Culler-Shalen 理論の概説

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1. Introduction - Outline -

$K \subset S^3$: a knot in S^3

$M = S^3 \setminus N(K)$: knot exterior
(more generally, cpt. ori. 3-mfd with torus boundary)

$X(M) = \{\text{homom. } \pi_1(M) \to \mathrm{SL}_2\mathbb{C} \text{ "up to conjugation"}\}$

is an affine algebraic set over $\mathbb{C}$, called the *character variety*.

$C \subset X(M)$: irreducible (affine) curve (possibly singular)

$\widetilde{C}$: smooth projective curve birationally equiv. to C
$(\varphi : \widetilde{C} \xrightarrow{\text{birat.}} C)$

A point $p \in \widetilde{C}$ at which φ is not defined is called an *ideal point*.

Culler-Shalen theory (in a word)

Construct an *incompressible surface* in M from an ideal point.

1. Introduction - incompressible surfaces -

M : a cpt. ori. 3-mfd with boundary

A surface $S \subset M$ is *properly embedded* if $S \cap \partial M = \partial S$, and S intersects ∂M transversely.

We assume that S is 2-sided in M and does not have S^2, D^2 components.

Definition

- A disk $D \subset M$ s.t. $D \cap S = \partial D$ is called a *compressing disk* if $D \cap S$ is an essential simple closed curve on S.
- S is *incompressible* if each component of S has no compressing disk.
- S is called *essential* if it is incompressible and not boundary parallel.

By the loop theorem, a 2-sided surface $S \subset M$ is incompressible if and only if $\pi_1(S) \to \pi_1(M)$ is injective (for each component).

1. Introduction - Application -

Today we focus on the application of Culler-Shalen theory to the Cyclic Surgery Theorem.

Let $M = S^3 \setminus N(K)$ be a knot exterior. We denote the intersection number of two slopes $\alpha, \beta \subset \partial M$ by $\Delta(\alpha, \beta)$.

The Dehn filling of M along a slope α is denoted by $M(\alpha)$.

Cyclic Surgery Theorem (Culler-Gordon-Luecke-Shalen)

Let K be a non-torus knot and M its exterior.
If $\pi_1(M(\alpha))$ and $\pi_1(M(\beta))$ are cyclic*, $\Delta(\alpha, \beta) \leq 1$.

* Including the trivial group and $\mathbb{Z}$.

We identify the set of slopes on ∂M with $\mathbb{Q} \cup \{1/0\}$.

Since $M(1/0) \cong S^3$ has a cyclic π_1, $\pi_1(M(\alpha))$ is cyclic only if α is integral. There are at most two such slopes, and if there are two they are consecutive.

(Ex: $(-2, 3, 7)$-pretzel has two cyclic slopes 18 and 19.)

Plan

1. Intro
2. Basics on the character variety $X(M)$ [§1, CS1]
3. Discrete valuations and algebraic curves
4. Tree actions and incompressible surfaces [§2, CS1]
5. Bass-Serre-Tits theory [§2, CS1]
6. Culler-Shalen's main construction [CS1]
7. Cyclic Surgery Theorem [CGLS]

References

[CS1] Culler-Shalen, "Varieties of group representations and splittings of 3-manifolds," Ann. of Math., 117(1983), 109-146.

[CGLS] Culler-Gordon-Luecke-Shalen, "Dehn surgery on knots," Ann. of Math., 125(1987), 237-300.

2. Basics on the character variety

$$\mathrm{SL}_2\mathbb{C} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C},\ ad - bc = 1\}$$

$\Gamma = \langle g_1, \cdots, g_n \mid r_1, \cdots, r_k \rangle$: a finitely presented group

$R(\Gamma) = \{\rho : \Gamma \to \mathrm{SL}_2\mathbb{C}$ homomorphisms$\}$

For a manifold M, we denote $R(M) := R(\pi_1(M))$.

$\rho \in R(\Gamma)$ is determined by

$$(\rho(g_1), \cdots, \rho(g_n)) \in \mathrm{SL}_2\mathbb{C}^n \subset \mathbb{C}^{4n}.$$

Conversely, any subset of $\mathrm{SL}_2\mathbb{C}^n$ satisfying $\rho(r_i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $(i = 1, \cdots, k)$ gives a point of $R(\Gamma)$.

Thus $R(\Gamma)$ is an affine algebraic set (possibly reducible), sometimes called the *representation variety*.

2. Basics on the character variety

Consider the set of *characters*

$$\{\operatorname{tr}\rho \mid \rho \in R(\Gamma)\}.$$

This has a structure of affine algebraic set as follows.

Let $\mathbb{C}[R(\Gamma)]$ be the set of regular functions* on $R(\Gamma)$.
(* functions $R(\Gamma) \to \mathbb{C}$ written as polynomials of affine coordinates of $R(\Gamma)$.)

For $\gamma \in \Gamma$,

$$\tau_\gamma(\rho) := \operatorname{tr}\rho(\gamma)$$

is a regular function on $R(\Gamma)$. Let T be the subring of $\mathbb{C}[R(\Gamma)]$ generated by τ_γ ($\gamma \in \Gamma$). We will see that T is the ring of regular functions on $\{\operatorname{tr}\rho \mid \rho \in R(\Gamma)\}$.

Proposition [Prop 1.4.1, CS1]

There exists a finite set $\{\gamma_1\}_{i=1}^N \subset \Gamma$ s.t. $\{\tau_{\gamma_i}\}$ generates T.

