The index of elliptic operators: I*

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Introduction

This is the first of a series of papers which will be devoted to a study of the index of elliptic operators on compact manifolds. The main result was announced in [6] and, for manifolds with boundary[1], in [5]. The long delay between these announcements and the present paper is due to several factors. On the one hand, a fairly detailed exposition has already appeared in [14]. On the other hand, our original proof, reproduced with minor modifications in [14], had a number of drawbacks. In the first place the use of cobordism, and the computational checking associated with this, were not very enlightening. More seriously, however, the method of proof did not lend itself to certain natural generalizations of the problem where appropriate cobordism groups were not known. The reader who is familiar with the Riemann-Roch theorem will realize that our original proof of the index theorem was modelled closely on Hirzebruch's proof of the Riemann-Roch theorem. Naturally enough we were led to look for a proof modelled more on that of Grothendieck. While we have not completely succeeded in this aim, we have at least found a proof which is much more natural, does not use cobordism, and lends itself therefore to generalization. In spirit, at least, it has much in common with Grothendieck's approach.

A further essential feature of our new approach is the elimination of cohomology from the picture, at least in the first instance. The algebraic topology that is really relevant to the study of elliptic operators is K-theory.[2] This is not surprising since K-theory may be roughly described as "the algebraic topology of linear algebra". Thus, in the present paper, no homology or cohomology is used, either explicitly or implicitly. Our main theorem, giving a formula for the index of an elliptic operator, is expressed purely in K-theoretical terms. This is especially significant for generalizations which will be dealt with in subsequent papers. The cohomological formula given in [6] can be obtained quite simply from the K-theory formula given here. This,

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[1] jointly with R. Bott.

[2] See for example [1] for an elementary treatment. See also [2] for a non-technical exposition of the relation between K-theory and elliptic operators in euclidean space.

of course, is simply an exercise in algebraic topology, translating from one set of topological invariants to another, and it will be dealt with in paper III of this series. Which formula actually provides the "best answer" is partly a matter of taste. It depends on which invariants one is more familiar with, or can most easily compute.

One generalization of the index problem plays a minor role in our proof and is therefore included in this first paper. This concerns an elliptic operator which is invariant under a compact Lie group G. The index in this case is a character of G. The appropriate algebraic topology here is K_G-theory[3] (or equivariant K-theory), and our formula for the index is expressed in those terms. In a subsequent paper, II of this series,[4] we shall show how this formula can be expressed in terms of the fixed-point sets of elements of G. The character formula obtained in this way is closely related to the generalized Lefschetz fixed-point formula of [4], and is a special case of it when the fixed-points are all isolated. As a particularly interesting special case of this character formula (for general fixed-point sets) one can derive the formula of Langlands [12] for the dimension of spaces of automorphic forms. This will be dealt with in a separate paper.[5]

Besides the topics referred to above, subsequent papers of this series will deal with

(1) Families of elliptic operators, parametrized by a space X. In this case one can define an index in $K(X)$.

(2) Real elliptic operators. For example, if D is a real skew-adjoint elliptic operator, dim Ker D (mod 2) is a deformation invariant. This can be interpreted as an index, and a general formula will be obtained which includes this as a special case.

(3) Manifolds with boundary.

(4) Operators which are "elliptic relative to a group action"; i.e., their symbols are elliptic in the directions transversal to the orbits.

In order to keep this paper to a reasonable and readable size, we shall not attempt to develop *ab initio* all the relevant topology and analysis. Instead we shall summarize the relevant basic material. Thus § 2 contains a review of K- and K_G-theory, with special emphasis on those parts of the theory which will be needed. In § 5 we review pseudo-differential operators, and in § 6 we review the basic analytical properties of the index of elliptic operators. All of this is essentially standard material.

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[3] See [16].

[4] by Atiyah and Segal.

[5] See also [7].

The main ideas of the proof are explained in a non-technical manner in § 1. In § 3 we define the *topological index* using the basic facts about K-theory explained in § 2. Then in § 4 we set up some axioms for "index functions" and prove a uniqueness theorem (4.6), so that any index function satisfying these axioms must coincide with the topological index of § 3. The elementary properties of the analytical index given in § 6 show that this is an index function. At this stage we can formulate our main theorem (6.7) which simply asserts that the analytical index is equal to the topological index. In view of the uniqueness theorem for index functions established in § 4 it is enough to show that the analytical index satisfies the axioms. This is done in § 8 and § 9. In § 7 there is a digression dealing with elliptic complexes, and we show how (6.7) implies a formula (7.1) for the Euler characteristic of an elliptic complex.

Many of the analytical ideas and devices which we use here are due to R. Seeley. He first introduced the "excision" property, and also carried through for us the technical treatment of operators on product manifolds. The final presentation of the analysis in §§ 5–9 owes a great deal to the help of L. Hörmander with whom we have had many illuminating discussions. Finally we are glad to acknowledge our indebtedness to R. Palais whose *Annals of Mathematics Study* [14] gave such a thorough presentation of our first proof, and has been of considerable assistance to us in preparing the present paper.

1. Idea of proof

In this section, we shall try to give some intuitive ideas of the nature of the proof. In our more formal treatment in the remaining sections, we shall present things in a slightly different manner for technical reasons.

Let X be a submanifold of a manifold Y, both being compact. Denote by $i: X \to Y$ the inclusion. Let A be an elliptic operator on X. Then the main step in our proof is the construction (on the symbolic level) of an elliptic operator $i_!(A)$ on Y such that

$$\text{index } A = \text{index } i_!(A) \,. \tag{1.1}$$

Once this has been done, we can take Y to be a sphere, and the general index problem is reduced to the case of operators on the sphere. For these the problem is easily solved. In fact we can go one step further. Thus, if $i: X \to S$ is an embedding, and if $j: P \to S$ denotes the inclusion of a point, we can show that there is an elliptic operator B on the point P so that $j_!(B) = i_!(A)$ (up to a suitable equivalence preserving the index). Thus

$$\text{index } A = \text{index } i_!(A) = \text{index } j_!(B) = \text{index } B \ ,$$

and the problem is reduced to a point, where of course it is trivial!

The construction of $i_!(A)$ and the verification of (1.1) is thus the heart of the problem. Now in a tubular neighborhood U of X in Y, we may consider the "transversal de Rham complex" C; i.e., in each normal plane of X we consider the de Rham complex of that plane. The "tensor product" $A \otimes C$ can then be defined as an elliptic complex over U, at least on the symbolic level. It turns out that there is a natural trivialization of the symbol of C on the boundary of U, and this enables us to extend the symbol of $A \otimes C$ to the whole of Y. This is the definition of $i_!(A)$.

To compute the index of $i_!(A)$, we use an excision property,[6] observed by Seeley [15], which shows that

$$\text{index } i_!(A) = \text{index } k_!(A) \ ,$$

where $k\colon X \to Z$ is the inclusion of X in "the double" of U. Thus Z is fibered over X by spheres. Moreover the symbol of $k_!(A)$ is the tensor product of the symbol of A on X and a symbol "elliptic along the fibres" which may be roughly described as "one-half" of the de Rham complex along the fibres. Now if Z were a product bundle $X \times S$, we could use the multiplicative property of the index [14; Ch. XIV] and obtain

$$\text{index } k_!(A) = \text{index } A \cdot \text{index } j_!(1) \ , \tag{1.2}$$

where $j\colon P \to S$ is the inclusion of a point, and 1 is an operator of index 1 on P. This would reduce us to proving (1.1) with $X = P$, $Y = S$, which is fairly simple by direct computation (and can moreover be reduced, by the multiplicative property, to the cases $\dim S = 1, 2$). In the general case when Z is not a product bundle, we have to generalize the multiplicative property of the index to deal with fibre bundles.[7] One does not always obtain a formula as simple as (1.2), but one does obtain this formula when the "operator along the fibres" is rigid in a certain sense. Fortunately the de Rham complex is rigid in this sense, and so (1.2) holds and the proof is then complete. This rigidity of the de Rham complex comes (in one point of view) from the fact that its cohomology groups are the cohomology groups of the underlying space and so have an *integral basis*. The space of solutions of a general elliptic operator does not normally have such a discrete structure, and it can therefore "vary" in a continuous fashion.

