

# Visualizing deformations of complex projective structures

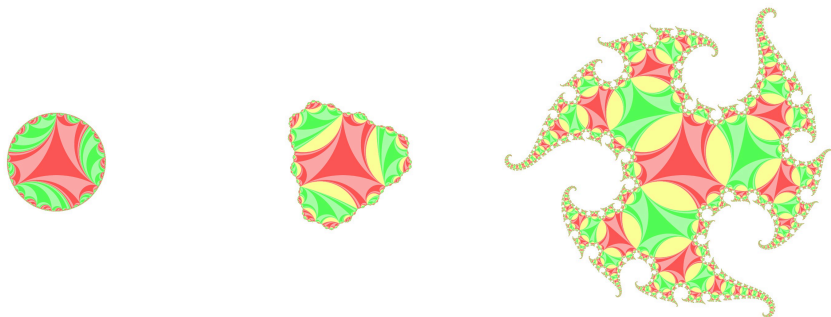
Yuichi Kabaya

Kitami Institute of Technology

Osaka City University, October 19 2019

# Outline

We give a visualization of grafting deformation of complex projective structures on a surface, such as:



What do these images mean? How to draw these?

## Notations

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$$

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$  : The projective line (Riemann sphere)

$\mathrm{PSL}(2, \mathbb{C})$  faithfully acts on  $\mathbb{C}P^1$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

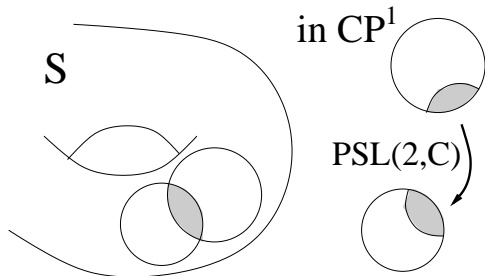
( $\mathrm{PSL}(2, \mathbb{C})$  is the group of holomorphic automorphisms of  $\mathbb{C}P^1$ .)

# Complex projective structures

$S$  : a surface (assume that the Euler characteristic  $\chi(S) < 0$ .)

## Definition

A **complex projective structure** or  **$\mathbb{C}P^1$ -structure** on  $S$  is a geometric structure locally modelled on  $\mathbb{C}P^1$  with transition functions in  $\text{PSL}(2, \mathbb{C})$ .



## Complex projective structures

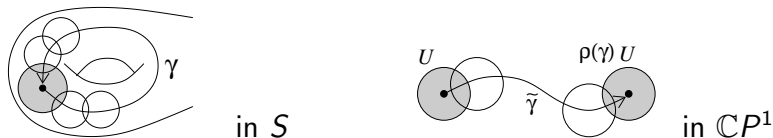
Let  $\tilde{S}$  be the universal cover of  $S$ .

By analytic continuation, we have a pair of maps

$$D : \tilde{S} \rightarrow \mathbb{C}P^1 \quad (\text{developing map}),$$

$$\rho : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C}) \quad (\text{holonomy})$$

satisfying  $D(\gamma \cdot x) = \rho(\gamma) \cdot D(x)$  ( $\gamma \in \pi_1(S)$ ,  $x \in \tilde{S}$ ).



Conversely  $(D, \rho)$  determines  $\mathbb{C}P^1$ -structure on  $S$ .

Today, we visualize the image of the developing map of a  $\mathbb{C}P^1$ -str.

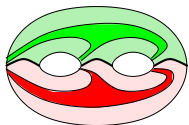
# Complex projective structures

## Example (Fuchsian uniformization)

A hyperbolic structure on  $S$  gives an identification  $\tilde{S} \cong \mathbb{H}^2$ . The Poincaré disk model  $\mathbb{H}^2 \subset \mathbb{C}P^1$  gives a  $\mathbb{C}P^1$ -structure on  $S$ .

A hyperbolic surface  $S$  can be decomposed into “pairs of pants” by cutting along disjoint simple closed geodesics.

Further, each pair of pants decomposed into two ideal triangles.