Idea. Using the trace identity $\operatorname{tr}A\operatorname{tr}B = \operatorname{tr}AB + \operatorname{tr}AB^{-1}$, any τ_g is written as a polynomial of finite τ_{γ_i}'s. □

2. Basics on the character variety

Proposition [Prop 1.4.1, CS1]

There exists a finite set $\{\gamma_i\}_{i=1}^N \subset \Gamma$ s.t. $\{\tau_{\gamma_i}\}$ generates T.

For such a finite set $\{\gamma_i\}_{i=1}^N \subset \Gamma$, define a map $t : R(\Gamma) \to \mathbb{C}^N$ by

$$t(\rho) = (\tau_{\gamma_1}(\rho), \cdots, \tau_{\gamma_N}(\rho)).$$

By the proposition above, $\{\operatorname{tr}\rho \mid \rho \in R(\Gamma)\}$ is identified with

$$X(\Gamma) := t(R(\Gamma))$$

Proposition [Prop 1.4.4, Cor 1.4.5, CS1]

$X(\Gamma) = t(R(\Gamma))$ is a (Zariski) closed subset in $\mathbb{C}^N$, thus $X(\Gamma)$ is an affine algebraic set.

Thus the set of the characters $\{\operatorname{tr}\rho \mid \rho \in R(\Gamma)\}$ has a structure of affine algebraic set via $X(\Gamma)$. $X(\Gamma)$ is called the *character variety* of Γ.

$t : R(\Gamma) \to X(\Gamma)$ is a regular map.

2. Basics on the character variety

- $\rho, \rho' \in R(\Gamma)$ are conjugate if $\exists A \in \mathrm{SL}_2\mathbb{C}$ s.t.

$$\rho'(\gamma) = A^{-1}\rho(\gamma)A \quad (\forall \gamma \in \Gamma)$$

- Easy to see that $\rho \sim \rho' \implies t(\rho) = t(\rho')$
- $\rho \in R(\Gamma)$ is *reducible* if there exists a line in $\mathbb{C}^2$ invariant under ρ. Otherwise, *irreducible*.

Proposition [Cor 1.2.2, Lem 1.4.2, CS1]

- ρ is reducible if and only if $\mathrm{tr}(\rho(\gamma)) = 2 \quad (\forall \gamma \in [\Gamma, \Gamma])$.
- The set of reducible representations in $R(\Gamma)$ has the form $t^{-1}(V)$ for some closed algebraic subset of $X(\Gamma)$.

Proposition [Prop 1.5.2, CS1]

Let $\rho, \rho' \in R(\Gamma)$ s.t. $t(\rho) = t(\rho')$. If ρ is irreducible, then ρ and ρ' are conjugate.

For an irreducible component $R_0 \subset R(\Gamma)$ containing an irreducible representation, $X_0 = t(R_0)$ is a closed set [Prop 1.4.4, CS1]. We have $\dim R_0 = \dim X_0 + 3$.

3. Discrete valuations and algebraic curves

K : a field, $K^* = K \setminus \{0\}$

$v : K^* \to \mathbb{Z}$ is called a *discrete valuation* if, for $\forall x, y \in K^*$

(i) $v(xy) = v(x) + v(y)$,

(ii) $v(x + y) \geq \min\{v(x), v(y)\}$.

(Assume that v is surjective. Define $v(0) = +\infty$.)

Example

$K = \mathbb{C}(t)$: 有理関数体, $p \in \mathbb{C}$ (also for $p = \infty$ in $\mathbb{C}P^1$)
$f(t) \in \mathbb{C}(t)$ is written by using $n \in \mathbb{Z}$, $c_i \in \mathbb{C}$, $c_0 \neq 0$ as

$$f(t) = (t - p)^n(c_0 + c_1(t - p) + \cdots + c_k(t - p)^k)$$

Then, $v_p(f) = n$ is a discrete valuation.

Example

$K = \mathbb{Q}$, p : prime
$r \in \mathbb{Q}$ is written as $r = p^n(c_0 + c_1 p + \cdots + c_k p^k)$
($n \in \mathbb{Z}$, $0 \leq c_i \leq p - 1$, $c_0 \neq 0$)
Then $v_p(r) = n$ is a discrete valuation.

3. Discrete valuations and algebraic curves

(i) $v(xy) = v(x) + v(y)$, (ii) $v(x+y) \geq \min\{v(x), v(y)\}$.

Easy facts

(1) $v(\pm 1) = 0$. (Thus $v(-x) = v(x)$.)

(2) If $v(x) < v(y)$, $v(x+y) = v(x)$.

Proof.

(1) By $v(1) = v(\pm 1 \cdot \pm 1) = v(\pm 1) + v(\pm 1)$.
(2) $v(x+y) \geq \min(v(x), v(y)) = v(x)$.
Conversely, $v(x) = v((x+y) - y) \geq \min(v(x+y), v(y))$.
If $v(x+y) > v(y)$, then $v(x) \geq v(y)$ contradicts $v(x) < v(y)$. Thus $\min(v(x+y), v(y)) = v(x+y)$, therefore $v(x) \geq v(x+y)$. □

$\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ is a PID, and a local ring (i.e. having a unique proper maximal ideal). $\mathcal{O}$ is called the discrete valuation ring (DVR).