It is perhaps worth making a remark about the construction of the

[6] A very similar idea was suggested to us independently by P. J. Cohen.

[7] It is here that it becomes helpful to introduce groups into the index problem.

operator $i_!(A)$. The trivialization of the de Rham complex used in extending the symbol of $A \otimes C$ from U to the whole of Y corresponds to the existence of a natural boundary-value problem.[8] The use of such a boundary-value problem in U may be more conceptual but, on the other hand, boundary questions involve more delicate considerations both analytical and topological. For this reason, we have kept away from boundary-value problems for the present.

Finally, it may be worth pointing out that the proof given in this paper could be technically simplified in a number of ways if one only wants the index theorem of [6], particularly if one is content to compute the index of differential operators only. However, since our aim is to present a proof which will generalize in various directions, we have not attempted to make these simplifications. The ideas in any case are the same.

2. Review of K-theory

A general reference for the results summarized here is [1]. The basic periodicity theorem is also in [3], while more detailed facts about K_G can be found in [16]. It should be emphasized that the only[9] non-trivial result which is involved in all this is the periodicity theorem.[10]

Let X be a compact space. The isomorphism classes of complex vector bundles over X form an abelian semi-group under $\oplus$, and the associated abelian group is denoted by $K(X)$. If E is a vector bundle over X, the corresponding element of $K(X)$ is denoted by $[E]$. $K(X)$ is a commutative ring under $\otimes$. A continuous map $f\colon X \to Y$ induces a natural ring homomorphism $f^*\colon K(Y) \to K(X)$ which depends only on the homotopy class of f. $K(\text{Point})$ is naturally isomorphic to the ring $\mathbf{Z}$ of integers. If X is a space with a given base point P, then $\tilde{K}(X)$ is defined as the kernel of the homomorphism

$$K(X) \longrightarrow K(P)$$

induced by the inclusion $P \to X$. We have, moreover, a natural decomposition

$$K(X) \cong \tilde{K}(X) \oplus K(P) \cong \tilde{K}(X) \oplus \mathbf{Z}\,.$$

If Y is a closed subspace of X, we denote by X/Y the space with base point obtained by collapsing Y to a point (if $Y = \varnothing$ we take $X/\varnothing = X + P$ disjoint sum). We define

[8] See [5] for the relation of boundary conditions to the trivialization of an elliptic symbol.

[9] In K_G-theory we also need the basic (Peter-Weyl) theorem about representations of G.

[10] Further comments on the periodicity theorem and the proofs in [1] and [3] will be made later in this section.

$$K(X, Y) = \tilde{K}(X/Y) .$$

If X is a locally compact space we define $K(X) = \tilde{K}(X^+)$ where X^+ is the one-point compactification of X: it will be a contravariant functor of X for *proper* maps, because only these extend to X^+. This is "K-theory with compact supports" and there is an alternative way to describe it which is very convenient. We take, as basic objects, complexes of vector bundles over X, that is sequences

$$0 \longrightarrow E^0 \xrightarrow{\alpha} E^1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} E^n \longrightarrow 0$$

of vector bundles E^i and homomorphisms α with $\alpha^2 = 0$. The *support* of such a complex is the set of points $x \in X$ for which

$$0 \longrightarrow E_x^0 \longrightarrow E_x^1 \longrightarrow \cdots \longrightarrow E_x^n \longrightarrow 0$$

is *not* exact. We consider only *complexes with compact support*. Two such complexes E, F are said to be homotopic if there is a complex G over $X \times I$ (with I the unit interval) such that $G \mid X \times \{0\} \cong E$ and $G \mid X \times \{1\} \cong F$. The homotopy classes of complexes over X, we denote by $C(X)$. This is an abelian semi-group under $\oplus$, and it contains a subsemigroup $C_\varphi(X)$ represented by complexes with empty support. As a second definition of $K(X)$ we can then take the quotient $C(X)/C_\varphi(X)$; although apparently only a semi-group, it turns out to be a group. The proof that our two definitions of $K(X)$ agree can be found in[11] [16]; in fact, we shall only use non-compact spaces in a very mild way (e.g. the total space of a vector bundle over a compact base) and the reader should have little difficulty in eliminating them entirely from the presentation. The formalism is, however, much simpler and more conceptual in the locally compact theory.

If E is a complex of vector bundles with compact support as above, we denote by $[E]$ its class in $K(X)$. If X is actually compact, then we have

$$[E] = \sum (-1)^i [E^i] . \tag{2.1}$$

Two complexes E and F which involve the same bundles, and whose homomorphisms α_E and α_F coincide outside a compact set, are homotopic, and so $[E] = [F]$. In fact we can define the class $[E]$ of E even if the homomorphisms α are not defined on some compact set L; we simply multiply by $1 - \varphi$ where φ is a continuous function with compact support and vanishing in a neighborhood of L.

Note that $K(X)$ is a ring without identity unless X is compact. If U is

[11] For X, Y compact the analogous definition of $K(X, Y)$ by complexes on X with support in $X - Y$ is treated in [1]. The version described here follows from this and the continuity property (2.2): we consider the groups $K(\bar{U}, \bar{U} - U)$.

an open set of the locally compact space X, we have a natural map

$$X^+ \longrightarrow X^+/(X^+ - U) = U^+$$

and so a natural homomorphism

$$i_*: K(U) \longrightarrow K(X) .$$

Thus if $\{U_\alpha\}$ denotes the directed family of all open relatively compact subsets of X, the groups $K(U_\alpha)$ form a direct system of groups (with the homomorphisms $i_*^{\alpha\beta}: K(U_\alpha) \to K(U_\beta)$ if $U_\alpha \subset U_\beta$). Then K has the following continuity property

$$K(X) = \operatorname{dir\,lim} K(U_\alpha) . \tag{2.2}$$

The proof of this is quite elementary (cf. [16]).

If we work only with complexes of fixed length $n \geqq 1$, then it is still true that $K(X)$ can be defined as $C^n(X)/C^n_\varphi(X)$. The advantage of using complexes of arbitrary finite length lies in the multiplicative structure. Thus if E is a complex on X, F a complex on Y, then $E \boxtimes F$ (the external tensor product) is a complex on $X \times Y$, and it has compact support if E and F have compact support. This induces a product

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y) .$$

As a simple example of multiplication of complexes, let $0 \longrightarrow E^0 \xrightarrow{\alpha} E^1 \longrightarrow 0$ and $0 \longrightarrow F^0 \xrightarrow{\beta} F^1 \longrightarrow 0$ be complexes with compact support on spaces X, Y respectively. Their external tensor product is the complex

$$0 \longrightarrow E^0 \boxtimes F^0 \xrightarrow{\varphi} E^1 \boxtimes F^0 \oplus E^0 \boxtimes F^1 \xrightarrow{\psi} E^1 \boxtimes F^1 \longrightarrow 0 ,$$

where $\varphi = \alpha \boxtimes 1 + 1 \boxtimes \beta$, $\psi = -1 \boxtimes \beta + \alpha \boxtimes 1$. Now, using metrics in the bundles, we can exhibit a complex of length one representing the same class in $K(X \times Y)$ as the complex $E \boxtimes F$, namely the complex

$$0 \longrightarrow E^0 \boxtimes F^0 \oplus E^1 \boxtimes F^1 \xrightarrow{\theta} E^1 \boxtimes F^0 \oplus E^0 \boxtimes F^1 \longrightarrow 0 ,$$

where

$$\theta = \begin{pmatrix} \alpha \boxtimes 1, & -1 \boxtimes \beta^* \\ 1 \boxtimes \beta, & \alpha^* \boxtimes 1 \end{pmatrix},$$

and α^*, β^* denote the adjoints of α, β with respect to the metrics (for the proof that this is equivalent to $E \boxtimes F$ see [1; 2.6.10]). This particular representative for the product of two complexes will be needed later in § 9.