$S$



$D(\tilde{S}) \cong \mathbb{H}^2$

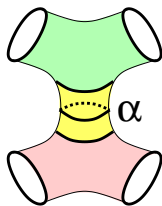
The developed image of these ideal triangle in  $\mathbb{C}P^1$  looks like above.

# Grafting

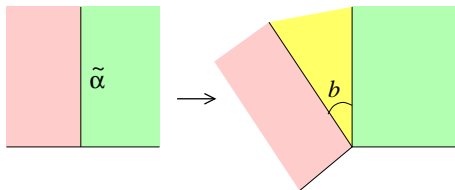
$X$  : a hyperbolic structure on  $S$ .

$\alpha \subset X$  : a simple closed geodesic.

$\text{Gr}_{b-\alpha}(X)$  is the  $\mathbb{C}P^1$ -str obtained by inserting an annulus of height  $b$  along  $\alpha$ .



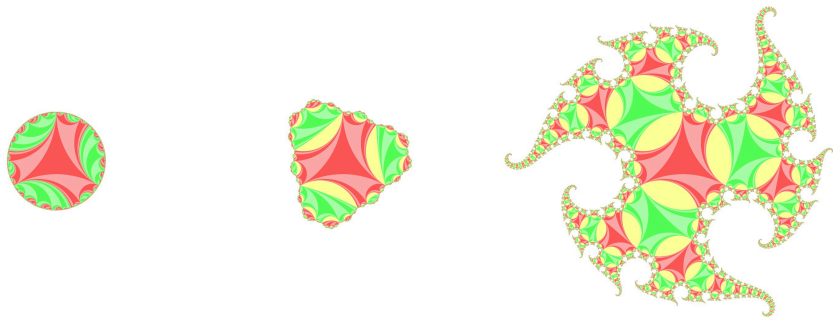
In the universal cover  $\tilde{X}$ , this looks like:



The grafting construction is very simple. But there are infinitely many lifts of  $\alpha$  to  $\tilde{X}$ , so the developing map might be very complicated.

# Grafting

The images in the first slide indicate the deformation of  $\mathbb{C}P^1$ -structures by grafting along some simple closed curve.



As these pictures show, the holonomy representation might be very complicated. ( $\rho(\pi_1(S)) \subset \text{PSL}(2, \mathbb{C})$  can be indiscrete.)



## Known facts

Theorem (Gallo-Kapovich-Marden, 2000)

*Every non-elementary representation  $\pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$  is realized as the holonomy of some  $\mathbb{C}P^1$ -structure on  $S$ .*

The grafting operation is also defined for weighted multicurves (weights correspond to heights of annuli).

Moreover, we can extend grafting operation to the measured geodesic laminations.

If we let

$\mathcal{T}(S)$  = Teichmüller space of  $S$ ,

$\mathcal{ML}(S)$  = {measured laminations on  $S$ },

$\mathcal{P}(S)$  = {marked  $\mathbb{C}P^1$ -structures on  $S$ },

the grafting operation gives a map  $\mathcal{T}(S) \times \mathcal{ML}(S) \rightarrow \mathcal{P}(S)$ .

## Known facts

Theorem (Thurston, Kamishima-Tan, 1992)

*The map*

$$\mathcal{T}(S) \times \mathcal{ML}(S) \rightarrow \mathcal{P}(S)$$

*is a homeomorphism.*

Since the set of simple closed curves is dense in  $\mathcal{ML}(S)$ , any  $\mathbb{C}P^1$ -structure on  $S$  is approximated by  $\mathbb{C}P^1$ -structures obtained by grafting along some s.c.c. (or disjoint s.c.c.'s).

So we want to draw the pictures of the developing maps of  $\mathbb{C}P^1$ -structures obtained by grafting along some disjoint s.c.c.'s.

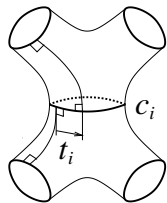
# Fenchel-Nielsen coordinates

Let  $S$  be a surface of genus  $g$ .