Actually, if we take $\pi \in K$ s.t. $v(\pi) = 1$, any non-trivial ideal has the form (π^n), thus (π) is the unique maximal ideal.

3. Discrete valuations and algebraic curves

Let X, Y be (affine, projective, or quasi-projective) variety over $\mathbb{C}$. Then the following are equivalent [Cor I.4.5, Har].

- X and Y are isomorphic on some non-empty Zariski open subsets.
- The function fields $\mathbb{C}(X)$, $\mathbb{C}(Y)$ are isomorphic.

In these cases, X and Y are called *birationally equivalent*.

A 1-dimensional variety C (possibly singular) gives the function field $K = \mathbb{C}(C)$, which is a fin. gen. field of trans. degree 1.

Conversely, for a given fin. gen. field $K/\mathbb{C}$ of trans. degree 1, the set of DVRs on K satisfying $v(\mathbb{C}^*) = 0$ has a structure of a smooth projective curve [§I.6, Har]. When applied to $K = \mathbb{C}(C)$, this gives a smooth projective curve $\widetilde{C}$ birat. equiv. to C.

[Har] Hartshorne, "Algebraic Geometry", GTM 52.

3. Discrete valuations and algebraic curves

Summary

For an algebraic curve C over $\mathbb{C}$, we can construct a smooth projective curve $\widetilde{C}$ birationally equivalent to C as the set of DVRs of $\mathbb{C}(C)/\mathbb{C}$.

Concisely,

$$\{ \text{ points on } \widetilde{C} \ \} \overset{1:1}{\longleftrightarrow} \{ \text{ discrete valuations on } \mathbb{C}(C)/\mathbb{C} \ \}$$

A point on $\widetilde{C}$ s.t. the birational map $\widetilde{C} \to C$ is not defined is called an *ideal point*.

The valuation v associated to an ideal point of C can be characterized by $\exists f \in \mathbb{C}[C] (\subset \mathbb{C}(C))$ s.t. $v(f) < 0$.

4. Tree actions and incompressible surfaces

Let T be a tree (a connected graph with no cycle).

M : a 3-mfd, $\widetilde{M}$: the universal cover of M.

$\pi_1(M) \curvearrowright T$: an action without inverting edges

Consider a $\pi_1(M)$-equivariant map $f : \widetilde{M} \to T$.

For each mid point m_e of an edge $e \subset T$, we assume that f is transverse to m_e. Then $\widetilde{S} := f^{-1}(m_e)$ is a surface in $\widetilde{M}$.

Since f is equivariant, $\widetilde{S}$ gives a surface $S \subset M$. In this case, we say that $S \subset M$ is associated with $\pi_1(M) \curvearrowright T$.

If $\pi_1(M)$ acts on T without inverting edges, S is 2-sided.

An action $\pi_1(M) \curvearrowright T$ is called non-trivial if it has no global fixed point of $\pi_1(M)$.

Proposition [Cor 1.3.7, CGLS], [Prop 2.3.1, CS1]

If $\pi_1(M) \curvearrowright T$ is non-trivial, we can deform f so that the associated surface is essential.

4. Tree actions and incompressible surfaces

Proposition [Cor 1.3.7, CGLS], [Prop 2.3.1, CS1]

If $\pi_1(M) \curvearrowright T$ is non-trivial, we can deform f so that the associated surface is essential.

In the beginning of talk, I wrote that the CS theory

constructs an incompressible surface in M from an ideal point.

But, actually,

the CS theory constructs a non-trivial tree action of $\pi_1(M)$.

The tree action (and the *translation length*) is uniquely determined by the ideal point, but the associated essential surface is not uniquely determined.

5. Bass-Serre-Tits theory

K : a field, $v : K^* \to \mathbb{Z}$: valuation

$\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ (discrete valuation ring)

Let $\pi \in K$ be an element with $v(\pi) = 1$.

We will construct a tree T associated with these data.

Let $V = K^2$. A lattice in V is a $\mathcal{O}$-submodule $L \subset V$ which spans V over K.

Two lattices L, L' are equivalent if $\exists \alpha \in K^*$ s.t $L' = \alpha L$.

Bass-Serre tree T

Define a tree T by

Vertices: equivalent classes of lattices $\Lambda = [L]$
Edges: Λ, Λ' are connected by an edge if $\exists$ lattices L, L' s.t.

$$\pi L \subset L' \subset L, \quad (\Lambda = [L], \Lambda' = [L'])$$

5. Bass-Serre-Tits theory

Bass-Serre tree T

Vertices: equivalent classes of lattices $\Lambda = [L]$
Edges: Λ, Λ' are connected by an edge if $\exists$ lattices L, L' s.t.

$$\pi L \subset L' \subset L, \quad (\Lambda = [L], \Lambda' = [L'])$$

Let $k = \mathcal{O}/(\pi)$ (called the residue field). The above $L' \subset L$ defines a line $L'/\pi L \subset L/\pi L \cong k^2$. Thus the link of a vertex can be regarded as $P^1(k)$ (the projective line $/k$).

$\mathrm{SL}_2 K$ naturally acts on the tree T.

It is easy to see that $\mathrm{SL}_2\mathcal{O}$ fixes the lattice $\mathcal{O}^2 \subset K^2$, thus fixing the vertex $[\mathcal{O}^2]$.

