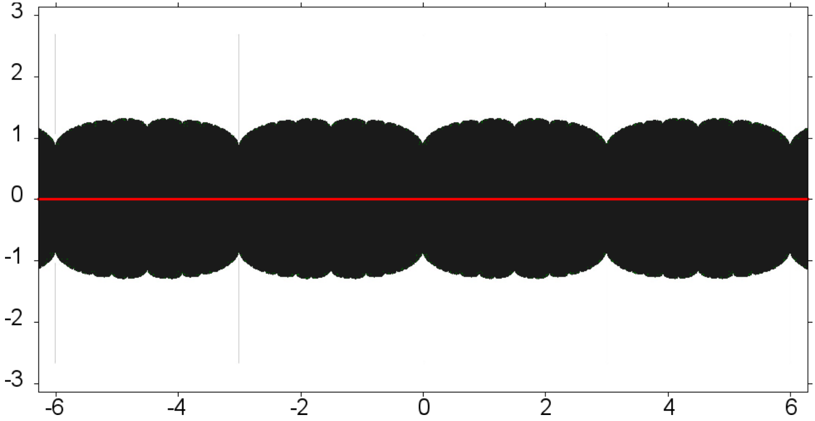
$QF(\ell)$ looks like this:



$\ell = 2.0$

Outline

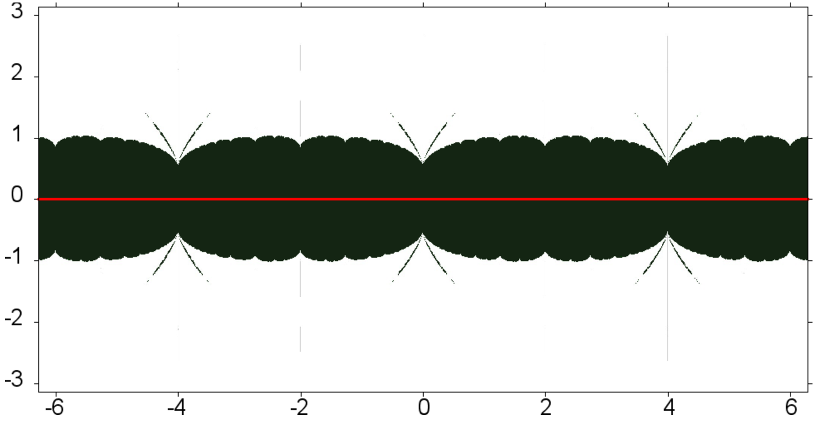
$QF(\ell)$ looks like this:



$\ell = 3.0$

Outline

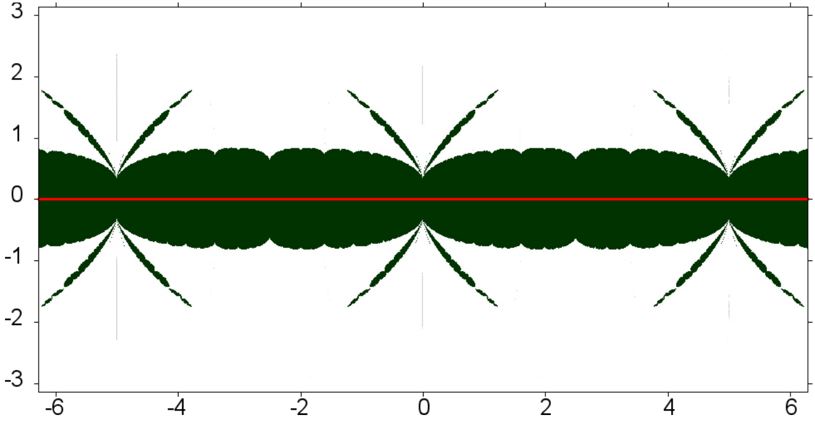
$QF(\ell)$ looks like this:



$\ell = 4.0$

Outline

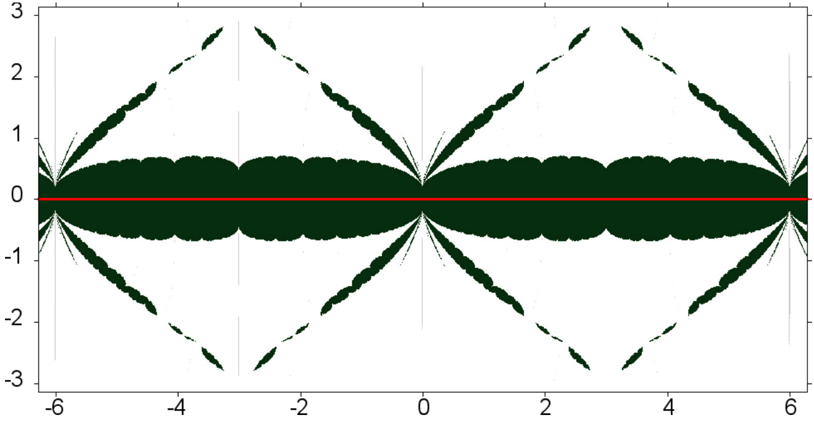
$QF(\ell)$ looks like this:



$\ell = 5.0$

Outline

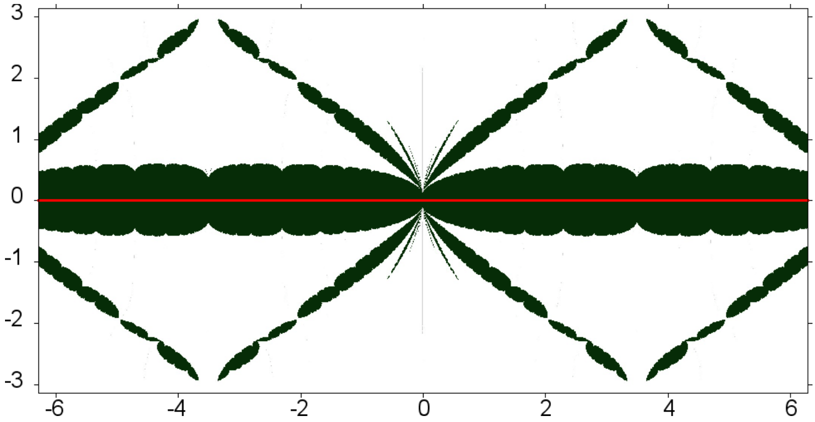
$QF(\ell)$ looks like this:



$\ell = 6.0$

Outline

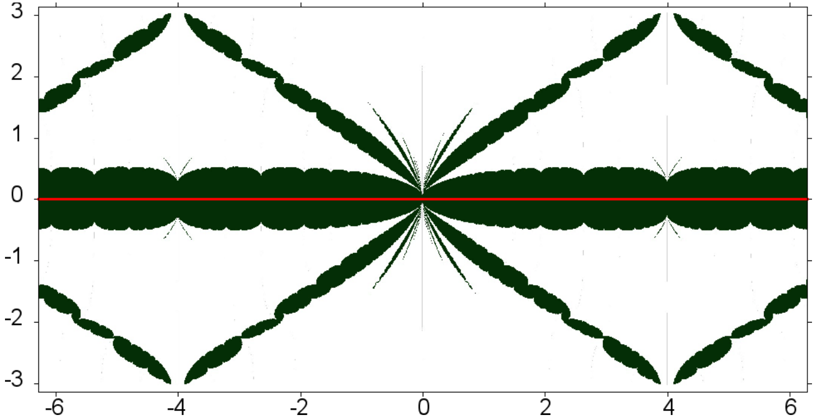
$QF(\ell)$ looks like this:



$\ell = 7.0$

Outline

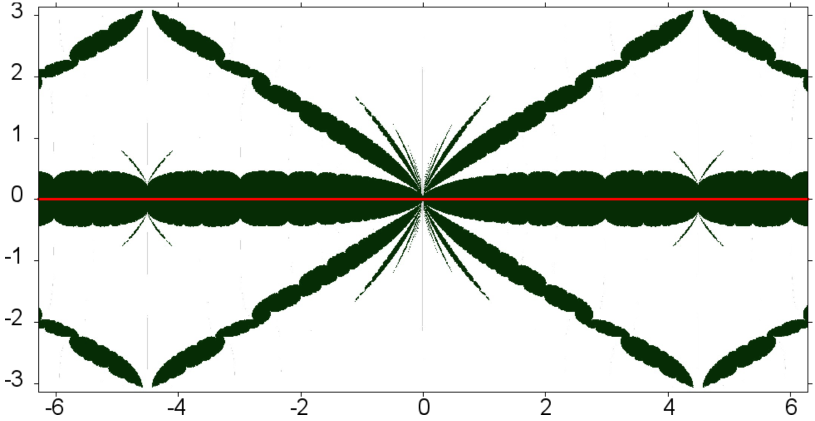
$QF(\ell)$ looks like this:



$\ell = 8.0$

Outline

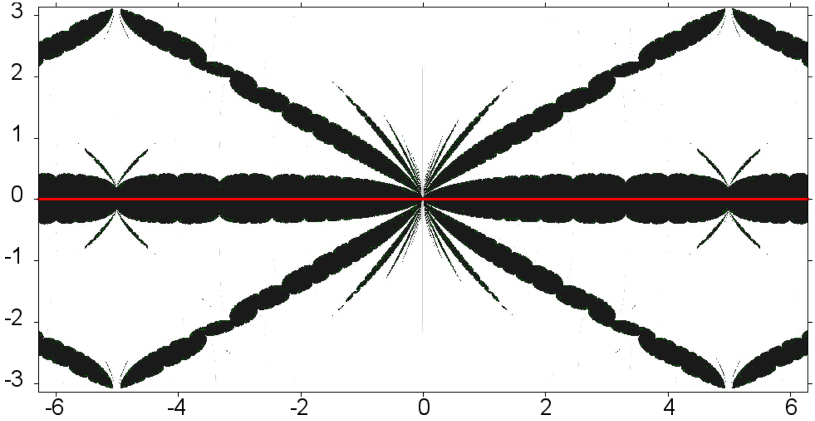
$QF(\ell)$ looks like this:



$\ell = 9.0$

Outline

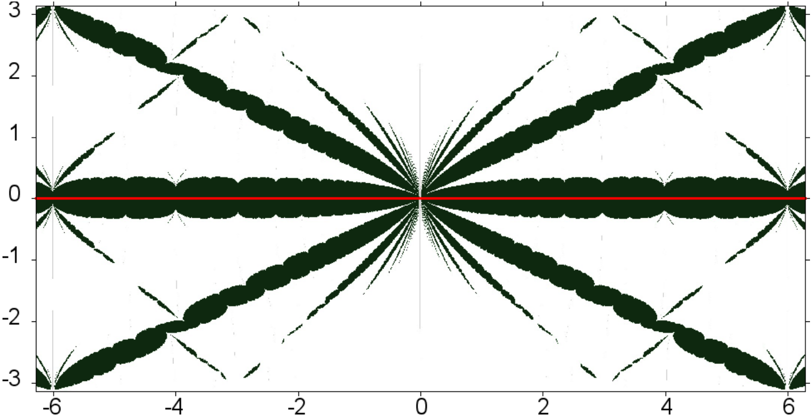
$QF(\ell)$ looks like this:



$\ell = 10.0$

Outline

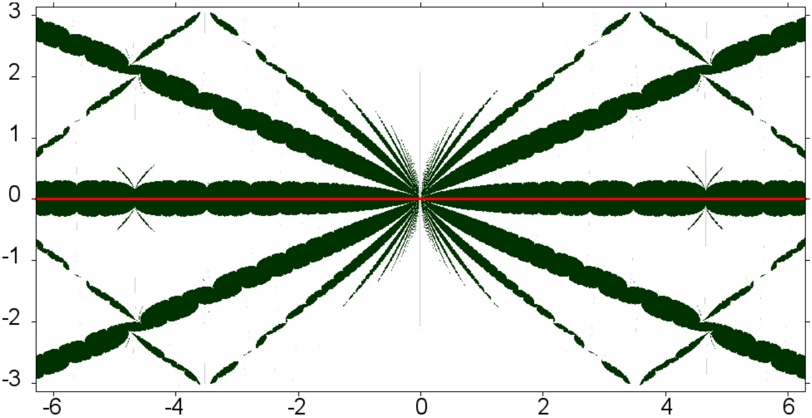
$QF(\ell)$ looks like this:



$\ell = 12.0$

Outline

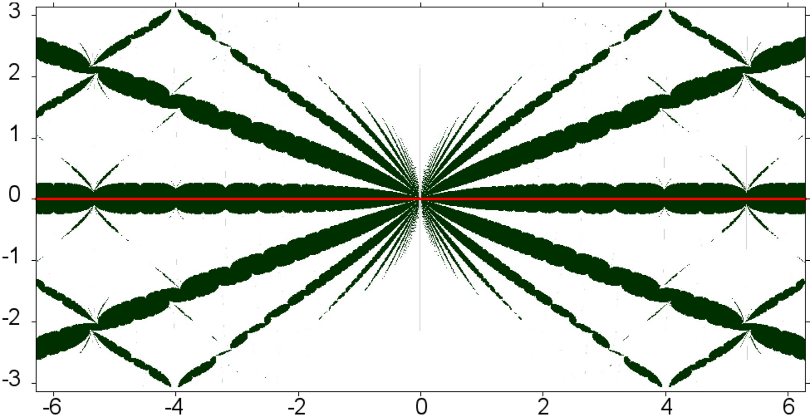
$QF(\ell)$ looks like this:



$\ell = 14.0$

Outline

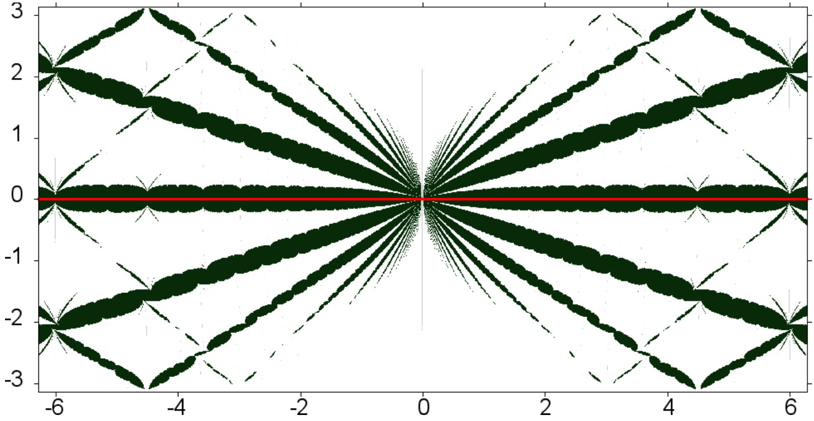
$QF(\ell)$ looks like this:



$\ell = 16.0$

Outline

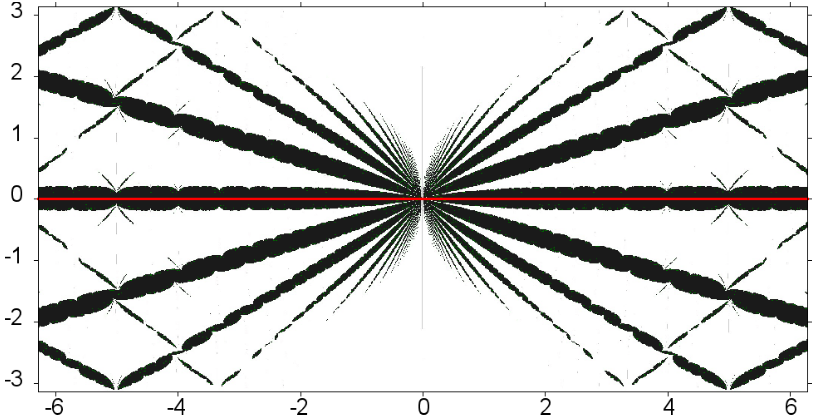
$QF(\ell)$ looks like this:



$\ell = 18.0$

Outline

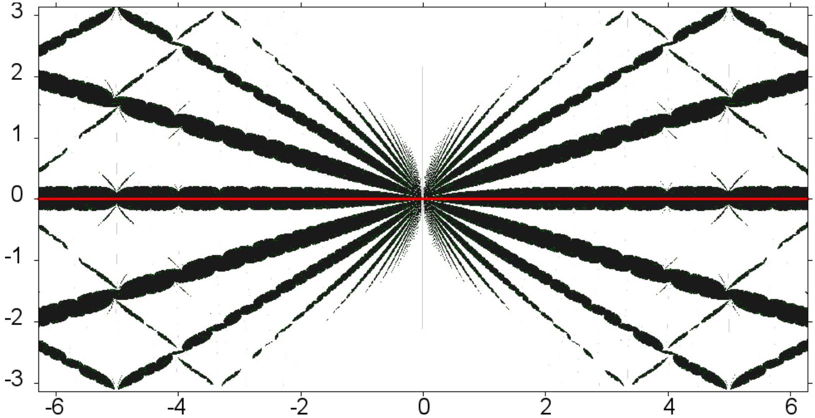
$QF(\ell)$ looks like this:



$\ell = 20.0$

Outline

$QF(\ell)$ looks like this:



$\ell = 20.0$

Interested in the shape of $QF(\ell)$ as ℓ getting longer.

Outline

1. Basics on Kleinian (once punctured torus) groups
2. Linear slices & Main theorem
3. Complex projective structures and complex earthquake
4. Proof of the main theorem

With many pictures ...

Basics (Hyperbolic space)

$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C},\ t \in \mathbb{R}_{>0}\}$: 3-dim hyperbolic space

Basics (Hyperbolic space)

$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C},\ t \in \mathbb{R}_{>0}\}$: 3-dim hyperbolic space

$\{t = 0\} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$: its boundary

Basics (Hyperbolic space)

$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C},\ t \in \mathbb{R}_{>0}\}$: 3-dim hyperbolic space

$\{t = 0\} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$: its boundary

$\mathrm{PSL}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}/\{\pm 1\}$ acts on $\mathbb{C}P^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \quad (z \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\})$$

Basics (Hyperbolic space)

$\mathbb{H}^3 = \{(z,t) \mid z \in \mathbb{C},\ t \in \mathbb{R}_{>0}\}$: 3-dim hyperbolic space

$\{t = 0\} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$: its boundary

$\mathrm{PSL}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}/\{\pm 1\}$ acts on $\mathbb{C}P^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \quad (z \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\})$$

This action extends to the interior $\mathbb{H}^3$ isometrically.

Basics (Hyperbolic space)

$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C},\ t \in \mathbb{R}_{>0}\}$: 3-dim hyperbolic space

$\{t = 0\} = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$: its boundary

$\mathrm{PSL}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}/\{\pm 1\}$ acts on $\mathbb{C}P^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \quad (z \in \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\})$$

This action extends to the interior $\mathbb{H}^3$ isometrically.

$\Gamma < \mathrm{PSL}_2\mathbb{C}$: torsion free discrete subgroup

$\Rightarrow M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold
s.t. $\pi_1(M) \cong \Gamma$

Basics (Deformation space)

$S = S_{g,n}$: genus g, n punctured surface ($\chi(S) < 0$)

Basics (Deformation space)

$S = S_{g,n}$: genus g, n punctured surface ($\chi(S) < 0$)

$$X(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{irreducible, preserving parabolics}\}/\sim_{\text{conj.}}$$

: the character variety

Basics (Deformation space)

$S = S_{g,n}$: genus g, n punctured surface ($\chi(S) < 0$)

$$X(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{irreducible, preserving parabolics}\}/\sim_{\text{conj.}}$$

: the character variety

$$AH(S) = \{[\rho] \in X(S) \mid \text{faithful, discrete image}\}$$

Basics (Deformation space)

$S = S_{g,n}$: genus g, n punctured surface ($\chi(S) < 0$)

$$X(S) = \{\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \mid \text{irreducible, preserving parabolics}\}/\sim_{\text{conj.}}$$

: the character variety

$$AH(S) = \{[\rho] \in X(S) \mid \text{faithful, discrete image}\}$$

If $\rho \in AH(S)$, then $\mathbb{H}^3/\rho(\pi_1(S))$ is a complete hyperbolic 3-manifold homotopy equivalent to S.