For the application to elliptic operators, we shall be particularly concerned with the groups $K(V)$ where V is a *real* vector bundle over a space[12] X.

[12] In the applications X will be a smooth manifold and $V = TX$ will be its cotangent bundle.

If W is another such bundle then composing the product

$$K(V) \otimes K(W) \longrightarrow K(V \times W)$$

with the map $K(V \times W) \to K(V \oplus W)$, induced by the diagonal inclusion $X \to X \times X$, we obtain a product

$$K(V) \otimes K(W) \longrightarrow K(V \oplus W) . \tag{2.3}$$

In particular, taking $W = X$, we see that $K(V)$ is a $K(X)$-module.

Let E^i, E^{i+1} be complex vector bundles over X, and let π^*E^i denote the induced bundle over V; the fibre $(\pi^*E^i)_v$ may then be identified with $E^i_{\pi(v)}$. A homomorphism $\alpha: \pi^*E^i \to \pi^*E^{i+1}$ is called (positively) homogeneous of degree m if

$$\alpha_{\lambda v} = \lambda^m \alpha_v \in \mathrm{Hom}\,(E^i_{\pi(v)}, E^{i+1}_{\pi(v)}) ,$$

for all $v \in V$ and all real positive scalars λ. Note that, if we fix a metric in V, and let $S(V)$ be the unit sphere bundle of V, then such a homogeneous α is determined by its restriction to $S(V)$. If all the homomorphisms α of the complex E over V are homogeneous of degree m, we say that E is homogeneous of degree m. If X is *compact* and if, outside the zero-section of V, E is exact, then it has compact support and so defines an element of $K(V)$. It is easy to show that $K(V)$ can in fact be defined using only homogeneous complexes (of fixed degree). More precisely let ${}^mC(V)$ denote the semigroup of (compactly supported) homogeneous complexes of degree m modulo homogenous (compactly supported) homotopies, and let ${}^mC_\varphi(V) \subset {}^mC(V)$ denote those elements represented by complexes whose restriction to the unit sphere $S(V)$ is induced by a complex on X, so that

$$\alpha_v = \| v \|^m \beta_{\pi(v)}$$

where

$$\cdots \longrightarrow E^i \xrightarrow{\beta} E^{i+1} \longrightarrow \cdots$$

is exact on X. Then

$${}^mC(V)/{}^mC_\varphi(V) \cong C(V)/C_\varphi(V) \cong K(V) .$$

This is proved as follows. Given a complex E on V with compact support L, choose a ball bundle $B_\rho(V)$ of radius ρ containing L. The class $[E]$ depends only on the restriction of E to $B_\rho(V)$. Since X is a deformation retract of $B_\rho(V)$, it follows that, over $B_\rho(V)$, $E^i \overset{\gamma_i}{\cong} \pi^*F^i$, where F^i is the restriction of E^i to the zero section. Moreover this isomorphism is unique up to homotopy if we choose γ_i to be the identity on the zero-section. Putting

$$\alpha'_i = \gamma_{i+1} \alpha_i \gamma_i^{-1}$$

on the boundary $S_\rho(V)$ of $B_\rho(V)$, and extending it to V as a homogeneous function of degree m, we obtain a homogeneous complex E'. Associating E' to E then defines a homomorphism $C(V) \to {}^mC(V)$, and this induces the required isomorphism $C(V)/C_\varphi(V) \to C^m(V)/{}^mC_\varphi(V)$.

Again we can restrict ourselves to complexes of length one so that we can write

$$K(V) = {}^mC^1(V)/{}^mC^1_\varphi(V) .$$

The degree m above plays, of course, a quite harmless role. Multiplication by $\|v\|^{s-m}$ will map ${}^mC(V) \to {}^sC(V)$ isomorphically. Thus it would be natural to consider only the case $m = 0$. However this is inconvenient for the explicit description of products. If E, F are compactly supported homogeneous complexes of degree m over vector bundles V, W respectively, then $E \boxtimes F$ over $V \times W$ is again compactly supported (i.e., exact outside the zero-section) provided $m > 0$; if $m \leqq 0$, there will be discontinuities in the homomorphisms of $V \boxtimes W$ at points $(v, 0)$ and points $(0, w)$. This fact will be of considerable analytical significance in § 9.

On the other hand, we cannot settle for $r > 0$ because in one situation we definitely must take $r = 0$. This occurs when X (the base of V) is non-compact. A homogeneous complex over V, as above, cannot now have compact support unless $r = 0$; when $r = 0$, it has compact support provided it is exact outside the zero-section of V and *in addition*, outside some compact set in X, the homomorphisms α are constant on the fibres of V. Actually, for non-compact X, we shall only explicitly need the following fact. Every element $a \in K(V)$ can be represented by a compactly supported homogeneous complex of degree zero

$$0 \longrightarrow \pi^*(E^0) \xrightarrow{\alpha} \pi^*(E^1) \longrightarrow 0 ,$$

with E^0 and E^1 trivial outside a compact set in X. This is proved much as before. Since $K(V) = \tilde{K}(V^+)$, we can represent a by a complex

$$0 \longrightarrow F^0 \xrightarrow{\varphi} F^1 \longrightarrow 0 \tag{2.4}$$

in which, outside some compact $L \subset V$, we have isomorphisms

$$\beta_i : F^i \mid V - L \longrightarrow (V - L) \times \mathbf{C}^n ,$$

and, over $V - L$, $\varphi = \beta_1^{-1}\beta_0$. Let Y be open relatively compact in X containing $\pi(L)$, and let ρ be chosen so that the compact set L lies in the ball bundle $B_\rho(V) \mid \bar{Y}$. Then a is already determined by the restriction of the complex (2.4) to $B_\rho(V) \mid \bar{Y}$.

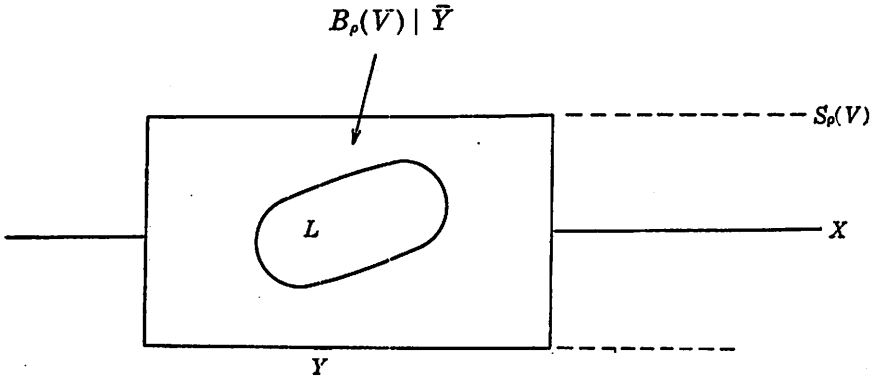


FIG. 1

Now on this compact set, since $B_\rho(V) \mid \bar{Y}$ retracts onto $\bar{Y}$, we can find isomorphisms θ_i: $F^i \to \pi^* E^i$ (where E^i is the restriction of F^i to the zero-section of V). Moreover we can assume the θ_i extend the isomorphisms over $\pi^{-1}(\bar{Y} - Y)$ induced by the β_i; more precisely this means that, if $x \in \bar{Y} - Y$ and $\pi(v) = x$, then the homomorphism $\theta_i(v)$: $F^i_v \to (\pi^* E^i)_v = F^i_x$ coincides with the composition $\beta_i(x)^{-1} \circ \beta_i(v)$. Now put $\alpha = \theta_1^{-1} \varphi \theta_0$ on

$$\partial(B_\rho(V) \mid \bar{Y}) = (S_\rho(V) \mid \bar{Y}) \cup (B_\rho(V) \mid (\bar{Y} - Y)) ,$$

extend it to all of $V \mid \bar{Y}$ so as to be homogeneous of degree zero and extend it outside $\bar{Y}$ in the obvious way (using the fact that the E^i are trivial there, and that α is isomorphic to the identity). We now have a representative for $a \in K(V)$ of the required type.