Moreover, it is known that if a subgroup $G \subset \mathrm{SL}_2 K$ fixing a vertex of the tree, then G is conjugate into $\mathrm{SL}_2\mathcal{O}$ by an element of $\mathrm{GL}_2 K$.

Further detail: [Se] Serre, "Trees", Springer.

6. Culler-Shalen's main construction

M : a cpt. ori. 3-mfd with torus boundary

$R(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2\mathbb{C})$

$\rho \in R(M)$ is written as

$$\rho(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} \quad (\gamma \in \pi_1(M)).$$

Thus we can regard $a(\gamma), b(\gamma), c(\gamma), d(\gamma) \in \mathbb{C}[R(M)]$. This gives a tautological representation $P : \pi_1 M \to \mathrm{SL}_2\mathbb{C}[R(M)]$

$$P(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} \in \mathrm{SL}_2\mathbb{C}[R(M)].$$

For any closed subset $D \subset R(M)$, the restriction of the tautological representation gives $P : \pi_1 M \to \mathrm{SL}_2\mathbb{C}[D]$.

6. Culler-Shalen's main construction

$X(M)$: the character variety

Take a curve $C \subset X(M)$.

We can take an affine curve $D \subset t^{-1}(C) \subset R(M)$ s.t. the restriction $t|_D : D \to C$ is not constant [proof of Prop 1.4.4, CS1]. We remark that $\mathbb{C}(D)/\mathbb{C}(C)$ is a finite extension.

An ideal point of C gives a valuation v on $\mathbb{C}(C)$, which gives a valuation w on the finite extension $\mathbb{C}(D)$.

This gives the tree associated with $\mathbb{C}(D)$ and the action

$$\pi_1(M) \xrightarrow{P} \mathrm{SL}_2\mathbb{C}[D] \subset \mathrm{SL}_2\mathbb{C}(D) \curvearrowright T.$$

(P : tautological rep.)

The Fundamental Theorem [Thm 2.2.1, CS1]

For an affine curve $C \subset X(M)$ and the ideal point p, the associated tree action is non-trivial.

Remark: A non-ideal point of $\widetilde{C}$ also gives a tree action, but it has a global fixed point, is not interesting.

6. Culler-Shalen's main construction

$$\begin{array}{ccccccccc} w & \leftrightarrow & q & \in & \widetilde{D} & \to & D & \subset & R(M) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ v & \leftrightarrow & p & \in & \widetilde{C} & \to & C & \subset & X(M) \end{array}$$

For $\gamma \in \pi_1(M)$, $v(\tau_\gamma) \geq 0$ if and only if γ fixes a vertex of T [Thm 2.2.1, CS1], [Prop 1.2.6, CGLS]. In this case, we can deform γ avoiding the associated surface S.

Thus if $\exists \alpha, \beta \in \pi_1(\partial M)$ s.t. $v(\tau_\alpha) \geq 0$ and $v(\tau_\beta) < 0$, α is a boundary slope.

Moreover, the translation length of $\gamma \in \pi_1(M)$ is given by

$$\min(0, -2w(\tau_\gamma))$$

[Prop II.3.15, MS1].

[MS1] Morgan-Shalen. "Valuations, trees, and degenerations of hyperbolic structures. I," Ann. of Math. (2) 120(1984), 401–476.

7. Cyclic Surgery Theorem

In §1 of [CGLS], the following theorem is proved as an application of the CS theory.

Theorem (CGLS, Thm 1.0.1)

Let M be a hyperbolic orientable 3-manifold with one torus boundary. Let α, β be slopes s.t. $\pi_1(M(\alpha))$, $\pi_1(M(\beta))$ are cyclic. If neither α nor β is strict boundary slope then $\Delta(\alpha, \beta) \leq 1$.

A slope ∂M is called a *boundary slope* if it is the boundary of some essential surface. A slope is called a *strict boundary slope* if it is the slope of some non-fiber essential surface.

Remark: The boundary slope detected by CS theory (more generally, detected by a tree action) is a strict boundary slope [CGLS, Prop 1.2.7].

The remaining part of the cyclic surgery theorem is proved in [§2, CGLS] by using different techniques.

7. Cyclic Surgery Theorem

Theorem (CGLS, Thm 1.0.1)

Let M be a hyperbolic orientable 3-manifold with one torus boundary. Let α, β be slopes s.t. $\pi_1(M(\alpha))$, $\pi_1(M(\beta))$ are cyclic. If neither r nor s is strict boundary slope then $\Delta(\alpha, \beta) \leq 1$.

A hyperbolic structure on M gives a discrete faithful representation $\rho_0 : \pi_1 M \to \mathrm{PSL}_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3)$, which is irreducible.

There exists a lift $\widetilde{\rho_0} : \pi_1 M \to \mathrm{SL}_2\mathbb{C}$ of ρ_0 [Prop 3.1.1, CS1]. (Since the obstruction to the lifting is living in $H^2(M; \mathbb{Z}/2\mathbb{Z})$, trivial for knot exteriors.)

7. Cyclic Surgery Theorem

Take an irreducible component $R_0 \subset R(M)$ containing $\widetilde{\rho}_0$ and $X_0 = t(R_0)$.

Proposition [Prop 1.1.1, CGLS]

$\dim X_0 = 1$. For any non-trivial $\gamma \in \pi_1(\partial M)$, τ_γ is non-constant. (Recall $\tau_\gamma([\rho]) = \operatorname{tr} \rho(\gamma)$ for $[\rho] \in X_0$.)