$AH(S)$ is the deformation space of such structures.

Basics (Limit sets)

$\Gamma < \mathrm{PSL}_2\mathbb{C}$: discrete subgroup

Fix a point $p \in \mathbb{H}^3$. The limit set of Γ is defined by

$$\Lambda(\Gamma) = \{\text{accumulation points of } \Gamma \cdot p \text{ on } \mathbb{C}P^1\}.$$

($\Lambda(\Gamma) \subset \mathbb{C}P^1$, not depend on the choice of p)

Basics (Limit sets)

$\Gamma < \mathrm{PSL}_2\mathbb{C}$: discrete subgroup

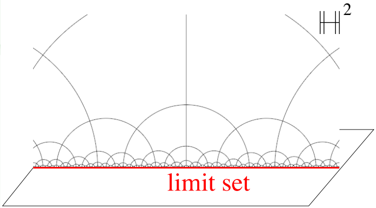
Fix a point $p \in \mathbb{H}^3$. The limit set of Γ is defined by

$$\Lambda(\Gamma) = \{\text{accumulation points of } \Gamma \cdot p \text{ on } \mathbb{C}P^1\}.$$

($\Lambda(\Gamma) \subset \mathbb{C}P^1$, not depend on the choice of p)

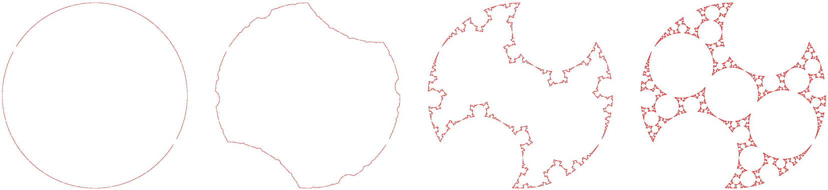
Example (Fuchsian groups)

If $\Gamma < \mathrm{PSL}_2(\mathbb{R})$, Γ preserves $\mathbb{H}^2(\subset \mathbb{H}^3)$, thus $\Lambda(\Gamma)$ is a subset of $\mathbb{R} \cup \{\infty\}$ (a 'round circle' in $\mathbb{C}P^1$).



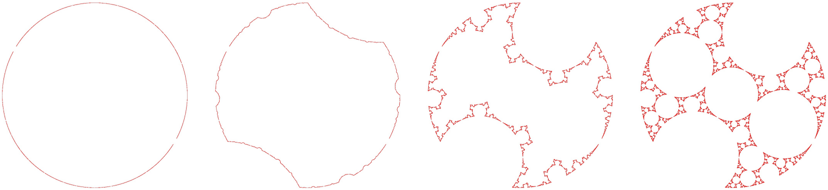
Basics (Quasi-Fuchsian representations)

We can deform a Fuchsian rep a little in $\mathrm{PSL}_2\mathbb{C}$. The limit set is no longer a round circle, but may be $\cong S^1$.



Basics (Quasi-Fuchsian representations)

We can deform a Fuchsian rep a little in $\mathrm{PSL}_2\mathbb{C}$. The limit set is no longer a round circle, but may be $\cong S^1$.



Definition

Let $\rho \in AH(S)$. If the limit set $\Lambda(\rho(\pi_1(S)))$ is homeomorphic to S^1, ρ is called quasi-Fuchsian.

$$QF(S) = \{\rho \in AH(S) \mid \rho \text{ is quasi-Fuchsian.}\}$$

Basics (Known properties)

Known facts

- $QF(S) \subset X(S)$ open subset

Basics (Known properties)

Known facts

- $QF(S) \subset X(S)$ open subset
- $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where $\mathcal{T}(S)$ is the Teichmüller space of S. ($\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$)

Basics (Known properties)

Known facts

- $QF(S) \subset X(S)$ open subset
- $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where $\mathcal{T}(S)$ is the Teichmüller space of S. $(\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n})$
- $\overline{QF(S)} = AH(S)$: density theorem

Basics (Known properties)

Known facts

- $QF(S) \subset X(S)$ open subset
- $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where $\mathcal{T}(S)$ is the Teichmüller space of S. ($\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$)
- $\overline{QF(S)} = AH(S)$: density theorem
- $AH(S)$ is parametrized by its end invariants (Ending Lamination Theorem).

Basics (Known properties)

Known facts

- $QF(S) \subset X(S)$ open subset
- $QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(\overline{S})$, where $\mathcal{T}(S)$ is the Teichmüller space of S. ($\mathcal{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$)
- $\overline{QF(S)} = AH(S)$: density theorem
- $AH(S)$ is parametrized by its end invariants (Ending Lamination Theorem).

But the shape of $QF(S)$ in $X(S)$ is very complicated! (e.g. self-bumping, $AH(S)$ is not locally connected.)

Basics (Complex length)

For $\gamma \in \pi_1(S)$, $\rho \in X(S)$, $\rho(\gamma)$ acts on $\mathbb{H}^3$.

Define the (complex) length by

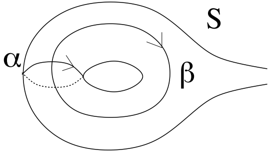
$$\lambda_\gamma(\rho) = (\text{translation length of } \rho(\gamma)) + \sqrt{-1}\,(\text{rotation angle of } \rho(\gamma))$$

mod $2\pi\sqrt{-1}\mathbb{Z}$. This is characterized by

$$\mathrm{tr}(\rho(\gamma)) = 2\cosh(\frac{\lambda_\gamma(\rho)}{2}).$$

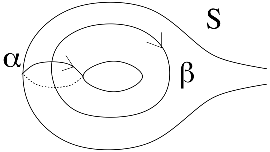
Character variety

$S = S_{1,1}$: once punctured torus
Fix $\alpha, \beta \in \pi_1(S)$ so that $[\alpha, \beta]$ is peripheral.



Character variety

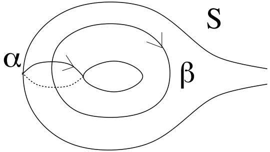
$S = S_{1,1}$: once punctured torus
Fix $\alpha, \beta \in \pi_1(S)$ so that $[\alpha, \beta]$ is peripheral.



The $SL_2\mathbb{C}$-character variety $X_{SL}(S)$ is defined similarly as $PSL_2\mathbb{C}$ case.

Character variety

$S = S_{1,1}$: once punctured torus
Fix $\alpha, \beta \in \pi_1(S)$ so that $[\alpha, \beta]$ is peripheral.



The $SL_2\mathbb{C}$-character variety $X_{SL}(S)$ is defined similarly as $PSL_2\mathbb{C}$ case. As affine varieties, we have

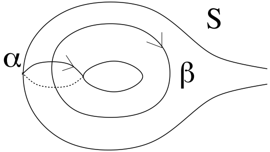
$$X_{SL}(S) \cong \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = xyz\}$$

via

$$[\rho] \mapsto (\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta)), \mathrm{tr}(\rho(\alpha\beta))).$$

Character variety

$S = S_{1,1}$: once punctured torus
Fix $\alpha, \beta \in \pi_1(S)$ so that $[\alpha, \beta]$ is peripheral.



The $\mathrm{SL}_2\mathbb{C}$-character variety $X_{SL}(S)$ is defined similarly as $\mathrm{PSL}_2\mathbb{C}$ case. As affine varieties, we have

$$X_{SL}(S) \cong \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = xyz\}$$

via

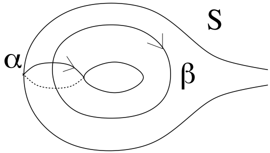
$$[\rho] \mapsto (\mathrm{tr}(\rho(\alpha)), \mathrm{tr}(\rho(\beta)), \mathrm{tr}(\rho(\alpha\beta))).$$

$X(S)$ is obtained as a quotient of $X_{SL}(S)$ by the action of $\mathbb{Z}/2\mathbb{Z}$ generated by

$$(x, y, z) = (-x, -y, z), \quad (x, y, z) = (x, -y, -z).$$

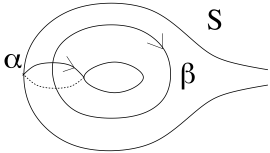
Linear slices

Any essential simple closed curve on $S = S_{1,1}$ is represented by a primitive element $p[\alpha] + q[\beta] \in H_1(S;\mathbb{Z})$. Regard it as $p/q \in \mathbb{Q} \cup \{\infty\}$.



Linear slices

Any essential simple closed curve on $S = S_{1,1}$ is represented by a primitive element $p[\alpha] + q[\beta] \in H_1(S;\mathbb{Z})$. Regard it as $p/q \in \mathbb{Q} \cup \{\infty\}$.

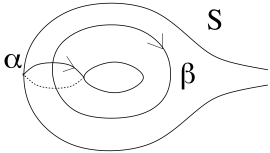


For p/q, take $\gamma_{p/q} \in \pi_1(S)$ freely homotopic to p/q.

Define the length function $\lambda_{p/q} : X(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ by $\lambda_{p/q}(\rho) = \lambda_{\gamma_{p/q}}(\rho)$.

Linear slices

Any essential simple closed curve on $S = S_{1,1}$ is represented by a primitive element $p[\alpha] + q[\beta] \in H_1(S;\mathbb{Z})$. Regard it as $p/q \in \mathbb{Q} \cup \{\infty\}$.



For p/q, take $\gamma_{p/q} \in \pi_1(S)$ freely homotopic to p/q.

Define the length function $\lambda_{p/q} : X(S) \to \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ by $\lambda_{p/q}(\rho) = \lambda_{\gamma_{p/q}}(\rho)$.

Definition

For $\ell > 0$, let

$$X(\ell) = \{\rho \in X(S) \mid \lambda_{1/0}(\rho) = \ell\}$$

$X(\ell)$ is a slice of $X(S)$ on which (cpx length of α) $\equiv \ell$.

Complex Fenchel-Nielsen coordinates

For $\ell > 0$, define a map

$$\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \xrightarrow{\cong} X(\ell)$$

by

$$\tau \mapsto \left(2\cosh(\ell/2), \frac{2\cosh(\tau/2)}{\tanh(\ell/2)}, \frac{2\cosh((\tau+\ell)/2)}{\tanh(\ell/2)}\right).$$

This gives a bijection. (Recall $\mathrm{tr}\,\rho(\alpha) = 2\cosh(\lambda_{1/0}/2)$.)