Remark. If the base X is a smooth manifold, then the isomorphism classes of continuous and smooth vector bundles over X coincide (by standard approximation arguments). Similarly, if V is also smooth, we get the same group $K(V)$ if we take only smooth bundles over V in the above definitions by complexes.

Suppose now that V is a complex vector bundle over X, and assume first that X is compact. Then the exterior algebra $\Lambda^*(V)$ defines in a natural way a complex of vector bundles over V, exact outside the zero-section (and homogeneous of degree 1). We shall call this the exterior complex of V, and denote it by $\Lambda(V)$. The class of $\Lambda(V)$ gives an element of $K(V)$ which we denote by λ_V. Since $K(V)$ is a $K(X)$-module, multiplication by λ_V induces a homomorphism

$$\varphi: K(X) \longrightarrow K(V) .$$

We call this the *Thom homomorphism*. Note that if i: $X \to V$ denotes the zero-section, then by (2.1) we obtain

$$i^* \varphi(x) = \{\sum (-1)^i \lambda^i(V)\} x \qquad \text{for } x \in K(X) . \tag{2.5}$$

For locally compact X, the exterior complex $\Lambda(V)$ does not have compact support. However for any complex E on X with compact support, the product $\Lambda(V) \otimes E$ has compact support, and

$$E \longmapsto \Lambda(V) \otimes E$$

defines again a Thom homomorphism $K(X) \to K(V)$.

Since for vector spaces V, W, we have a natural isomorphism of algebras

$$\Lambda^*(V \otimes W) \cong \Lambda^*(V) \otimes \Lambda^*(W) ,$$

it follows that, for vector bundles, V, W, over a compact space X, we have the multiplicative formula

$$\lambda_V \cdot \lambda_W = \lambda_{V \oplus W} . \tag{2.6}$$

Here we use the product (2.3). The formula (2.6) implies that the Thom homomorphism

$$K(X) \longrightarrow K(V \oplus W)$$

coincides with the composition of the two Thom homomorphisms

$$K(X) \longrightarrow K(V)$$
$$K(V) \longrightarrow K(V \oplus W)$$

the latter being obtained by regarding $V \oplus W$ as a bundle over V.

The fundamental result of K-theory, the *Bott periodicity theorem*,[13] asserts that φ is an *isomorphism*. Note in particular the special case $X =$ point; $V = \mathbf{C}^n$. Then we have an isomorphism

$$\varphi: K(\text{Point}) \longrightarrow K(\mathbf{C}^n) .$$

Thus $K(\mathbf{C}^n) \cong \mathbf{Z}$ and is generated by $\lambda_n = \lambda_{\mathbf{C}^n}$. The multiplicative property of λ shows that $\lambda_n = (\lambda_1)^n$.

There is a fairly straightforward generalization of K-theory to the category of G-spaces where G is a compact Lie group. Thus let X be a compact G-space, i.e., a compact space with a given action of G on X. By a G-vector bundle over X, we mean a G-space E which is also a vector bundle over X, the projection $E \to X$ being compatible with the group action, and such that for each $g \in G$ the map

$$E_x \longrightarrow E_{g(x)}$$

defined by g is linear. Note that if $X =$ Point, then a G-vector bundle over X is just a complex representation space of G.

Starting from G-vector bundles over a G-space X we form a ring $K_G(X)$

[13] This is also called the Thom isomorphism theorem for K-theory; the Bott periodicity being restricted to the special case where V is a trivial bundle.

just as in the case where there is no group. All the elementary part of K-theory goes over without essential change to K_G (see [16]). The Bott periodicity theorem in the general case presents some new features, and not all proofs generalize automatically to K_G-theory. Thus the proof given in [1] applies only when the vector bundle V decomposes into a sum of line-bundles. This decomposable case would be adequate for our purposes because we shall explicitly need only the case when X is a point and G is an abelian group. However this is a rather artificial restriction and, in [3], a more fundamental proof of the periodicity theorem is given which applies directly to K_G. Since the proof in [3] uses indices of certain classical elliptic operators (on the sphere and projective space) it might appear that we were involved in a vicious circle. However this is not the case; the main theorem (6.7) of this paper is not used in [3]. In a sense, Theorem (6.7) expresses the index of a general elliptic operator in terms of the index of certain classical operators and, for these, one has explicit methods of computation which yield ultimately the concrete formulas in papers II and III of this series.

We shall in fact only use the special case of the Bott periodicity when X is a point, and so we state this explicitly:

(2.7) *Let G be a compact Lie group, V a complex G-module. Then the homomorphism*

$$\varphi\colon K_G(\text{point}) \longrightarrow K_G(V)\ ,$$

given by $\varphi(x) = x \cdot \lambda_V$, is an isomorphism.

Note that K_G (point) is the character or representation ring $R(G)$, and so is determined by its restriction to all abelian subgroups. It is for this reason that we could restrict ourselves to abelian groups and hence use only the periodicity theorem as proved in [1].

If G acts freely on X so that $X \to X/G$ is a principal fibre bundle, then we have a natural isomorphism

$$K_G(X) \cong K(X/G)\ .$$

More generally if $G \times H$ acts on X with H acting freely, then

$$K_{G\times H}(X) \cong K_G(X/H)\ .$$

In addition to the results reviewed so far, we shall need a small technical lemma which is not given explicitly in the literature, and which we shall therefore prove here.

Suppose W is a real G-module, $V = W \otimes_{\mathbf{R}} \mathbf{C}$ its complexification. Then the map

$$\psi\colon V \longrightarrow V$$

given by complex conjugation is a G-map. We want to investigate the induced homomorphism

$$\psi^*: K_G(V) \longrightarrow K_G(V) .$$

By (2.7) $K_G(V)$ is a free $R(G)$-module generated by the element λ_V, so it is sufficient to compute $\psi^*\lambda_V$. This can be done quite easily in general, but for our purposes the two cases given in following lemma will suffice.

LEMMA (2.8). *Let ψ: $V \to V$ be complex conjugation, where $V = W \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of the real G-module W. Then if ψ^* is the induced homomorphism of $K_G(V)$ we have*

(1) *If $W = \mathbf{R}^1$, $G = O(1)$, $\psi^* a = -a[V]$.*

(2) *If $W = \mathbf{R}^2$, $G = SO(2)$, $\psi^* a = a$.*

PROOF. Case (2) follows from the fact that ψ is G-homotopic to the identity. We define

$$\psi_t(u + iv) = u + ig_t(v) \qquad u, v \in W, 0 \leqq t \leqq 1$$

where $g_t \in G = SO(2)$ is rotation through πt. In case (1) we observe that the element $\psi^*\lambda_V + \lambda_V[V] \in K_{O(1)}(V)$ is represented by the complex

$$\mathbf{C} \oplus V \xrightarrow{\alpha_z} V \oplus \mathbf{C}$$

where

$$\alpha_z = \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} .$$

Here $\mathbf{C} = \Lambda^0(V)$ is a trivial $O(1)$-module, and we have used the natural isomorphisms $V \cong \bar{V}$, $V \otimes V \cong \mathbf{C}$. Let $g_t \in GL(2, \mathbf{C})$ be a path connecting the identity to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} \bar{z} & 0 \\ 0 & 1 \end{pmatrix} g_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

gives a homotopy from α_z to $\begin{pmatrix} 0 & z\bar{z} \\ 1 & 0 \end{pmatrix}$. Thus, on the unit circle $S(V)$, α_z is homotopic to a constant and so

$$\psi^*\lambda_V + \lambda_V[V] = 0 .$$

In view of (2.7), this completes the proof.