$\dim X_0 \geq 1$ is rather elementary [Prop 3.2.1, CS1], which is the same line of argument as Theorem 5.6 of Thurston's Lecture Notes.

$\dim X_0 \leq 1$ is shown in [Prop 2, CS2] using some local rigidity result. The second assertion also follows from [Prop 2, CS2].

[CS2] Culler-Shalen, "Bounded, separating, incompressible surfaces in knot manifolds," Invent., 75(1984), 537-545.

Remark: For the (p, q)-torus knot exterior, $\exists \gamma \in \pi_1(\partial M)$ s.t. τ_γ is constant on the component $X_0 \subset X(M)$ containing irreducible characters.

7. Cyclic Surgery Theorem

For $\alpha \in \pi_1(\partial M) \cong H_1(\partial M; \mathbb{Z}) \cong \mathbb{Z}^2$, we consider the trace function $\tau_\alpha \in \mathbb{C}[X_0]$. ($\tau_\alpha([\rho]) = \operatorname{tr} \rho(\gamma)$ for $[\rho] \in X_0$.) Define

$$f_\alpha := \tau_\alpha^2 - 4 \in \mathbb{C}[X_0].$$

If $f_\alpha(\rho) = 0$, then $\operatorname{tr} \rho(\alpha) = \pm 2$, thus $\rho(\alpha)$ is conjugate to

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The former case gives a representation $\pi_1(M(\alpha)) \to \mathrm{PSL}_2\mathbb{C}$. ($M(\alpha)$: Dehn filling along α)

If the image is "large", $\pi_1(M(\alpha))$ could not be cyclic.

So it is important to study zeros of f_α for $\alpha \in H_1(\partial M; \mathbb{Z})$.

We denote the order of zero of $f \in \mathbb{C}(X_0) = \mathbb{C}(\widetilde{X}_0)$ at $x \in \widetilde{X}_0$ by $Z_x(f)$. ($\widetilde{X}_0$: smooth projective model of X_0)

7. Cyclic Surgery Theorem

Theorem (CGLS, Thm 1.0.1)

Let α, β be slopes s.t. $\pi_1(M(\alpha))$, $\pi_1(M(\beta))$ are cyclic. If neither α nor β is strict boundary slope then $\Delta(\alpha, \beta) \leq 1$.

The proof divided into the following 2 propositions.

Proposition [Prop 1.1.2, CGLS]

There exists a norm $|| \cdot || : H_1(\partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ s.t.

- For $\alpha \in H_1(\partial M, \mathbb{Z})$, $||\alpha|| = \deg f_\alpha$.
- The unit ball is a finite-sided polygon whose vertices are rational multiple of strict boundary slopes.

Proposition [Prop 1.1.3, CGLS]

Let $\alpha \in H_1(\partial M; \mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic, then for any $x \in \widetilde{X}_0$, we have

$$Z_x(f_\alpha) \leq Z_x(f_\delta) \quad (\forall \delta \in H_1(\partial M; \mathbb{Z}), \delta \neq 0).$$

7. Cyclic Surgery Theorem

Proposition [Prop 1.1.2, CGLS]

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Proposition [Prop 1.1.3, CGLS]

Let $\alpha \in H_1(\partial M;\mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic, then for any $x \in \widetilde{X}_0$, we have

$$Z_x(f_\alpha) \leq Z_x(f_\delta) \quad (\forall \delta \in H_1(\partial M;\mathbb{Z}),\ \delta \neq 0).$$

Since $\sum_{x\in\widetilde{X}_0} Z_x(f) = \deg f$, we deduce the following.

Corollary [Cor 1.1.4, CGLS]

Let $\alpha \in H_1(\partial M;\mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic,

$$||\alpha|| \leq ||\delta|| \quad (\forall \delta \in H_1(\partial M;\mathbb{Z}),\ \delta \neq 0).$$

7. Cyclic Surgery Theorem

Corollary [Cor 1.1.4, CGLS]

Let $\alpha \in H_1(\partial M; \mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic,

$$||\alpha|| \leq ||\delta|| \quad (\forall \delta \in H_1(\partial M; \mathbb{Z}),\ \delta \neq 0).$$

Theorem (CGLS, Thm 1.0.1)

Let α, β be slopes s.t. $\pi_1(M(\alpha))$, $\pi_1(M(\beta))$ are cyclic. If neither α nor β is strict boundary slope then $\Delta(\alpha, \beta) \leq 1$.

Corollary to Theorem

Let $L := H_1(\partial M; \mathbb{Z}) \cong \mathbb{Z}^2$, $V := H_1(\partial M; \mathbb{R}) \cong \mathbb{R}^2$.

Let $m = \min_{0 \neq \delta \in L} ||\delta||$, B the ball of radius m in V w.r.t. $||\cdot||$.

B is a finite sided balanced ($B = -B$) convex polygon [Prop 1.1.2, CGLS], contains no integral point in the interior by the definition of m.

Since $\operatorname{Int} B$ is mapped to $V/2L$ injectively, Area $B \leq 4$.