Complex Fenchel-Nielsen coordinates

For $\ell > 0$, define a map

$$\{\tau \in \mathbb{C} \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \xrightarrow{\cong} X(\ell)$$

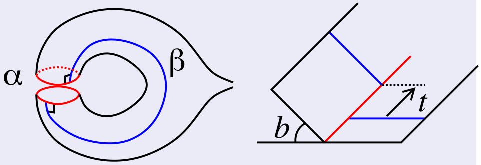
by

$$\tau \mapsto \left(2\cosh(\ell/2), \frac{2\cosh(\tau/2)}{\tanh(\ell/2)}, \frac{2\cosh((\tau+\ell)/2)}{\tanh(\ell/2)}\right).$$

This gives a bijection. (Recall $\mathrm{tr}\,\rho(\alpha) = 2\cosh(\lambda_{1/0}/2)$.)

Note

If we let $\tau = t + \sqrt{-1}b$, t is the twisting distance and b is the bending angle along α.

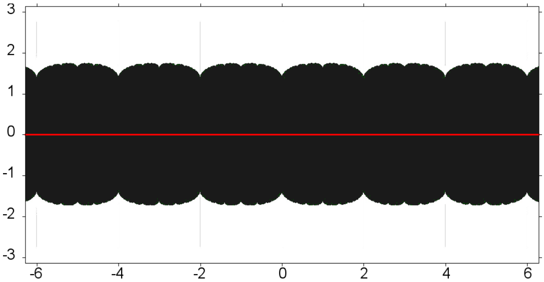


Linear slices of $QF(S)$

Definition

For $\ell > 0$, define

$$QF(\ell) = QF(S) \cap X(\ell)$$



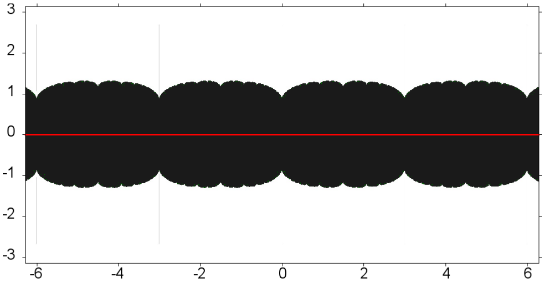
$QF(2.0) \subset X(2.0)$ (black region)

Linear slices of $QF(S)$

Definition

For $\ell > 0$, define

$$QF(\ell) = QF(S) \cap X(\ell)$$



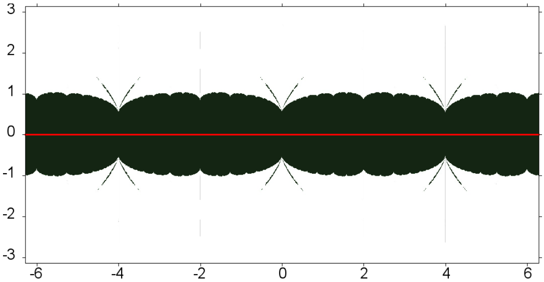
$QF(3.0) \subset X(3.0)$ (black region)

Linear slices of $QF(S)$

Definition

For $\ell > 0$, define

$$QF(\ell) = QF(S) \cap X(\ell)$$



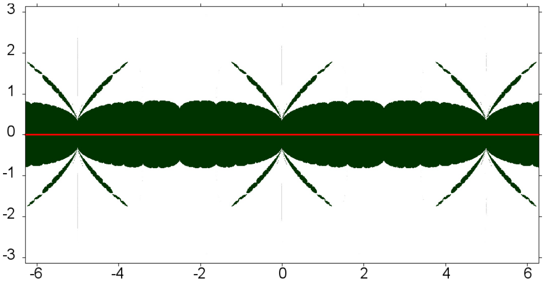
$QF(4.0) \subset X(4.0)$ (black region)

Linear slices of $QF(S)$

Definition

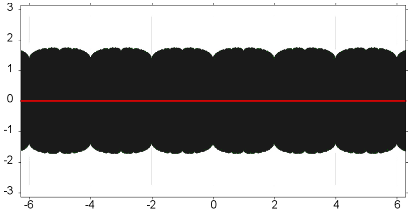
For $\ell > 0$, define

$$QF(\ell) = QF(S) \cap X(\ell)$$



$QF(5.0) \subset X(5.0)$ (black region)

Linear slices of $QF(S)$



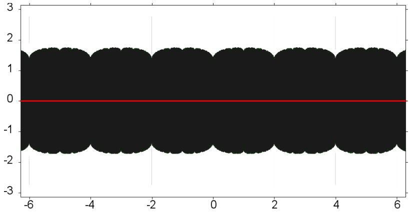
$QF(2.0)$

Facts

- The Dehn twist along α acts on $X(\ell)$ as

$$\tau \mapsto \tau + \ell. \quad \text{(translation)}$$

Linear slices of $QF(S)$

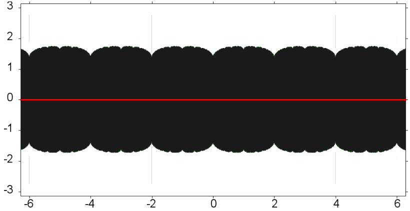


$QF(2.0)$

Facts

- The Dehn twist along α acts on $X(\ell)$ as
$$\tau \mapsto \tau + \ell. \quad \text{(translation)}$$
- The real line $\{\tau \mid \mathrm{Im}(\tau) = 0\}$ corresponds to the Fuchsian representations satisfying $\lambda_\alpha = \ell$.

Linear slices of $QF(S)$

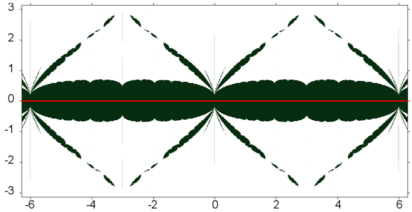


$QF(2.0)$

Facts

- The Dehn twist along α acts on $X(\ell)$ as
$$\tau \mapsto \tau + \ell. \quad \text{(translation)}$$
- The real line $\{\tau \mid \mathrm{Im}(\tau) = 0\}$ corresponds to the Fuchsian representations satisfying $\lambda_\alpha = \ell$.
- By McMullen's disk convexity of $QF(S)$, $QF(\ell)$ is a union of (open) disks.

Linear slices of $QF(S)$



$QF(6.0)$

Facts

- The Dehn twist along α acts on $X(\ell)$ as
$$\tau \mapsto \tau + \ell. \quad \text{(translation)}$$
- The real line $\{\tau \mid \text{Im}(\tau) = 0\}$ corresponds to the Fuchsian representations satisfying $\lambda_\alpha = \ell$.
- By McMullen's disk convexity of $QF(S)$, $QF(\ell)$ is a union of (open) disks.

Linear slices of $QF(S)$

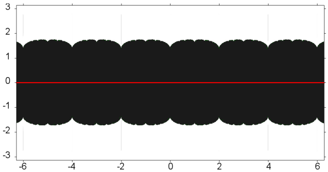
For any $\ell > 0$, there exists a unique standard component containing Fuchsian representations. As pictures suggest;

Linear slices of $QF(S)$

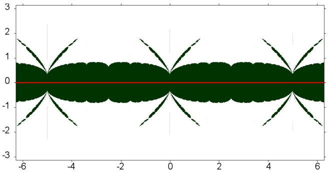
For any $\ell > 0$, there exists a unique standard component containing Fuchsian representations. As pictures suggest;

Theorem (Komori-Yamashita, 2012)

$QF(\ell)$ has only one component if ℓ is sufficiently small, has more than one component if ℓ is sufficiently large.



$QF(2.0)$



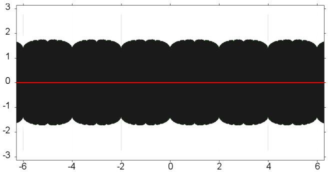
$QF(5.0)$

Linear slices of $QF(S)$

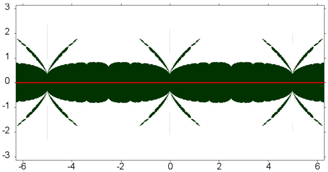
For any $\ell > 0$, there exists a unique standard component containing Fuchsian representations. As pictures suggest;

Theorem (Komori-Yamashita, 2012)

$QF(\ell)$ has only one component if ℓ is sufficiently small, has more than one component if ℓ is sufficiently large.



$QF(2.0)$



$QF(5.0)$

Today, we will give another proof for the latter part, and give refined results.

More on $QF(S)$

The standard component was extensively studied by Keen-Series, they called it the BM-slice (denote BM).

More on $QF(S)$

The standard component was extensively studied by Keen-Series, they called it the BM-slice (denote BM).

For $\rho \in QF(S)$, let $pl^{\pm}(\rho) \in \mathcal{ML}(S)$ be the bending measures on the convex hull boundary.

More on $QF(S)$

The standard component was extensively studied by Keen-Series, they called it the BM-slice (denote BM).

For $\rho \in QF(S)$, let $pl^{\pm}(\rho) \in \mathcal{ML}(S)$ be the bending measures on the convex hull boundary.

Theorem (Keen-Series, 2004)

$\rho \in BM$ iff one of $[pl^{\pm}]$ coincides with α in $\mathcal{PML}(S)$.

More on $QF(S)$

The standard component was extensively studied by Keen-Series, they called it the BM-slice (denote BM).

For $\rho \in QF(S)$, let $pl^{\pm}(\rho) \in \mathcal{ML}(S)$ be the bending measures on the convex hull boundary.