3. Symbols and the topological index

Let X be a differentiable manifold, TX its tangent bundle.[14] This is a

[14] Using a riemannian metric (G-invariant when appropriate) we will usually, in the topological sections, identify the tangent and cotangent bundles. In the analytical sections TX will denote the cotangent bundle.

real vector bundle. If X is a differentiable G-manifold (i.e. if the compact Lie group G acts differentiably on X) then so is TX and we may consider the group $K_G(TX)$.

Suppose now that X, Y are G-manifolds with $X \subset Y$ and X compact. Let $i: X \to Y$ denote the inclusion. Then we shall first define an $R(G)$-homomorphism

$$i_! : K_G(TX) \longrightarrow K_G(TY) .$$

Choose a G-invariant riemannian metric on Y (in the neighborhood of X), and let N be an open tubular neighborhood of X in Y. Then N is a G-manifold, and it may be identified with the normal bundle of X in Y (which is, of course, a real G-vector bundle). The tangent bundle TX is then a closed G-submanifold of TY, and the tubular neighborhood TN of TX in TY may be identified with the vector bundle over TX obtained by lifting $N \oplus N$. The simple verification is left to the reader, but since there are two copies of N involved, we must be explicit about our identification. We shall agree that the first factor corresponds to a point of Y, and the second factor to tangents to Y (along the fibres of N). Moreover we shall identify $N \oplus N$ with $N \oplus iN = N \otimes_{\mathbf{R}} \mathbf{C}$, so that the neighborhood TN of TX in TY is identified with $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$, $\pi: TX \to X$ denoting the projection. Since this is a complex G-vector bundle, we have the Thom homomorphism

$$\varphi: K_G(TX) \longrightarrow K_G(TN) .$$

Since TN is open in TY, we have a natural homomorphism

$$k_* : K_G(TN) \longrightarrow K_G(TY)$$

induced by the inclusion. Combining φ and k_* we obtain the homomorphism

$$i_! : K_G(TX) \longrightarrow K_G(TY)$$

which we wanted to construct. It is easy to check that it is independent of the metric used in the construction and of the choice of tubular neighborhood. The fact that it is functorial (i.e., that if $X \xrightarrow{i} Y \xrightarrow{j} Z$ are inclusions, then $(ji)_! = j_! i_!$) is a consequence of the transitivity of the Thom homomorphism (see § 2). Formula (2.5) implies the following

$$i^* i_!(x) = \{\textstyle\sum (-1)^i \lambda^i (N \otimes_{\mathbf{R}} \mathbf{C})\} x \qquad \text{for } x \in K_G(TX) \tag{3.1}$$

where $i^*: K_G(TY) \to K_G(TX)$ is the restriction homomorphism, and $K_G(TX)$ is regarded as a $K_G(X)$-module in the usual way.

Using the homomorphisms $i_!$ just constructed we shall now proceed to define a homomorphism

$$K_G(TX) \longrightarrow R(G) .$$

This homomorphism will be called the *topological index* and will be written as t-ind.

Thus let X be a compact differentiable G-manifold, and let $i: X \to E$ be a differentiable G-embedding of X in a real representation space E of G. The existence of such embeddings is essentially a consequence of the Peter-Weyl theorem (for a proof see [13]). Let $j: P \to E$ denote the inclusion of the origin P in E. Then we have homomorphisms

$$K_G(TX) \xrightarrow{i_!} K_G(TE) \xleftarrow{j_!} K_G(TP) = R(G) .$$

But $j_!$ is just the Thom homomorphism for the vector space $E \otimes_{\mathbf{R}} \mathbf{C}$ (regarded as G-bundle over the point P) and hence, by (2.7), it is an isomorphism. Thus we may define

$$\text{t-ind}: K_G(TX) \longrightarrow R(G)$$

by

$$\text{t-ind} = (j_!)^{-1} \circ i_! .$$

To see that this is independent of the choice of the embedding $i: X \to E$, let $i': X \to E'$ be another embedding and consider the diagonal embedding

$$k: X \longrightarrow E \oplus E', \quad k(x) = i(x) \oplus i'(x) .$$

It will be sufficient to show that i and k give the same answer for t-ind (the same will then be true for i' and k). Now we have a G-homotopy of embeddings $X \to E \oplus E'$ given by

$$k_s(x) = i(x) \oplus si'(x) \qquad 0 \leqq s \leqq 1$$

and t-ind depends only on the G-homotopy class. Thus it will be enough to compare the embedding $i: X \to E$ with $k_0: X \to E \oplus E'$. If N is the normal bundle of $i(X)$, then the normal bundle of $k_0(X)$ is $N \oplus E'$ A similar result holds, rather trivially with X replaced by the origin P. The transitivity of the Thom homomorphism then gives rise to the commutative diagram

$$\begin{array}{ccc}
 & K_G(TX) & \\
 {\scriptstyle i_!}\swarrow & & \searrow{\scriptstyle (k_0)_!} \\
K_G(TE) & \xrightarrow{\psi} & K_G(T(E \oplus E')) \\
 {\scriptstyle j_!}\nwarrow & & \nearrow{\scriptstyle l_!} \\
 & K_G(TP) & \\
 & \| & \\
 & R(G) &
\end{array}$$

where ψ is the Thom homomorphism for $E' \otimes_{\mathbf{R}} \mathbf{C}$ (as G-bundle over TE), and j, l denote the inclusions $P \to E, P \to E \oplus E'$. By (2.7), $j_!$ and $l_!$ are isomorphisms, and so therefore is ψ. It now follows from the diagram that the two ways from $K_G(TX)$ down to $R(G)$ coincide, i.e., formally

$$l_!^{-1}(k_0)_! = j_!^{-1}\psi^{-1}\psi i_! = j_!^{-1} i_! .$$

Thus i and k_0 yield the same value of t-ind. We have then established that t-ind is independent of the choice of the embedding $X \to E$ used in its definition.

An important element $\rho_X \in K(TX)$ on any compact manifold X is the class ρ_X of the "de Rham symbol." This is defined as follows.[15] We consider the exterior algebra $\Lambda^*(T)$ of the tangent bundle $T = TX$. Lifted to T this gives a complex of real vector bundles, exact outside the zero section. The complexification of this thus defines an element $\rho_X \in K(TX)$. If X is a G-manifold, $\Lambda^*(T)$ is acted on naturally by G and so $\rho_X \in K_G(TX)$.

Remark. The element ρ_X is closely related to (but not to be confused with) the elements λ_V of § 2. Their relation is as follows. If $T^c = T \otimes_{\mathbf{R}} \mathbf{C}$, the element

$$\lambda_{T^c} \in K_G(T^c X) .$$

If $i: TX \to T^c X$ is the inclusion, then

$$\rho_X = i^* \lambda_{T^c} .$$

If we take $X = S^n = \mathbf{R}^n \cup \infty$, $G = O(n)$, then, in particular, we get an element $\rho_{S^n} \in K_{O(n)}(TS^n)$. This element will play a fundamental role in our proof. Actually a more fundamental object is the element $j_!(1) \in K_{O(n)}(TR^n)$ where $j: P \to \mathbf{R}^n$ is the inclusion of the origin, but, as the following lemma shows, this element is in some sense just "half" of ρ_X.