7. Cyclic Surgery Theorem

Corollary [Cor 1.1.4, CGLS]

Let $\alpha \in H_1(\partial M; \mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic,

$$||\alpha|| \leq ||\delta|| \quad (\forall \delta \in H_1(\partial M; \mathbb{Z}),\ \delta \neq 0).$$

Theorem (CGLS, Thm 1.0.1)

Let α, β be slopes s.t. $\pi_1(M(\alpha))$, $\pi_1(M(\beta))$ are cyclic. If neither α nor β is strict boundary slope then $\Delta(\alpha, \beta) \leq 1$.

Corollary to Theorem (continued)

By Cor, if γ is not a strict boundary slope and $\pi_1(M(\gamma))$ is cyclic, γ is on the boundary of B. Let α, β as in Thm, and P the parallelogram spanned by 4 pts $\pm\alpha$, $\pm\beta$

$$\Delta(\alpha, \beta) = \frac{\text{Area } P}{2} \leq \frac{\text{Area } B}{2} \leq 2.$$

If $\Delta(\alpha, \beta) = 2$, then $P = B$, which implies that α and β are vertices of B, thus they are strict boundary slopes. $\square$

7. Cyclic Surgery Theorem

We have left to show

Proposition [Prop 1.1.2, CGLS]

There exists a norm $||\cdot|| : H_1(\partial M;\mathbb{R}) \to \mathbb{R}_{\geq 0}$ s.t.

- For $\alpha \in H_1(\partial M,\mathbb{Z})$, $||\alpha|| = \deg f_\alpha$.
- The unit ball is a finite-sided polygon whose vertices are rational multiple of strict boundary slopes.

Proposition [Prop 1.1.3, CGLS]

Let $\alpha \in H_1(\partial M;\mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic, then for any $x \in \widetilde{X}_0$, we have

$$Z_x(f_\alpha) \leq Z_x(f_\delta) \quad (\forall \delta \in H_1(\partial M;\mathbb{Z}),\ \delta \neq 0).$$

The norm is called the *Culler-Shalen norm*.

7. Cyclic Surgery Theorem

We denote the order of pole of $f \in \mathbb{C}(X_0) = \mathbb{C}(\widetilde{X}_0)$ at $x \in \widetilde{X}_0$ by $\Pi_x(f)$. We have

$$\deg f = \sum_{x \in \widetilde{X}_0} Z_x(f) = \sum_{x \in \widetilde{X}_0} \Pi_x(f).$$

Furthermore, if f is a regular function on X_0 ($f \in \mathbb{C}[X_0]$),

$$\deg f = \sum_{x \in \widetilde{X}_0} \Pi_x(f) = \sum_{x\text{: ideal}} \Pi_x(f).$$

Lemma [Lem 1.4.1, CGLS]

For each ideal point $x \in \widetilde{X}_0$, there exists a homomorphism $\phi_x : L \to \mathbb{Z}$ s.t.

$$\Pi_x(f_\alpha) = |\phi_x(\alpha)|.$$

We use

Theorem [Thm 1.2.3, CGLS], [Lem II.4.4, MS1]

A valuation v on $\mathbb{C}(X_0)^*$ is extended to a valuation w on $\mathbb{C}(R_0)^*$ s.t. $w|_{\mathbb{C}(X_0)^*} = d \cdot v$ for some $d \in \mathbb{N}$.

7. Cyclic Surgery Theorem

For each ideal point $x \in \widetilde{X}_0$, there exists a homomorphism $\phi_x : L \to \mathbb{Z}$ s.t.

$$\Pi_x(f_\alpha) = |\phi_x(\alpha)|.$$

Proof. Fix a basis $\alpha_1, \alpha_2 \in L$. If $\rho(\alpha_i) \sim \begin{pmatrix} \lambda_i & * \\ 0 & \lambda_i^{-1} \end{pmatrix}$,

$$f_{\alpha_i} = (\lambda_i + \lambda_i^{-1})^2 - 4 = (\lambda_i - \lambda_i^{-1})^2.$$

In general, for $f \neq 0, \pm 1$, we have

$-\min(0, v(f - f^{-1})) = |v(f)|$. (E.g. if $v(f) > 0$, $v(f^{-1}) = -v(f) < 0$, thus $v(f - f^{-1}) = -v(f) = -|v(f)|$.)

Thus, for $\alpha = \alpha_1^p \alpha_2^q \in L$,

$$\begin{aligned}
\Pi_x(f_{\alpha_1^p \alpha_2^q}) &= \Pi_x(\lambda_1^p \lambda_2^q - \lambda_1^{-p} \lambda_2^{-q})^2 \\
&= -\min(0, v((\lambda_1^p \lambda_2^q - \lambda_1^{-p} \lambda_2^{-q})^2)) \\
&= -\frac{2}{d} \min(0, w(\lambda_1^p \lambda_2^q - \lambda_1^{-p} \lambda_2^{-q})) \\
&= \frac{2}{d} |w(\lambda_1^p \lambda_2^q)| = \frac{2}{d} |p\, w(\lambda_1) + q\, w(\lambda_2)|.
\end{aligned}$$

So set $\phi_x(\alpha) = \frac{2}{d} |p\, w(\lambda_1) + q\, w(\lambda_2)|$. □

7. Cyclic Surgery Theorem

Lemma [Lem 1.4.1, CGLS]

For each ideal point $x \in \widetilde{X}_0$, there exists a homomorphism $\phi_x : L \to \mathbb{Z}$ s.t.

$$\Pi_x(f_\alpha) = |\phi_x(\alpha)|.$$

Proposition [Prop 1.1.2, CGLS]

There exists a norm $|| \cdot || : H_1(\partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$ s.t.