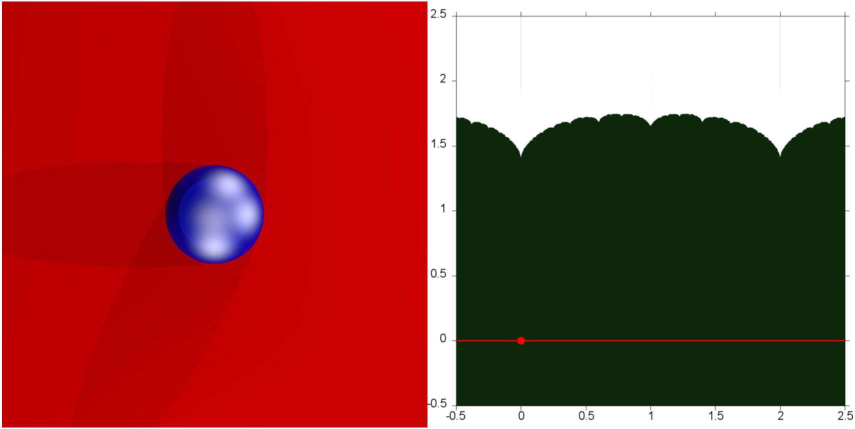
Theorem (Keen-Series, 2004)

$\rho \in BM$ iff one of $[pl^{\pm}]$ coincides with α in $\mathcal{PML}(S)$.

Roughly, a representation in BM is obtained from a Fuchsian one by bending along α continuously.

More on $QF(S)$

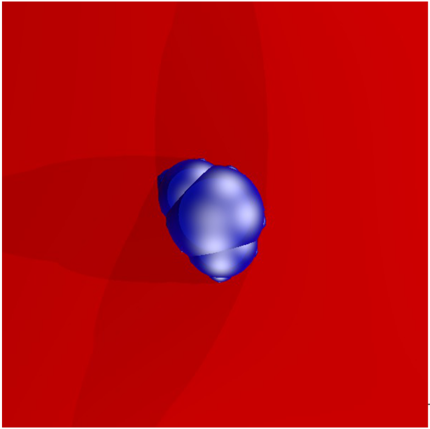
Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

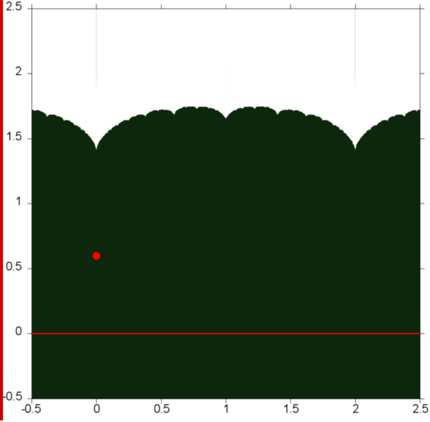


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

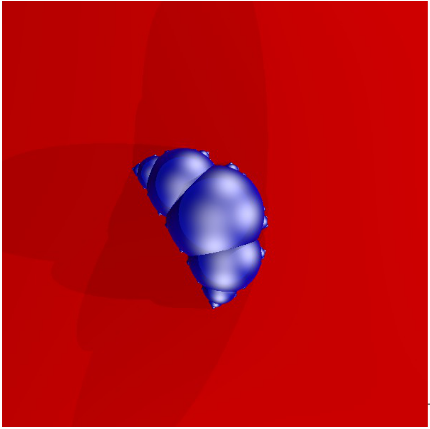


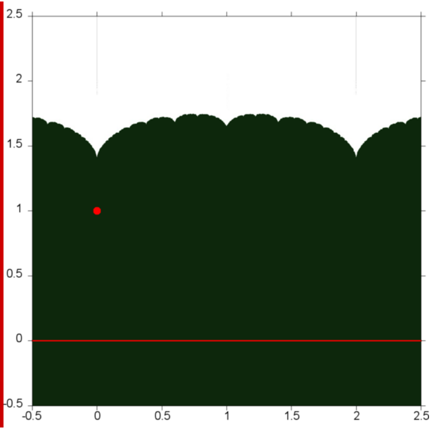


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = \text{(twisting dist.)} + \sqrt{-1}\text{(bending angle)}.$

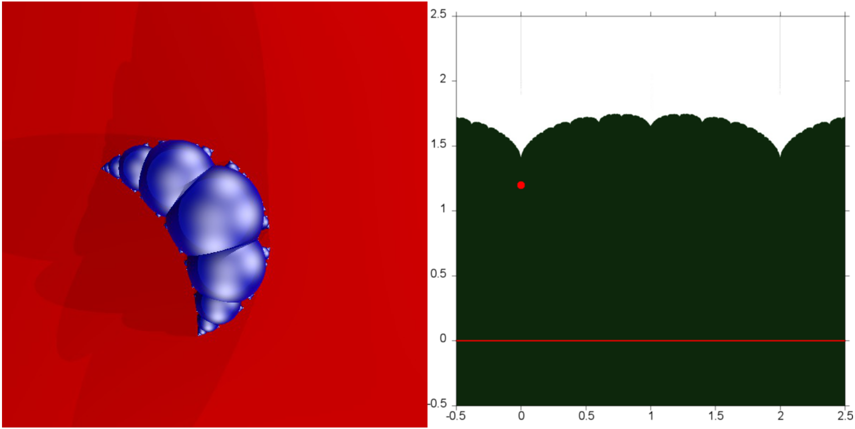




In the BM-slice of $QF(2.0)$.

More on $QF(S)$

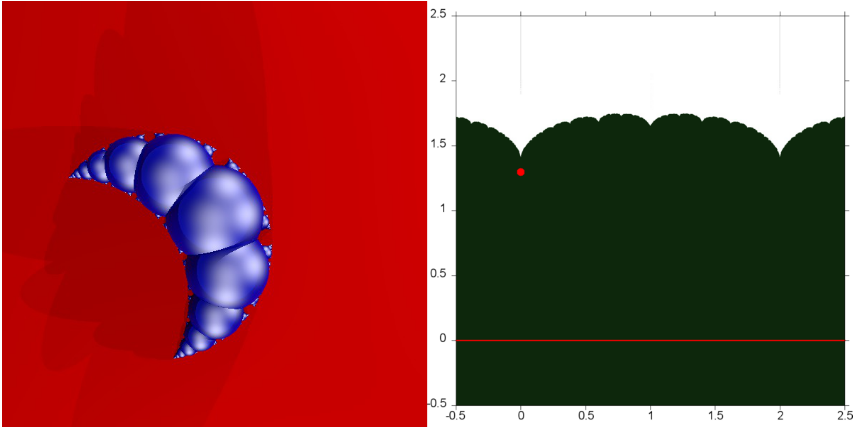
Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$



In the BM-slice of $QF(2.0)$.

More on $QF(S)$

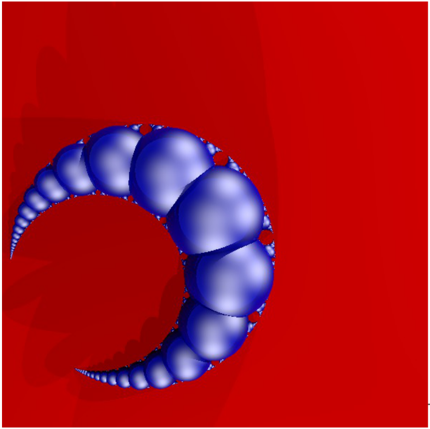
Recall $\tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

