On the possible solution(s) of the Continuum Problem

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Abstract

Consider the following type of statements:

(1) For the class $\mathcal{C}$ of structures, if $\mathfrak{A} \in \mathcal{C}$ then there is a substructure $\mathfrak{B} \in \mathcal{C}$ of $\mathfrak{A}$ of cardinality $< \kappa$.

Note that Löwenheim-Skolem theorems (LSTs) can be seen as of this type. For example, the usual LST for first-order logic can be formulated as (1) with:

$\mathcal{C}$ = structures in countable language with built-in Skolem-functions, and $\kappa = \aleph_1$.

Let us call a statement of the form (1) a *reflection principle* and κ the *reflection point* of the reflection principle. Some reflection principles are independent from ZFC and have large cardinal consistency strength.

(2) Any graph with uncountable coloring number has a subgraph of cardinality $< \aleph_2$ with uncountable coloring number.

is such an example — It can be forced starting from a model of ZFC with a supercompact cardinal and it implies the existence of an inner model with class many Woodin cardinals.

In contrast,

(3) Any graph with uncountable chromatic number has a subgraph of cardinality $< \aleph_2$ with uncountable chromatic number.

is simply wrong since the reflection point of uncountable chromatic number $\geq \beth_\omega$ (if it exists) by a theorem of Erdős and Hajnal.

It is known that some reflection principles have influence on the size of the continuum. For example, the reflection principle called RP in Jech's Millennium Book ([5], Definition 37.17) implies that the continuum is $\leq \aleph_2$ (the principle (2) follows from RP but (2) does not restrict the size of the continuum).

In a joint work with André Ottenbreit Maschio Rodrigues, and Hiroshi Sakai, I obtained results which can be summarized as

(4) If very strong (reasonable) reflection principle with reflection point around the continuum should hold, then the continuum is either $\aleph_1$ or $\aleph_2$ or extremely large ([1], [2]).

In this talk, I shall give an overview of this theory as well as proofs of some of the related theorems which can be done assuming only the basics of forcing.

References

[1] S.F., André Ottenbreit Maschio Rodrigues, and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, Archive for Mathematical Logic, Vol.60, 1-2, (2021), 17–47.
`https://fuchino.ddo.jp/papers/SDLS-x.pdf`

[2] ________________________, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Vol.60, 3-4, (2021), 495–523.
`https://fuchino.ddo.jp/papers/SDLS-III-x.pdf`

[3] ________________________, Strong downward Löwenheim-Skolem theorems for stationary logics, III — mixed support iteration, to appear in the Proceedings of the Asian Logic Conference 2019.

[4] S.F., and André Ottenbreit Maschio Rodrigues, Reflection principles, generic large cardinals, and the Continuum Problem, Proceedings of the Symposium on Advances in Mathematical Logic 2018, Springer (2022).

[5] T. Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006).

[6] Bernhard König, Generic compactness reformulated, Archive of Mathematical Logic 43, (2004), 311 – 326.