- For $\alpha \in H_1(\partial M, \mathbb{Z})$, $||\alpha|| = \deg f_\alpha$.
- The unit ball is a finite-sided polygon whose vertices are rational multiple of strict boundary slopes.

Sketch. Define

$$||\alpha|| = \sum_{x:\text{ ideal}} |\phi_x(\alpha)| \quad (\alpha \in V).$$

It is easy to see $|| \cdot ||$ is a semi-norm. Since f_α is non-const. for $0 \neq \alpha \in L$, this is a norm. The first assertion follows from Lem. Since a vertex of the unit ball is on the line $\phi_x = 0$ for some x, thus the 2nd assertion follows. □

7. Cyclic Surgery Theorem

We have left to show

Proposition [Prop 1.1.3, CGLS]

Let $\alpha \in H_1(\partial M; \mathbb{Z})$ be a primitive, not a strict boundary slope. If $\pi_1(M(\alpha))$ is cyclic, then for any $x \in \widetilde{X}_0$, we have

$$Z_x(f_\alpha) \leq Z_x(f_\delta) \quad (\forall \delta \in H_1(\partial M; \mathbb{Z}),\ \delta \neq 0).$$

This is further divided into two cases:

- x is non-ideal [§1.5, p.254∼260, CGLS], and
- x is ideal [§1.6, p.260∼264, CGLS].

(By the way, §1.1∼1.4 (p.242∼254).)

We show

$$0 \neq \exists \delta \in L,\ Z_x(f_\alpha) > Z_x(f_\delta) \Longrightarrow \pi_1(M(\alpha)) \text{ is not cyclic.}$$

7. Cyclic Surgery Theorem

$0 \neq \exists \delta \in L,\ Z_x(f_\alpha) > Z_x(f_\delta) \Longrightarrow \pi_1(M(\alpha))$ is not cyclic.

- If x is non-ideal, find $\rho \in R_0$ s.t.
  (i) $t(\rho) = x$,
  (ii) $\rho(\pi_1(M))$ is non-cyclic in $\mathrm{PSL}_2\mathbb{C}$,
  (iii) $\rho(\alpha) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

  (Recall $t : R(M) \to X(M)$.) So $\pi_1(M(\alpha))$ is non-cyclic.
- If x is ideal, let S be the associated essential surface.
  Since $Z_x(f_\alpha) > 0$, τ_α is finite at x. Thus α is boundary slope of S, or S is closed.
  But we assume that α is not a strict boundary slope, S is closed.
  We show that S is incompressible in $M(\alpha)$. (Technical part of §1.6.) In particular, $\pi_1(M(\alpha))\,(\supset \pi_1(S))$ is non-cyclic.

7. Cyclic Surgery Theorem

Technical points in §1.5 of CGLS

We have to take the normalizations X_0^ν, R_0^ν of X_0, R_0 (taking integral closure of the coordinate rings) to avoid singularities.

$$\begin{array}{ccc} R_0^\nu & \xrightarrow{\nu} & R_0 \\ {\scriptstyle t^\nu}\downarrow & & \downarrow{\scriptstyle t} \\ X_0^\nu & \xrightarrow{\nu} & X_0 \end{array}$$

We ignore these technical details.

Proposition [Prop 1.5.2, CGLS]

For $x \in X_0^\nu$ (non-ideal point), assume that $0 \neq \exists\delta \in L$ s.t. $Z_x(f_\alpha) > Z_x(f_\delta)$. Then $\exists\rho \in R_0$ s.t.

- (i) $t(\rho) = \nu(x)$,
- (ii) $\rho(\pi_1(M))$ is non-cyclic in $\mathrm{PSL}_2\mathbb{C}$,
- (iii) $\rho(\alpha) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

7. Cyclic Surgery Theorem

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(iii) $\rho(\alpha) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{array}{ccc} R_0^\nu & \xrightarrow{\nu} & R_0 \\ {\scriptstyle t^\nu}\downarrow & & \downarrow{\scriptstyle t} \\ X_0^\nu & \xrightarrow{\nu} & X_0 \end{array}$$

It is shown that

(i) t^ν is surjective [Prop 1.5.6, CGLS].

(ii) $\exists$ dense $U \subset (t^\nu)^{-1}(x)$ s.t. $\rho \in \nu(U)$ has non-cyclic image [Prop 1.5.5, CGLS].

(iii) For $\widetilde{\rho} \in (t^\nu)^{-1}(x)$, $\nu(\widetilde{\rho})(\alpha) = \pm 1$ [Prop 1.5.4, CGLS].

(i) is technical. We give sketches of (ii) and (iii).

7. Cyclic Surgery Theorem

(ii) Since $\dim X_0^\nu = 1$, $\dim R_0^\nu = \dim X_0^\nu + 3 = 4$. Thus each component of $(t^\nu)^{-1}(x)$ has dimension at least 3.

On the other hand, let

$$Z = \{\rho \in t^{-1}(\nu(x)) \mid \rho(\pi_1(M)) \text{ is cyclic in } \mathrm{PSL}_2\mathbb{C}\} \subset R_0,$$

and $\mathcal{N} = \{\ker \rho \mid \rho \in Z\}$. Since the set of finite index subgroups of $\pi_1(M)$ is countable, $\mathcal{N}$ is countable. For each $N \in \mathcal{N}$, let $Y_N := \{\rho \in t^{-1}(x) \mid \rho(N) = \{1\}\}$. We have $Z \subset \bigcup_{N \in \mathcal{N}} Y_N$, and $\dim Y_N \leq 2$ since $\rho \in Y_N$ is (almost) determined by the image of the cyclic generator.