LEMMA (3.2). *Let $j^0: P^0 \to S^n$, $j^\infty: P^\infty \to S^n$ be the inclusions of the origin and the point at infinity, and let $\theta: TS^n \to TS^n$ be multiplication (of tangent vectors) by -1. Then we have*

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1) \in K_{O(n)}(TS^n) .$$

PROOF. S^n may be identified with the union $B_0^n \cup B_\infty^n$ of two copies of the unit ball $B^n \subset \mathbf{R}^n$, and this identification is compatible with the action of $O(n)$. Then we have an $O(n)$-isomorphism

$$TS^n \cong (B_0^n \times \mathbf{R}^n) \cup (B_\infty^n \times \mathbf{R}^n) ,$$

where we identify points over the equator $S^{n-1} = \partial B_0^n = \partial B_\infty^n$ by

[15] We will omit a factor i, which is more natural from the point of view of differential operators but for our present purposes it is irrelevant.

$$(x, v) \mapsto (x, h_x v) \qquad x \in S^{n-1}, v \in \mathbf{R}^n ,$$

h_x denoting the reflection in the hyperplane of $\mathbf{R}^n$ orthogonal to x. Passing to exterior algebras, we then have an $O(n)$-isomorphism of complexes

$$\pi^*\Lambda^* T^c \cong \left(B_0^n \times \mathbf{R}^n \times \Lambda^*(\mathbf{C}^n)\right) \cup \left(B_\infty^n \times \mathbf{R}^n \times \Lambda^*(\mathbf{C}^n)\right) ,$$

where π: $TS^n \to S^n$ is the projection and, on the right, we make the identification

(i) $(x, v, w) \mapsto (x, h_x v, h_x(w))$,

$h_x(\ )$ denoting the action on $\Lambda^*(\mathbf{C}^n)$ induced by the reflection h_x. Suppose now that $0 \leqq s \leqq 1$, and define a new complex A_s of vector bundles over TS^n by changing the bundle homomorphisms as follows. We define

$$B_0^n \times \mathbf{R}^n \times \Lambda^i(\mathbf{C}^n) \longrightarrow B_0^n \times \mathbf{R}^n \times \Lambda^{i+1}(\mathbf{C}^n)$$

by

(ii) $(x, v, w) \mapsto (x, v, (v - isx) \wedge w)$ and

$$B_\infty^n \times \mathbf{R}^n \times \Lambda^i(\mathbf{C}^n) \longrightarrow B_\infty^n \times \mathbf{R}^n \times \Lambda^{i+1}(\mathbf{C}^n)$$

by

(iii) $(x, v, w) \mapsto (x, v, (v + isx) \wedge w)$.

Since $h_x(x) = -x$, (i) implies that (ii) and (iii) agree over S^{n-1}, and thus define a complex of vector bundles over TS^n as asserted. Now observe the following facts about the complex A_s

(a) for all s, it is exact outside the zero section of TS^n;

(b) for $s = 0$, it is the original complex $\pi^*\Lambda^* T^c$;

(c) for $s = 1$, it is exact outside P_0 and P_∞.

By (a) and (b), the element $\rho_{S^n} \in K_{O(n)}(TS^n)$ is equally well defined by A_s, for any value of s. By (c), the complex A_1 defines an element $a = a^0 + a^\infty$ in

$$K_{O(n)}\left(T(S^n - S^{n-1})\right) = K_{O(n)}(T^0) \oplus K_{O(n)}(T^\infty) ,$$

where $T^0 = T(B_0^n - S^{n-1})$, $T^\infty = T(B_\infty^n - S^{n-1})$. From their definitions[16] we see that

$$a^0 = k_!^0(1) \qquad a^\infty = \theta^* k_!^\infty(1) ,$$

where k^0: $P^0 \to B_0 - S^{n-1}$, k^∞: $P^\infty \to B_\infty - S^{n-1}$ are the inclusions. Applying the natural homomorphism

$$K_{O(n)}\left(T(S^n - S^{n-1})\right) \longrightarrow K_{O(n)}(TS^n)$$

a becomes ρ_{S^n}, $k_!^0(1)$, and $k_!^\infty(1)$ become $j_!^0(1)$, $j_!^\infty(1)$ respectively, and so we obtain

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1)$$

as required.

[16] There is actually a scalar factor i between the complexes involved, but this does not affect the class in K.

4. Axioms for the index

In the preceding section we constructed an $R(G)$-homomorphism

$$\text{t-ind}: K_G(TX) \longrightarrow R(G) .$$

In this section we shall give axioms which will uniquely characterize this homomorphism. In the analytical part of the paper we shall then introduce the *analytical index* and show that it verifies the axioms.

We suppose now that, for every compact differentiable G-manifold X, we are given an $R(G)$-homomorphism

$$\text{ind}_G^X: K_G(TX) \longrightarrow R(G) .$$

(We shall write ind instead of ind_G^X when there is no possibility of confusion.) We assume this is functorial with respect to diffeomorphisms in X and homomorphisms in G. More precisely if $f: X \to Y$ is a G-diffeomorphism, then the diagram

$$\begin{array}{ccc} K_G(TX) & \xrightarrow{f^*} & K_G(TY) \\ & {\scriptstyle \text{ind}_G^X} \searrow \quad \swarrow {\scriptstyle \text{ind}_G^Y} & \\ & R(G) & \end{array}$$

commutes, and if $\varphi: G' \to G$ is a homomorphism, the diagram

$$\begin{array}{ccc} K_G(TX) & \xrightarrow{\varphi^*} & K_{G'}(TX) \\ {\scriptstyle \text{ind}_G^X}\downarrow & & \downarrow{\scriptstyle \text{ind}_{G'}^X} \\ R(G) & \xrightarrow{\varphi^*} & R(G') \end{array}$$

commutes. Such a functorial homomorphism, we will refer to briefly as an *index function*.

We introduce the following two axioms for index functions.

(A1) *If X is a point,* ind *is the identity of $R(G)$.*

(A2) ind *commutes with the homomorphism $i_!$ of* § 3.

The meaning of (A1) is clear; when X is a point, $TX = X$, and $K_G(TX)$ is naturally isomorphic to $R(G)$. In (A2) we mean that, for any inclusion $i: X \to Y$ with X, Y compact G-manifolds, the diagram

$$\begin{array}{ccc} K_G(TX) & \xrightarrow{i_!} & K_G(TY) \\ & {\scriptstyle \text{ind}_G^X} \searrow \quad \swarrow {\scriptstyle \text{ind}_G^Y} & \\ & R(G) & \end{array}$$

commutes.

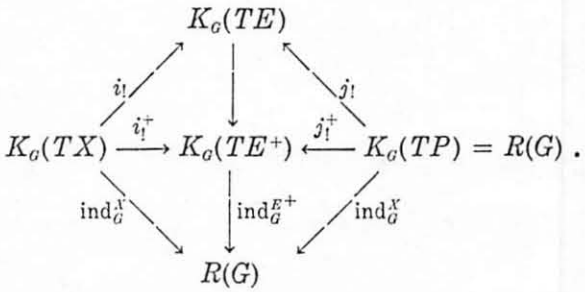
The topological index t-ind satisfies (A1) and (A2). The first is trivial,

and the second follows from the transitivity of $i_!$.

The following is then a rather trivial consequence of § 3:

PROPOSITION (4.1). *Let* ind *be an index function satisfying* (A1) *and* (A2). *Then we have* ind = t-ind.