Fix a pants decomposition of  $S$ . We need  $3g - 3$  curves (say  $c_1, \dots, c_{3g-3}$ ) to decompose  $S$  into pairs of pants.

The set of marked hyperbolic structures on  $S$  (i.e. Teichmüller space of  $S$ ) is parametrized by

- $3g - 3$  lengths  $\ell_i$  of  $c_i$ , and
- $3g - 3$  twist parameters  $t_i$  along  $c_i$ .



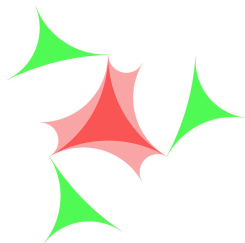
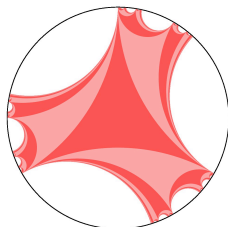
Inserting height  $b_i \geq 0$  curves along  $c_i$ 's, we obtain a complex projective structure on  $S$ .

## Drawing developing maps

From given parameters  $(\ell_1, \dots, \ell_{3g-3}, t_1, \dots, t_{3g-3}, b_1, \dots, b_{3g-3})$ , we can explicitly construct the developing map and the holonomy rep.

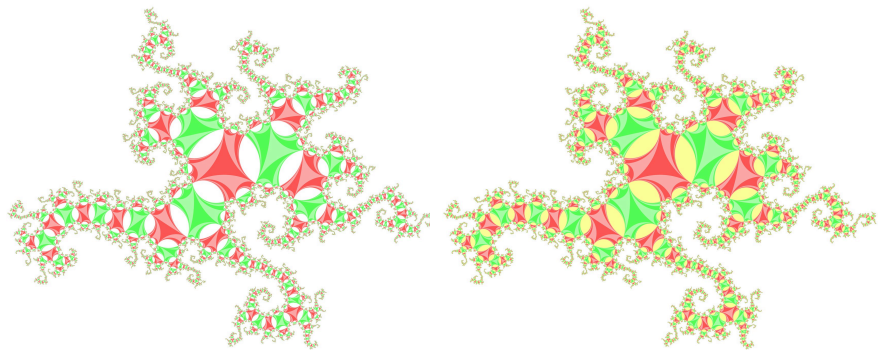
To draw the developing map from given parameters  $\ell_i, t_i, b_i$ , it is enough to compute:

- relative position of two ideal triangles belonging to one pair of pants, in  $\mathbb{C}P^1$ . (See left below.)
- relative position of two ideal triangles belonging to two adjacent pairs of pants, in  $\mathbb{C}P^1$ . (See right below.)



## Drawing developing maps

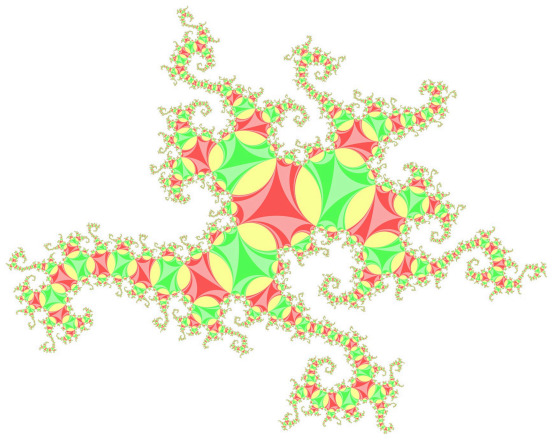
Draw these ideal triangles repeatedly, then draw lifts of the inserted annuli (light yellow region right below), we obtain the image of the developing map.



Formulas to draw these ideal triangles are found in [arXiv:1110.6674](https://arxiv.org/abs/1110.6674).