Set $U = \underbrace{(t^\nu)^{-1}(x)}_{\dim \geq 3} - \underbrace{\nu^{-1}(\bigcup_{N \in \mathcal{N}} Y_N)}_{\dim \leq 2}$.

(iii) Since $Z_x(f_\alpha) > Z_x(f_\delta) \geq 0$, $0 = f_\alpha(x) = \mathrm{tr}\,\rho(\alpha)^2 - 4$, so $\mathrm{tr}\,\rho(\alpha) = \pm 2$. Thus $\rho(\alpha) \sim \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
Using the assumption, we can show that the former holds.

7. Cyclic Surgery Theorem

Proposition [Prop 1.5.4, CGLS]

Let $0 \neq \alpha, \delta \in H_1(\partial M; \mathbb{Z})$ and $x \in X_0^\nu$.
Assume $Z_x(f_\alpha) > Z_x(f_\delta) \geq 0$ (thus $\operatorname{tr} \rho(\alpha) = \pm 2$).

$$\widetilde{\rho} \in (t^\nu)^{-1}(x) \Longrightarrow \nu(\widetilde{\rho})(\alpha) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We use the following lemma.

Lemma [Lem 1.5.7, CGLS]

K : a filed, $v : K^* \to \mathbb{Z}$: a discrete valuation
$\mathcal{O} = \{f \in K \mid v(f) \geq 0\}$: the DVR.
$\mathcal{M} = \{f \in K \mid v(f) > 0\}$: its maximal ideal
For $A, B \in \mathrm{SL}_2(\mathcal{O})$ s.t. $[A, B] = 0$,

$$v((\operatorname{tr} A)^2 - 4) > v((\operatorname{tr} B)^2 - 4) \Longrightarrow A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \mathcal{M}.$$

7. Cyclic Surgery Theorem

Lemma [Lem 1.5.7, CGLS]

For $A, B \in \mathrm{SL}_2(\mathcal{O})$ s.t. $[A, B] = 0$,
$$v((\operatorname{tr} A)^2 - 4) > v((\operatorname{tr} B)^2 - 4) \Longrightarrow A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \mathcal{M}.$$

Sketch.

(After taking a quadratic field extension) A and B are simultaneously upper triangulable:

$$A = \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix},\ B = \begin{pmatrix} b & y \\ 0 & b^{-1} \end{pmatrix}\ (x, y, a^{\pm 1}, b^{\pm 1} \in \mathcal{O}).$$

Since $(\operatorname{tr} A)^2 - 4 = (a + a^{-1})^2 - 4 = (a - a^{-1})^2$,
$v((\operatorname{tr} A)^2 - 4) = 2 \cdot v(a - a^{-1})$.
Thus $v((\operatorname{tr} A)^2 - 4) > v((\operatorname{tr} B)^2 - 4)$ implies $v(a - a^{-1}) > v(b - b^{-1})$. Thus $v(a - a^{-1}) > 0$, which implies $a \equiv \pm 1 \mod \mathcal{M}$.

Since A and B commute, $(b - b^{-1})x = (a - a^{-1})y$, thus
$$v(x) \geq v(x) - v(y) = v(a - a^{-1}) - v(b - b^{-1}) > 0.$$
$u(x) > 0$ means $x \in \mathcal{M}$, i.e. $x \equiv 0 \mod \mathcal{M}$. □

7. Cyclic Surgery Theorem

Proposition [Prop 1.5.4, CGLS]

Let $0 \neq \alpha, \delta \in H_1(\partial M; \mathbb{Z})$ and $x \in X_0^\nu$.
Assume $Z_x(f_\alpha) > Z_x(f_\delta) \geq 0$ (thus $\operatorname{tr} \rho(\alpha) = \pm 2$).

$$\widetilde{\rho} \in (t^\nu)^{-1}(x) \Longrightarrow \nu(\widetilde{\rho})(\alpha) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Sketch. For each component $Q \subset (t^\nu)^{-1}(x) \subset R_0^\nu$, since $Q \subset R_0^\nu$ is a codimension 1 subvariety, Q determines a discrete valuation w on $F = \mathbb{C}(R_0^\nu) = \mathbb{C}(R_0)$. Let v be the valuation corresponding to $x = t^\nu(Q) \in X_0^\nu$. Then $\exists d \in \mathbb{N}$ s.t. $w|_{\mathbb{C}(X_0)^*} = d \cdot v$. Since $x \in X_0^\nu$ is non-ideal, we have

$$Z_x(f_\alpha) = v(f_\alpha) = \frac{1}{d} w((\operatorname{tr} P(\alpha))^2 - 4)$$

where $P : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C}(R_0))$ is the tautological rep. Likewise for δ. Thus the assumption implies $w((\operatorname{tr} P(\alpha))^2 - 4) > w((\operatorname{tr} P(\delta))^2 - 4)$, thus by Lem 1.5.7, $P(\alpha) = \pm I \mod \mathcal{M}_w$ This means that, for $\rho \in \nu(Q)$, $\rho(\alpha) = \pm I$. □

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