PROOF. Given X we take an embedding $i: X \to E$ where E is a real G-module. Let E^+ be the one-point compactification of E. Since we can assume G acts orthogonally on E, it is clear that G acts differentiably on the sphere E^+; i.e., E^+ is a G-manifold. Now let $i^+: X \to E^+$ be the embedding given by i. Similarly if P is the origin of E, we have $j: P \to E$, $j^+: P \to E^+$. Consider the following diagram



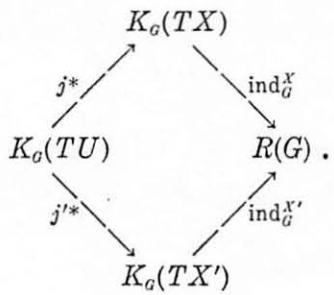
The top two triangles commute because of the way $i_!$ and $j_!$ are defined. The bottom two triangles commute by (A2). By (A1), ind_G^P is the identity. Now $j_!$ is an isomorphism, and t-ind: $K_G(TX) \to R(G)$ is defined by t-ind $= j_!^{-1} i_!$. The diagram then shows that it coincides with ind_G^X.

Axiom (A2) is not easy to verify, and our next aim is to show that it follows from a number of other more elementary axioms. First we describe an *excision axiom*.

(B1) *Let U be a (non-compact) G-manifold,*

$$j: U \longrightarrow X, \qquad j': U \longrightarrow X'$$

two open G-embeddings into compact G-manifolds X, X'. Then the following diagram commutes



When (B1) is satisfied, we can define

$$\text{ind}: K_G(TU) \longrightarrow R(G)$$

as the composition $\text{ind} \circ j^*$, and it is independent of the choice of $j: U \to X$ provided that one such X exists[17]. In particular if E is a real G-module then

$$\text{ind}: K_G(TE) \longrightarrow R(G)$$

is defined. Our next axiom is, like (A1), a normalization axiom[18].

(B2) *Let* $j_!: R(O(n)) \to K_{O(n)}(T\mathbf{R}^n)$ *be induced by the inclusion* $j: P \to \mathbf{R}^n$, *where* P *is the origin. Then* $\text{ind}\, j_!(1) = 1$.

Actually we shall also want to consider a weaker axiom, involving only the abelian groups $O(1)$ and $SO(2)$.

(B2′) $\text{ind}\, j_!(1) = 1$ *where* $j_!$ *is either*

$$R(O(1)) \longrightarrow K_{O(1)}(T\mathbf{R}^1)$$

or

$$R(SO(2)) \longrightarrow K_{SO(2)}(T\mathbf{R}^2)\,.$$

Finally we want to introduce a *multiplicative axiom*. This is the most significant one, and it requires some care in setting up. The simplest kind of multiplicative axiom would be concerned with product manifolds $X \times Y$, but this is not enough. We need to consider not just products but fibre bundles.

Suppose then that $P \to X$ is a compact differentiable principal fibre bundle with group H (a compact Lie group). Thus H acts freely on P, on the right, and $X = P/H$. If F is a compact differentiable H-manifold (H acting on the left), we can form the associated fibre bundle Y over X, given by

$$Y = P \times_H F;$$

i.e., Y is the quotient of $P \times F$ by the action of H: $h(p, f) = (ph^{-1}, hf)$. Since H acts on F, it also acts on TF, and $P \times_H TF$ is then a vector bundle over Y. This is usually called the tangent bundle along the fibres, and denoted by $T(Y/X)$; it is a sub-bundle of TY and (using a metric), we have a decomposition

$$TY = T(Y/X) \oplus \pi^* TX$$

where $\pi: Y \to X$ is the projection. Thus we have a multiplication

$$K(TX) \otimes K(T(Y/X)) \longrightarrow K(TY)\,.$$

On the other hand, we have homomorphisms

$$\begin{aligned} K_H(TF) \longrightarrow K_H(P \times TF) &\cong K(P \times_H TF) \\ &= K(T(Y/X))\,. \end{aligned}$$

[17] In fact this assumption can be avoided, but we do not need this.

[18] In (B2) we either assume (B1) so that ind is unambiguously defined on $K_{O(n)}(T\mathbf{R}^n)$, or else we define it by the standard embedding $\mathbf{R} \subset (\mathbf{R}^n)^+ = S^n$.

Combining these we thus obtain a mutiplication

$$K(TX) \otimes K_H(TF) \longrightarrow K(TY) .$$

Suppose now that a second compact Lie group G acts on all the preceding situation. Thus we assume that G acts (on the left) on P and F commuting with the action of H and so inducing an action on $X = P/H$ and $Y = P \times_H F$. Then we get a multiplication

$$K_G(TX) \otimes K_G(T(Y/X)) \longrightarrow K_G(TY)$$

and a homomorphism

$$\begin{aligned} K_{G\times H}(TF) \longrightarrow K_{G\times H}(P \times TF) &\cong K_G(P \times_H TF) \\ &= K_G(T(Y/X)) , \end{aligned}$$

which combine to give

$$K_G(TX) \otimes K_{G\times H}(TF) \longrightarrow K_G(TY) . \tag{4.2}$$

On the other hand, if V is any complex $G \times H$-module, $P \times_H V$ is a G-vector bundle over X. This extends to an $R(G)$-homomorphism

$$\mu_P \colon R(G \times H) \longrightarrow K_G(X) . \tag{4.3}$$

Finally let us recall that $K_G(TX)$ is a $K_G(X)$-module. We are now ready to formulate our multiplicative axiom.

(B3) *For all G, H, P, F, as above we have*

$$\operatorname{ind}_G^Y(ab) = \operatorname{ind}_G^X(a \cdot \mu_P(\operatorname{ind}_{G\times H}^F(b))$$

where $a \in K_G(TX)$, $b \in K_{G\times H}(TF)$ the product ab is taken as in (4.2) *and μ_P is the homomorphism of* (4.3).

Suppose, in particular, that $\operatorname{ind}_{G\times H}^F(b)$ lies in the subring $R(G)$ of $R(G\times H)$. Then, since μ_P and ind_G^X are both $R(G)$-homomorphisms, (B3) simplifies to

$$\operatorname{ind} ab = \operatorname{ind} a \cdot \operatorname{ind} b \in R(G) .$$

It is only this special case of (B3) which we shall need, and so we state this as

(B3′) *If* $\operatorname{ind}_{G\times H}^F(b) \in R(G) \subset R(G \times H)$, *then, in the notation of* (B3),

$$\operatorname{ind}_G^Y(ab) = \operatorname{ind}_G^X a \cdot \operatorname{ind}_{G\times H}^F b .$$

Remark. Taking $F =$ point, $H = 1$, we see that (B3′) implies (A1) unless $\operatorname{ind} \equiv 0$; but, of course, this is a trivial formal observation.

Axioms (B3) and (B3′) are for fibre bundles with group H. In particular therefore they apply to products, taking $H = 1$. Thus (B3′) implies

(B3″) *If X, F are G-manifolds and $a \in K_G(TX)$, $b \in K_G(TF)$, then*

$$\operatorname{ind}_G^{X\times F}(ab) = \operatorname{ind}_G^X a \cdot \operatorname{ind}_G^F b .$$

It is of course possible to reformulate (B3″) in terms of "external products"

for different groups G. Thus, if X_i is a G_i-manifold and $a_i \in K_{G_i}(TX_i)$, we can form the product $a_1 a_2 \in K_G(TX)$ where $X = X_1 \times X_2$, $G = G_1 \times G_2$. Then applying (B3″) with $X_1 = X$, $X_2 = F$ we see that

$$\operatorname{ind}_G^X a_1 a_2 = \operatorname{ind}_{G_1}^{X_1} a_1 \cdot \operatorname{ind}_{G_2}^{X_2} a_2 ,$$

where the latter product is given by:

$$R(G_1) \otimes R(G_2) \longrightarrow R(G) .$$

Our aim now will be to show that Axioms (B1), (B2′) and (B3′) imply (A2). First we prove

PROPOSITION (4.4). *Axioms* (B1), (B2′) *and* (B3″) *imply* (B2).

