解析学 I 演習に対する追加説明#8

- 通常の x-y 座標と極座標との間の変換も大切である。演習時間にはやっていない演習問題であるが解説しておく。

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

はラプラシアン (*Laplacian*) と呼ばれる重要な作用素である。これは x-y 座標で書かれているが，極座標に変換することを考える。

- x-y 座標と極座標の関係は

$$x = r\cos\theta, \qquad y = r\sin\theta$$

なので，ヤコビ行列は $\dfrac{D(x,y)}{D(r,\theta)} = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

である。

- ラプラシアンを関数 $z = z(x, y)$ に作用させると

$$\Delta z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

である。

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad (1)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta \qquad (2)$$

- 式 (1) を r で微分すると

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right) \\ &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\sin\theta \qquad (3)\end{aligned}$$

となる。

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} \\
&= \frac{\partial^2 z}{\partial x^2}\cos\theta+\frac{\partial^2 z}{\partial x\partial y}\sin\theta \\
\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} \\
&= \frac{\partial^2 z}{\partial x\partial y}\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin\theta
\end{aligned}
$$

を (3) に代入して

$$
\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta \quad (4)
$$

を得る。計算の途中で $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$ を使った。

- 同様に式 (2) を θ で微分すると

$$
\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-\frac{\partial z}{\partial x}r\sin\theta+\frac{\partial z}{\partial y}r\cos\theta\right) \\
&= -\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)r\sin\theta-\frac{\partial z}{\partial x}r\cos\theta+\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)r\cos\theta-\frac{\partial z}{\partial y}r\sin\theta \quad (5)
\end{aligned}
$$

となる。

$$
\begin{aligned}
\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial \theta} \\
&= -\frac{\partial^2 z}{\partial x^2}r\sin\theta+\frac{\partial^2 z}{\partial x\partial y}r\cos\theta \\
\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial \theta}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial \theta} \\
&= -\frac{\partial^2 z}{\partial x\partial y}r\sin\theta+\frac{\partial^2 z}{\partial y^2}r\cos\theta
\end{aligned}
$$

を (5) に代入して

$$
\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial^2 z}{\partial x^2}r^2\sin^2\theta-2\frac{\partial^2 z}{\partial x\partial y}r^2\sin\theta\cos\theta \\
&\quad +\frac{\partial^2 z}{\partial y^2}r^2\cos^2\theta-\frac{\partial z}{\partial x}r\cos\theta-\frac{\partial z}{\partial y}r\sin\theta \quad (6)
\end{aligned}
$$

を得る。

- よって (4) と (6) より

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta \\
&\quad +\frac{1}{r^2}\left\{\frac{\partial^2 z}{\partial x^2}r^2\sin^2\theta-2\frac{\partial^2 z}{\partial x\partial y}r^2\sin\theta\cos\theta\right. \\
&\quad \left.+\frac{\partial^2 z}{\partial y^2}r^2\cos^2\theta-\frac{\partial z}{\partial x}r\cos\theta-\frac{\partial z}{\partial y}r\sin\theta\right\} \\
&= \frac{\partial^2 z}{\partial x^2}\left(\cos^2\theta+\sin^2\theta\right)+\frac{\partial^2 z}{\partial y^2}\left(\sin^2\theta+\cos^2\theta\right) \\
&\quad -\frac{1}{r}\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right) \\
&= \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}-\frac{1}{r}\frac{\partial z}{\partial r}
\end{aligned}$$

  を得る。

- $\Delta z=\dfrac{\partial^2 z}{\partial r^2}+\dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2}+\dfrac{1}{r}\dfrac{\partial z}{\partial r}$ よりラプラシアンを極座標で表現すると

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r}\frac{\partial}{\partial r}$$

  となる。

- 3 変数関数は本質的に 2 変数関数と同様であるが，計算は複雑になる。演習時間には時間をとってやっていないので，1 題解説しておく。

$$w=x^3+y^3+z^3, x+y+z=s, xy+yz+zx=t, xyz=u$$

  とする。

$$\frac{D(s,t,u)}{D(x,y,z)}=\begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ yz & zx & xy \end{pmatrix}$$

  なので基本変形を実行する。

$$
\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ y+z & z+x & x+y & 0 & 1 & 0 \\ yz & zx & xy & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & x-y & x-z & -z-y & 1 & 0 \\ 0 & z(x-y) & y(x-z) & -yz & 0 & 1 \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \dfrac{x-z}{x-y} & -\dfrac{y+z}{x-y} & \dfrac{1}{x-y} & 0 \\ 0 & 0 & (z-x)(z-y) & z^2 & -z & 1 \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \dfrac{x-z}{x-y} & -\dfrac{y+z}{x-y} & \dfrac{1}{x-y} & 0 \\ 0 & 0 & 1 & \dfrac{z^2}{(z-x)(z-y)} & -\dfrac{z}{(z-x)(z-y)} & \dfrac{1}{(z-x)(z-y)} \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \dfrac{y^2}{(y-z)(y-x)} & -\dfrac{y}{(y-z)(y-x)} & \dfrac{1}{(y-z)(y-x)} \\ 0 & 0 & 1 & \dfrac{z^2}{(z-x)(z-y)} & -\dfrac{z}{(z-x)(z-y)} & \dfrac{1}{(z-x)(z-y)} \end{pmatrix}
$$

$$
\to \begin{pmatrix} 1 & 0 & 0 & \dfrac{x^2}{(x-y)(x-z)} & -\dfrac{x}{(x-y)(x-z)} & \dfrac{1}{(x-y)(x-z)} \\ 0 & 1 & 0 & \dfrac{y^2}{(y-z)(y-x)} & -\dfrac{y}{(y-z)(y-x)} & \dfrac{1}{(y-z)(y-x)} \\ 0 & 0 & 1 & \dfrac{z^2}{(z-x)(z-y)} & -\dfrac{z}{(z-x)(z-y)} & \dfrac{1}{(z-x)(z-y)} \end{pmatrix}
$$

$$
\begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial s} & \dfrac{\partial z}{\partial t} & \dfrac{\partial z}{\partial u} \end{pmatrix} = \frac{D(x,y,z)}{D(s,t,u)} = \left(\frac{D(s,t,u)}{D(x,y,z)}\right)^{-1}
$$

$$
= \begin{pmatrix} \dfrac{x^2}{(x-y)(x-z)} & -\dfrac{x}{(x-y)(x-z)} & \dfrac{1}{(x-y)(x-z)} \\ \dfrac{y^2}{(y-z)(y-x)} & -\dfrac{y}{(y-z)(y-x)} & \dfrac{1}{(y-z)(y-x)} \\ \dfrac{z^2}{(z-x)(z-y)} & -\dfrac{z}{(z-x)(z-y)} & \dfrac{1}{(z-x)(z-y)} \end{pmatrix}
$$

となる。

$$\frac{\partial w}{\partial x} = 3x^2, \quad \frac{\partial w}{\partial y} = 3y^2, \quad \frac{\partial w}{\partial z} = 3z^2$$

なので

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= 3x^2 + 3y^2 + 3z^2 + 3xy + 3zx + 3yz\end{aligned}$$

となる。$X = \dfrac{\partial w}{\partial s}$ とおき，これに合成関数の微分法を適用すると

$$\begin{aligned}\frac{\partial^2 w}{\partial s^2} = \frac{\partial X}{\partial s} &= \frac{\partial X}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial s} \\ &= 6x + 6y + 6z\end{aligned}$$

が得られる。