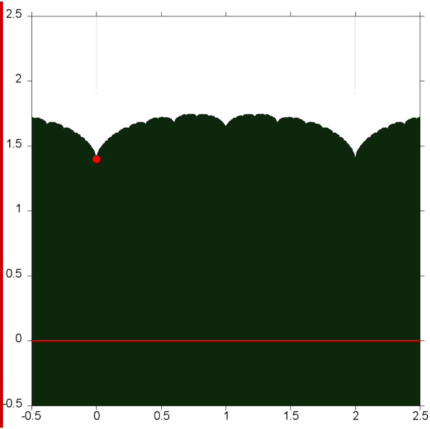


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

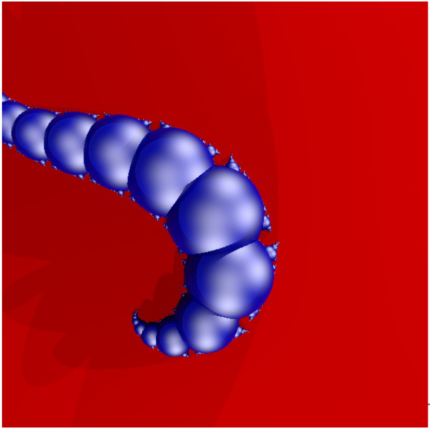


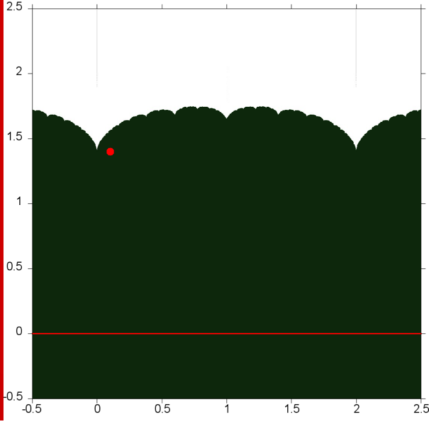


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

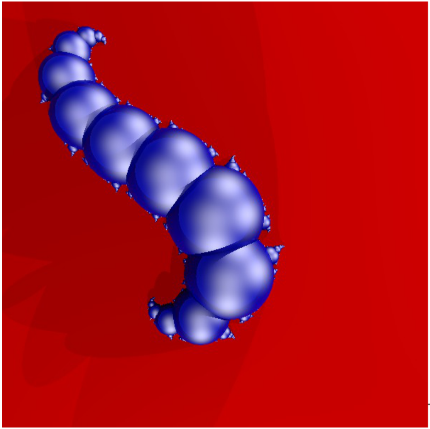


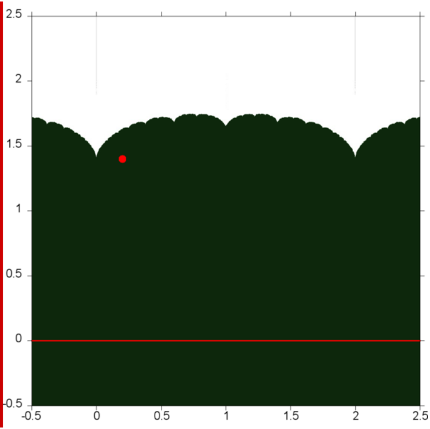


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

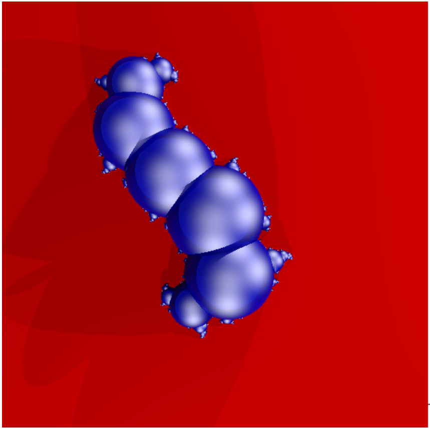


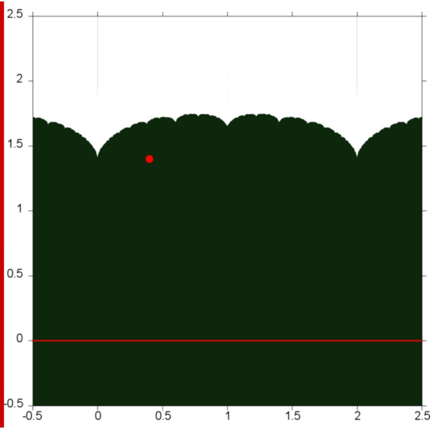


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

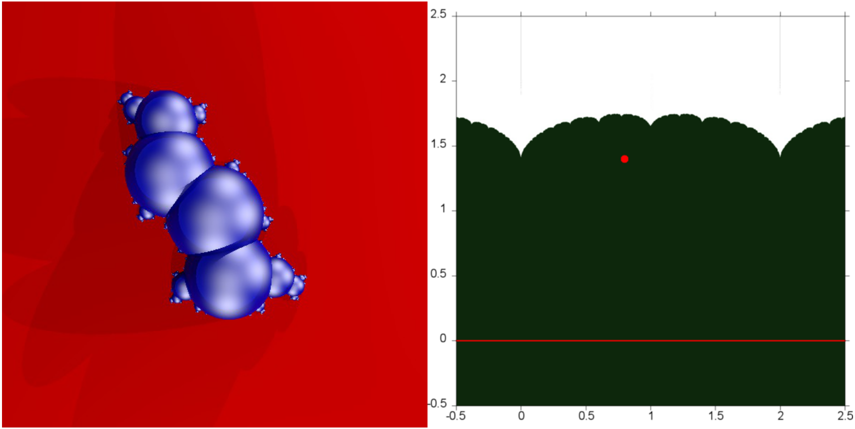




In the BM-slice of $QF(2.0)$.

More on $QF(S)$

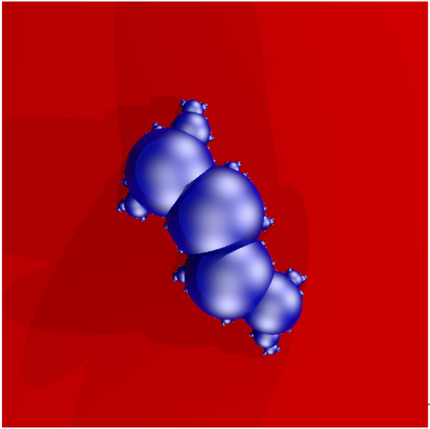
Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

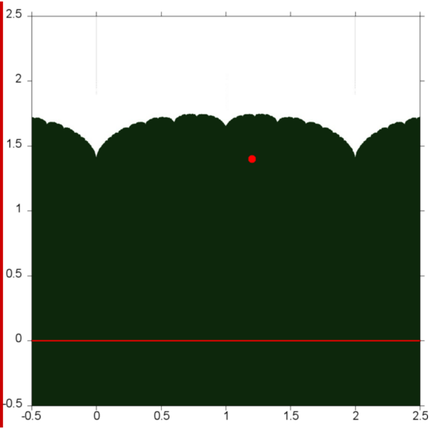


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

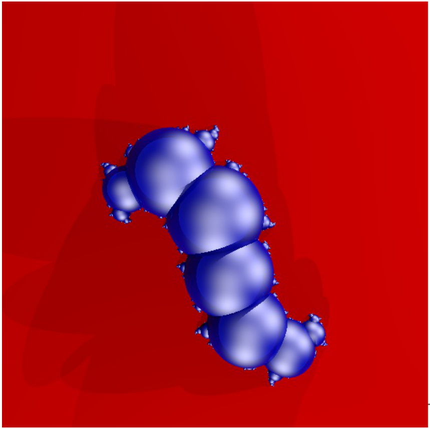


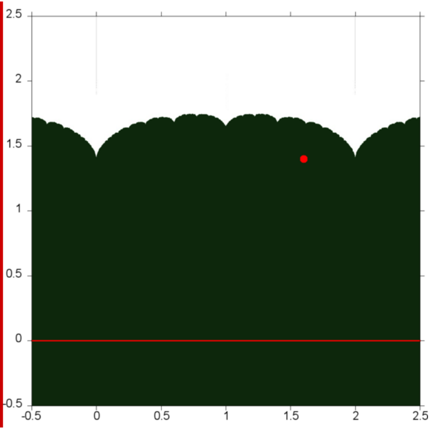


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

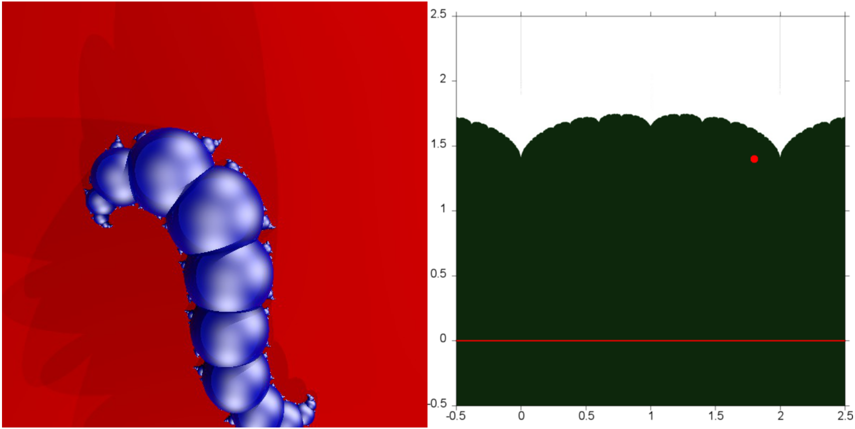




In the BM-slice of $QF(2.0)$.

More on $QF(S)$

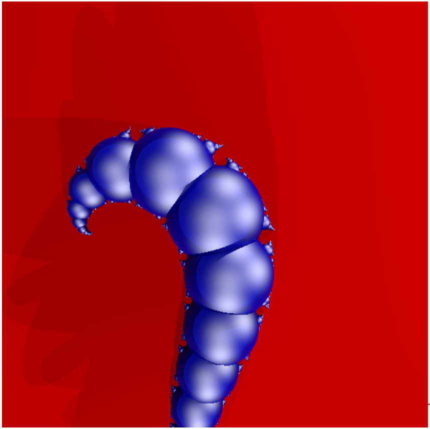
Recall $\quad \tau = \text{(twisting dist.)} + \sqrt{-1}\text{(bending angle)}.$

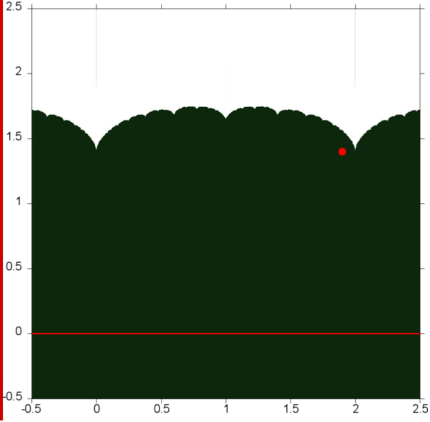


In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\tau = (\text{twisting dist.}) + \sqrt{-1}(\text{bending angle}).$

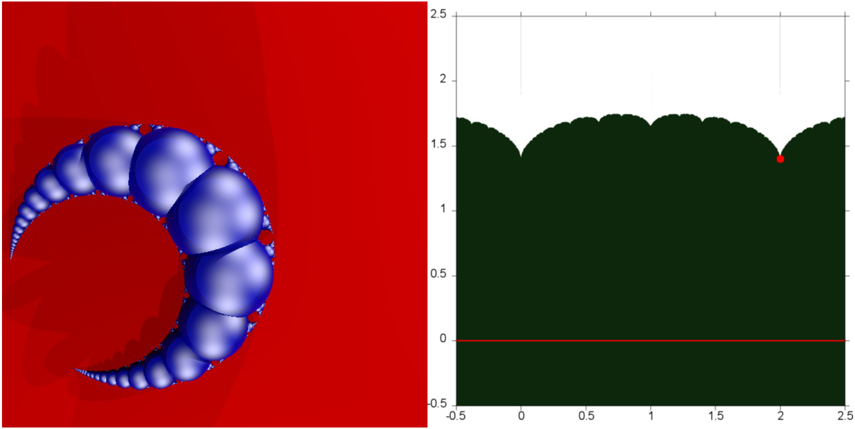




In the BM-slice of $QF(2.0)$.

More on $QF(S)$

Recall $\quad \tau = \text{(twisting dist.)} + \sqrt{-1}\text{(bending angle)}.$



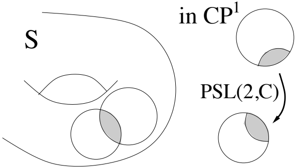
In the BM-slice of $QF(2.0)$.

Complex projective structures

S : surface ($\chi(S) < 0$)

Definition

A complex projective structure or $\mathbb{C}P^1$-structure on S is a geometric structure locally modelled on $\mathbb{C}P^1$ with transition functions in $\mathrm{PSL}_2\mathbb{C}$.