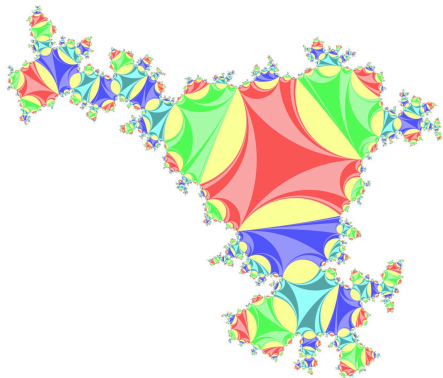
## Example 1

Genus 2 ( $3g - 3 = 3$ )  $\ell_0 = 2.3, \ell_1 = 1.5, \ell_2 = 2.0,$   
 $t_0 = 0.1, t_1 = -0.2, t_2 = 1.1, \quad b_0 = 1.40, b_1 = 1.91, b_2 = 1.54$



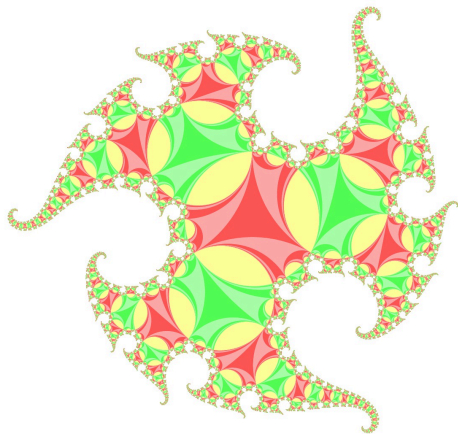
## Example 2

Genus 3 ( $3g - 3 = 6$ )  $\ell_0 = \dots = \ell_5 = 2.0$ ,  
 $t_0 = t_1 = t_2 = 0.2 \cdot \ell_0$ ,  $t_3 = 0.3 \cdot \ell_3$ ,  $t_4 = t_5 = 0.8 \cdot \ell_4$ ,  
 $b_0 = b_1 = b_2 = 0.3\pi$ ,  $b_3 = 0.7\pi$ ,  $b_4 = 0.3\pi$ ,  $b_5 = 0.5\pi$



## Example 3

Genus 2     $l_0 = l_1 = l_2 = 2.0$ ,  
 $t_0 = t_2 = t_2 = 0.02$ ,     $b_0 = b_1 = b_2 = 0.490\pi$     (very symmetric)





# Tips

We need to draw a huge number of ideal triangles.

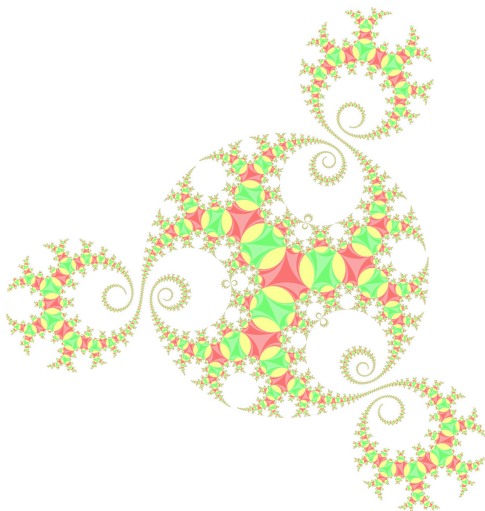
Once a sufficiently 'small' ideal triangle appeared, we don't continue the iteration process. This greatly reduces the number of ideal triangles to draw.

But if there is a thin part, we may miss some important features.

## Example 4

Genus 2  $l_0 = l_1 = l_2 = 2.0,$

$t_0 = t_2 = t_2 = 0.2503 \cdot 2.0,$   $b_0 = b_1 = b_2 = 0.507\pi$



# Useful tools

- PostScript : Programming language, easy to draw polygons and circular arcs. For example, an ideal triangle is drawn by

```
0 setlinewidth
newpath
450.0 -8.1 440.0 120.0 60.0 arcn
890.0 754.0 440.0 -120.0 180.0 arcn
10.0 754.0 440.0 0.0 -60.0 arcn
0.996094 0.332031 0.332031 setrgbcolor
fill
```

- ImageMagick : Sometimes PS files might be too large (10MB~1GB). ImageMagick converts PS file into other formats (jpeg, png, etc.). This substantially reduces the file size.
- FFmpeg : Create a movie (gif, mp4, etc.) from static images.