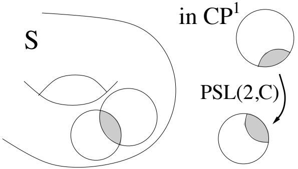


Complex projective structures

S : surface ($\chi(S) < 0$)

Definition

A complex projective structure or $\mathbb{C}P^1$-structure on S is a geometric structure locally modelled on $\mathbb{C}P^1$ with transition functions in $\mathrm{PSL}_2\mathbb{C}$.



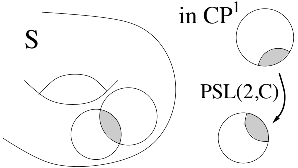
(If S has punctures, assume some boundary conditions.)

Complex projective structures

S : surface ($\chi(S) < 0$)

Definition

A complex projective structure or $\mathbb{C}P^1$-structure on S is a geometric structure locally modelled on $\mathbb{C}P^1$ with transition functions in $\mathrm{PSL}_2\mathbb{C}$.



(If S has punctures, assume some boundary conditions.)

Example (Fuchsian uniformization)

A hyperbolic str on S gives an identification $\widetilde{S} \cong \mathbb{H}^2$. Since $\mathbb{H}^2 \subset \mathbb{C}P^1$, this gives a $\mathbb{C}P^1$-str.

Complex projective structures

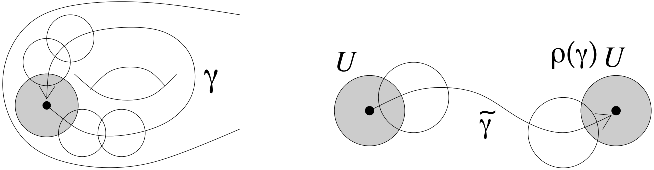
Similarly as Teichmüller space, we can define $P(S)$ the set of marked $\mathbb{C}P^1$-structures on S.

Complex projective structures

Similarly as Teichmüller space, we can define $P(S)$ the set of marked $\mathbb{C}P^1$-structures on S.

By analytic continuation, we have a holonomy map

$$\text{hol} : P(S) \to X(S).$$



This is known to be a local homeomorphism.

Grafting

We can construct another $\mathbb{C}P^1$-str from a Fuchsian uniformization.

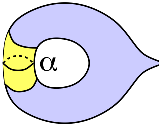
X : a hyp str on S, $\quad \alpha \subset X$: a simple closed geodesic.

Grafting

We can construct another $\mathbb{C}P^1$-str from a Fuchsian uniformization.

X : a hyp str on S, $\alpha \subset X$: a simple closed geodesic.

Let $\mathrm{Gr}_{b\cdot\alpha}(X)$ be the $\mathbb{C}P^1$-str obtained from X by inserting a height b annulus along α.

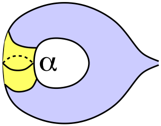


Grafting

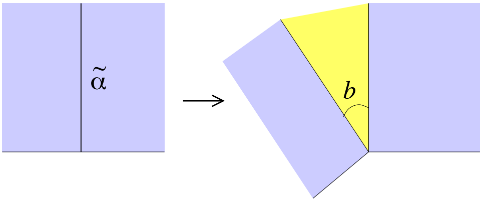
We can construct another $\mathbb{C}P^1$-str from a Fuchsian uniformization.

X : a hyp str on S, $\alpha \subset X$: a simple closed geodesic.

Let $\mathrm{Gr}_{b\cdot\alpha}(X)$ be the $\mathbb{C}P^1$-str obtained from X by inserting a height b annulus along α.



In the universal cover $\widetilde{X}$, the local picture looks like:

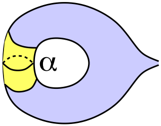


Grafting

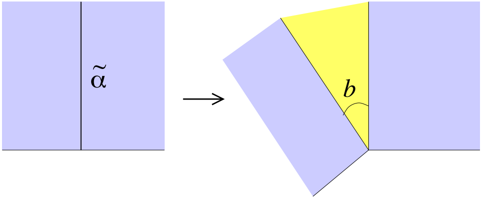
We can construct another $\mathbb{C}P^1$-str from a Fuchsian uniformization.

X : a hyp str on S, $\alpha \subset X$: a simple closed geodesic.

Let $\mathrm{Gr}_{b \cdot \alpha}(X)$ be the $\mathbb{C}P^1$-str obtained from X by inserting a height b annulus along α.

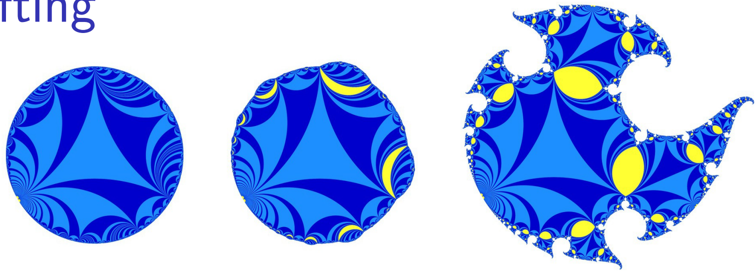


In the universal cover $\widetilde{X}$, the local picture looks like:

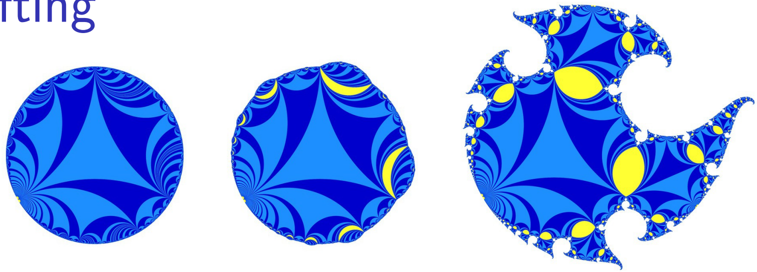


But there are infinitely many lifts of $\alpha \cdots$

Grafting



Grafting



The grafting operation $\mathrm{Gr}_{b\cdot\alpha} : \mathcal{T}(S) \to P(S)$ can be generalized for measured laminations.

Theorem (Thurston, Kamishima-Tan)

$$\begin{array}{rcl} \mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) & \to & P(S) \\ (\mu, X) & \mapsto & \mathrm{Gr}_\mu(X) \end{array}$$

is a homeomorphism (Thurston coordinates).

$\mathbb{C}P^1$-structures with q-F holonomy

$$Q_0 = \{ \text{ marked } \mathbb{C}P^1\text{-strs with q-F holonomy and injective developing maps } \} \subset P(S)$$

$\mathbb{C}P^1$-structures with q-F holonomy

$Q_0 = \{$ marked $\mathbb{C}P^1$-strs with q-F holonomy and injective developing maps $\} \subset P(S)$

$\mathcal{ML}_{\mathbb{Z}}(S)$: the set of multicurves with integral weights.

$\mathbb{C}P^1$-structures with q-F holonomy

$Q_0 = \{$ marked $\mathbb{C}P^1$-strs with q-F holonomy and injective developing maps $\} \subset P(S)$

$\mathcal{ML}_{\mathbb{Z}}(S)$: the set of multicurves with integral weights.

For $\mu \in \mathcal{ML}_{\mathbb{Z}}(S)$, let Q_μ be the set of $\mathbb{C}P^1$-strs obtained from Q_0 by $2\pi\mu$-grafting. (**Remark** $Q_\mu \cong Q_0$.)

$\mathbb{C}P^1$-structures with q-F holonomy

$Q_0 = \{$ marked $\mathbb{C}P^1$-strs with q-F holonomy and injective developing maps $\} \subset P(S)$

$\mathcal{ML}_{\mathbb{Z}}(S)$: the set of multicurves with integral weights.

For $\mu \in \mathcal{ML}_{\mathbb{Z}}(S)$, let Q_μ be the set of $\mathbb{C}P^1$-strs obtained from Q_0 by $2\pi\mu$-grafting. (**Remark** $Q_\mu \cong Q_0$.)

Theorem (Goldman)

$$\mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} Q_\mu$$

The component Q_0 is called standard, Q_μ $(\mu \neq 0)$ exotic.

Complex Earthquake

Let $\quad \overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}.$

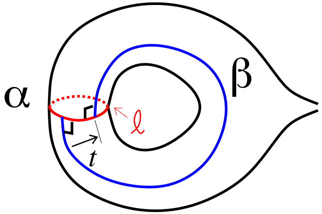
Complex Earthquake

Let $\overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}$. Fix $\ell > 0$.

Complex Earthquake

Let $\overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}$. Fix $\ell > 0$.

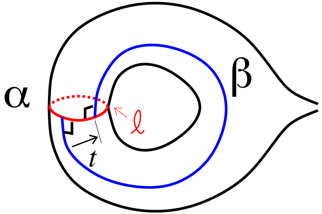
Let $\mathrm{tw}_{t\cdot\alpha}(X_\ell) = \Bigg($



$\Bigg) \in \mathcal{T}(S)$.

Complex Earthquake

Let $\overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}$. Fix $\ell > 0$.

Let $\mathrm{tw}_{t\cdot\alpha}(X_\ell) = \Bigg($ 

$\Bigg) \in \mathcal{T}(S)$.

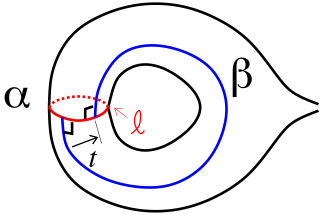
Define $\mathrm{Eq} : \overline{\mathbb{H}} \to P(S)$ by

$$\mathrm{Eq}(t + \sqrt{-1}b) = \mathrm{Gr}_{b\cdot\alpha}(\mathrm{tw}_{t\cdot\alpha}(X_\ell)) \in P(S)$$

By Thurston coords, we can regard $\overline{\mathbb{H}} \subset P(S)$.

Complex Earthquake

Let $\overline{\mathbb{H}} = \{\tau = t + \sqrt{-1}b \in \mathbb{C} \mid b \geq 0\}$. Fix $\ell > 0$.

Let $\mathrm{tw}_{t\cdot\alpha}(X_\ell) = \Big($ 

$\Big) \in \mathcal{T}(S)$.

Define $\mathrm{Eq} : \overline{\mathbb{H}} \to P(S)$ by

$$\mathrm{Eq}(t + \sqrt{-1}b) = \mathrm{Gr}_{b\cdot\alpha}(\mathrm{tw}_{t\cdot\alpha}(X_\ell)) \in P(S)$$

By Thurston coords, we can regard $\overline{\mathbb{H}} \subset P(S)$.

Simply denote the image of $\overline{\mathbb{H}}$ by $\mathrm{Eq}(\ell)$.

Complex Earthquake

By construction, hol is the natural projection:

$$\begin{array}{ccc}
P(S) & \xrightarrow{\text{hol}} & X(S) \\
\cup & & \cup \\
\mathrm{Eq}(\ell) & \to & X(\ell) \\
\| & & \| \\
\{\tau \mid \mathrm{Im}(\tau) \geq 0\} & & \{\tau \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \\
\Cup & & \Cup \\
\tau & \mapsto & \tau \mod 2\pi\sqrt{-1}
\end{array}$$

Complex Earthquake

By construction, hol is the natural projection:

$$\begin{array}{ccc}
P(S) & \xrightarrow{\text{hol}} & X(S) \\
\cup & & \cup \\
\mathrm{Eq}(\ell) & \to & X(\ell) \\
\| & & \| \\
\{\tau \mid \mathrm{Im}(\tau) \geq 0\} & & \{\tau \mid -\pi < \mathrm{Im}(\tau) \leq \pi\} \\
\Downarrow & & \Downarrow \\
\tau & \mapsto & \tau \mod 2\pi\sqrt{-1}
\end{array}$$

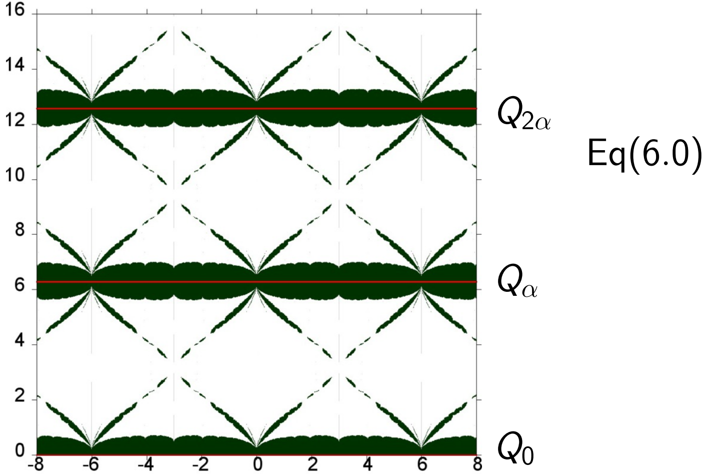
We are interested in $QF(\ell) \subset X(\ell)$, so consider

$$\begin{aligned}
\mathrm{hol}^{-1}(QF(\ell)) &= \mathrm{hol}^{-1}(X(\ell) \cap QF(S)) \\
&= \mathrm{Eq}(\ell) \cap \mathrm{hol}^{-1}(QF(S)).
\end{aligned}$$

Complex Earthquake

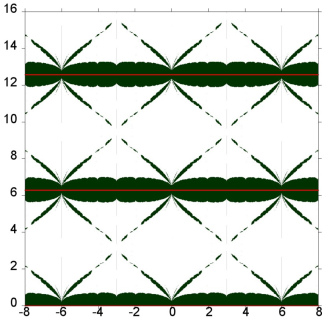
By Goldman's Theorem, we have

$$\mathrm{Eq}(\ell) \cap \mathrm{hol}^{-1}(QF(S)) = \bigsqcup_{\mu \in \mathcal{ML}_{\mathbb{Z}}(S)} \mathrm{Eq}(\ell) \cap Q_\mu.$$



Complex Earthquake

hol maps each component of $\mathrm{Eq}(\ell) \cap Q_\mu$ into a comp of $QF(\ell)$.

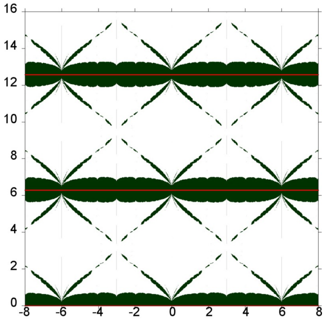


Complex Earthquake

hol maps each component of $\text{Eq}(\ell) \cap Q_\mu$ into a comp of $QF(\ell)$.
Thus if

$$\text{Eq}(\ell) \cap Q_\mu \neq \emptyset$$

for some $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$, $QF(\ell)$ has a comp other than the standard one BM.

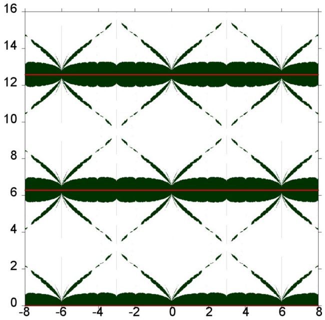


Complex Earthquake

hol maps each component of $\mathrm{Eq}(\ell) \cap Q_\mu$ into a comp of $QF(\ell)$. Thus if

$$\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$$

for some $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$, $QF(\ell)$ has a comp other than the standard one BM. Moreover,



Prop (K.)

$$Eq(\ell) \cap \mathrm{hol}^{-1}(BM) = \bigsqcup_{k \geq 0} Eq(\ell) \cap Q_{k \cdot \alpha}$$

for any $\ell > 0$.

Existence of exotic components in Eq(ℓ)

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Existence of exotic components in $\mathrm{Eq}(\ell)$

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

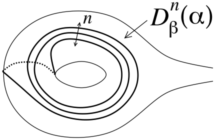
Existence of exotic components in Eq(ℓ)

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

$$\left(\frac{2\pi}{n} D_\beta^n(\alpha),\, X\right)$$



which converges to $(2\pi\beta,\, X) \in Q_\beta$ as $n \to \infty$.

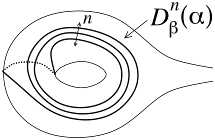
Existence of exotic components in $\mathrm{Eq}(\ell)$

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

$$\left(\frac{2\pi}{n} D_\beta^n(\alpha),\ X\right)$$



which converges to $(2\pi\beta,\ X) \in Q_\beta$ as $n \to \infty$.

Thus $(\frac{2\pi}{n} D_\beta^n(\alpha),\ X) \in Q_\beta$ for large n.

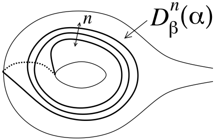
Existence of exotic components in $\mathrm{Eq}(\ell)$

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

$$\left(\frac{2\pi}{n} D_\beta^n(\alpha),\, X\right)$$



which converges to $(2\pi\beta,\, X) \in Q_\beta$ as $n \to \infty$.

Thus $(\frac{2\pi}{n} D_\beta^n(\alpha),\, X) \in Q_\beta$ for large n.

Apply D_β^{-n}, then $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in Q_\beta$ for large n.

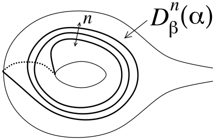
Existence of exotic components in $\mathrm{Eq}(\ell)$

We need to find $\mu \notin \{0, \alpha, 2\alpha, \cdots\}$ s.t. $\mathrm{Eq}(\ell) \cap Q_\mu \neq \emptyset$ for sufficiently large $\ell > 0$. Consider the case $\mu = \beta$.

Let D_β be the Dehn twist along β. Fix $X \in \mathcal{T}(S)$.

Consider a sequence in $P(S) \cong \mathcal{ML}(S) \times \mathcal{T}(S)$

$$\left(\frac{2\pi}{n} D_\beta^n(\alpha),\, X\right)$$



which converges to $(2\pi\beta,\, X) \in Q_\beta$ as $n \to \infty$.

Thus $(\frac{2\pi}{n} D_\beta^n(\alpha),\, X) \in Q_\beta$ for large n.

Apply D_β^{-n}, then $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in Q_\beta$ for large n.

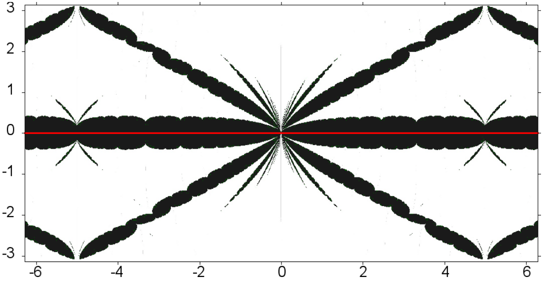
But if we let $\ell = \ell_\alpha(D_\beta^{-n}(X))$, $(\frac{2\pi}{n}\alpha,\, D_\beta^{-n}(X)) \in \mathrm{Eq}(\ell)$.

Final Remarks

- For $k \in \mathbb{N}$, we can show $\mathrm{Eq}(\ell) \cap Q_{k \cdot \beta} \neq \emptyset$ similarly for large ℓ by considering

$$\left(\frac{2\pi k}{n} D_\beta^n(\alpha),\, X \right) \xrightarrow{n \to \infty} (2\pi k \beta,\, X) \in Q_{k \cdot \beta}.$$

$\ell = 10.0$

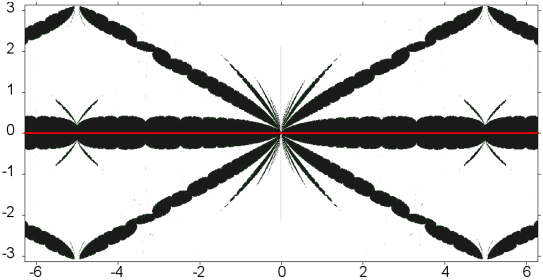


Final Remarks

- For $k \in \mathbb{N}$, we can show $\mathrm{Eq}(\ell) \cap Q_{k\cdot\beta} \neq \emptyset$ similarly for large ℓ by considering

$$\left(\frac{2\pi k}{n} D_\beta^n(\alpha),\ X\right) \xrightarrow{n\to\infty} (2\pi k\beta,\ X) \in Q_{k\cdot\beta}.$$

$\ell = 10.0$



- Moreover we can use $\mu \in \mathcal{ML}(S)_{\mathbb{Z}}$ instead of β provided $i(\mu, \alpha) \neq 0$.