PROOF. Axiom (B1) enables us to extend (B3″) to open sets in compact manifolds. Thus if $a_i \in K_{G_i}(TU_i)$ with U_i open in X_i $(i = 1, \cdots, k)$, then

$$\operatorname{ind} \prod a_i = \prod \operatorname{ind} a_i .$$

(The point of (B1) is that $\operatorname{ind} \prod a_i$ may be computed by any manifold compactifying $\prod U_i$, not necessarily $\prod X_i$.) In particular, this multiplicative property holds for $U_i = R^{n_i}$ and $G_i \subset O(n_i)$

$$a_i = j_!^i(1) \qquad j^i \colon P \longrightarrow R^{n_i} .$$

If all $n_i = 1$ or 2 and $G_i = O(1)$ or $SO(2)$, then (B2′) asserts that $\operatorname{ind} a_i = 1$. Hence by (B3′) $\operatorname{ind} \prod a_i = \prod \operatorname{ind} a_i = 1 \in R(\prod G_i)$. Now, since $j_!$ is multiplicative, it follows that $\prod a_i$ is the restriction of $a = j_!(1) \in K_{O(n)}(TR^n)$. Thus $\operatorname{ind} a \in R(O(n))$ gives 1 when restricted to any subgroup $\prod G_i$ of $O(n)$ (with $G_i = O(1)$ or $SO(2)$). But these subgroups contain all cyclic subgroups of $O(n)$, and so are sufficient to determine a character of $O(n)$. Hence $\operatorname{ind} a = 1 \in R(O(n))$, establishing (B2).

Remark. The verification (for the analytical index) of (B2) is not in fact much more difficult than (B2′), but it seems relevant to observe (as we have done in (4.4)) that (B2′) is sufficient.

We can now prove

PROPOSITION (4.5). *Axioms* (B1), (B2) *and* (B3′) *imply* (A2).

If we have an index function satisfying (B1), then as observed above (B3′) or (B3″) implies a corresponding result for open sets in compact manifolds. Thus (B3′) continues to hold when F is an open set (stable under $G \times H$) of some $G \times H$- compact manifold $\tilde{F}$, the space $Y = P \times_H F$ is then an open set of the compact manifold $P \times_H \tilde{F}$. In particular, we can take $F = \mathbf{R}^n$, $\tilde{F} = (\mathbf{R}^n)^+ = S^n$, $H = O(n)$, $b = j_!(1)$ where $j \colon A \to \mathbf{R}^n$ is the inclusion of the origin A. Then P is a principal $O(n)$-bundle over the compact manifold X, and G acts on P, commuting with $O(n)$. We make G act trivially on $\mathbf{R}^n$. The space $Y = P \times_{O(n)} \mathbf{R}^n$ is then the

associated real vector bundle over X; it is a G-bundle. The homomorphism

$$K_G(TX) \longrightarrow K_G(TY)$$

given by $a \mapsto ab$ is just the homomorphism $i_!$ induced by the zero-section inclusion $i\colon X \to Y$. If ind satisfies (B2), then

$$\mathrm{ind}_{O(n)}(b) = 1 \in R(O(n)) ,$$

and if we regard G as acting trivially on $\mathbf{R}^n$, the same formula holds with $G \times O(n)$ instead of $O(n)$ (using the functoriality of ind for the projection homomorphism $G \times O(n) \to O(n)$). Now applying (B3'), we obtain

$$\begin{aligned} \mathrm{ind}\, i_!(a) = \mathrm{ind}\, ab &= \mathrm{ind}\, a \cdot \mathrm{ind}\, b \\ &= \mathrm{ind}\, a \in R(G) . \end{aligned}$$

This establishes (A2) in the special case when Y is a real vector bundle over X. But for a general embedding

$$k\colon X \longrightarrow Z ,$$

the homomorphism

$$k_!\colon K_G(TX) \longrightarrow K_G(TZ)$$

is defined as the composition of

$$j_!\colon K_G(TX) \longrightarrow K_G(TN)$$

and the natural homomorphism

$$K_G(TN) \longrightarrow K_G(TZ) ,$$

where N is the normal bundle of X in Z. In the following diagram

$$\begin{array}{ccccc} K_G(TX) & \longrightarrow & K_G(TN) & \longrightarrow & K_G(TZ) \\ & {\scriptstyle \mathrm{ind}^X} \searrow & \downarrow {\scriptstyle \mathrm{ind}^N} & \swarrow {\scriptstyle \mathrm{ind}^Z} & \\ & & R(G) & & \end{array}$$

the first triangle commutes by what we have just proved, and the second commutes by (B1). Thus

$$\mathrm{ind}^Z k_!(a) = \mathrm{ind}^X a$$

and (A2) is established in general.

Putting together Propositions (4.1), (4.4), and (4.5), we then obtain the following uniqueness theorem.

THEOREM (4.6). *Let* ind *be an index function satisfying* (A1), (B1), (B2') *and* (B3'). *Then* ind = t-ind.

Remark. It will be noticed that (B3) has not been used—only (B3'). It will

actually turn out in § 9 that the stronger axiom (B3) holds for the analytical index.

Axiom (B2′) can be replaced by another which is, in practice, more convenient to check.[19] Let us recall (cf. § 3) that, for any compact G-manifold X, we have the "de Rham symbol class" $\rho_X \in K_G(TX)$. We then introduce the following variant of (B2′):

(B2″). (i) $\operatorname{ind} \rho_{S^2} = 2 \in R(SO(2))$;

(ii) $\operatorname{ind} \rho_{S^1} = 1 - \xi \in R(O(1))$, where $\xi: O(1) \to U(1)$ is the standard representation.

(iii) $\operatorname{ind} j_!(1) = 1 \in \mathbf{Z}$, where $j: P \to S^1$ is the inclusion of the origin.

Remark. The elements occurring here all belong to $K_{O(n)}$ ($n = 1$ or 2), but in (i) and (iii) we choose to restrict to the smaller groups $SO(n)$.

We shall now show that (B2′), can be replaced by (B2″).

LEMMA (4.7). *Let* ind *be an index function satisfying* (B2″). *Then it also satisfies* (B2′).

PROOF. By Lemma (3.2), we have

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1) \in K_{O(n)}(TS^n) ,$$

where $j^0: P^0 \to S^n$, $j^\infty \to S^n$ are the inclusions of the origin and the point at infinity, and θ is multiplication by -1 on tangent vectors. Let $f: S^n \to S^n$ be reflection in the equator so that f interchanges P^0 and P^∞ and commutes with θ. Then

$$f^*\big(\theta^* j_!^\infty(1)\big) = \theta^* j_!^0(1) .$$

By the functorial properties of an index function, this implies that

$$\operatorname{ind} \theta^* j_!^\infty(1) = \operatorname{ind} \theta^* j_!^0(1) \in R\big(O(n)\big) .$$

Thus

$$\operatorname{ind} \rho_{S^n} = \operatorname{ind} (1 + \theta^*) j_!^0(1) .$$

Now, by definition, $j_!^0$ (which we now write simply as $j_!$) factors through the group $K_{O(n)}(T\mathbf{R}^n)$ and, on $T\mathbf{R}^n = \mathbf{C}^n$, θ coincides with complex conjugation. Applying Lemma (2.8) therefore we deduce

$$\operatorname{ind} \rho_{S^2} = \operatorname{ind} 2j_!(1) = 2 \operatorname{ind} j_!(1) \in R\big(SO(2)\big)$$
$$\operatorname{ind} \rho_{S^1} = \operatorname{ind} (1 - \xi) j_!(1) = (1 - \xi) \operatorname{ind} j_!(1) \in R\big(O(1)\big) .$$

Parts (i) and (ii) of (B2″) then give

[19] Verifying (B2′) for the analytical index involves computing the index of a certain operator in euclidean space (see a forthcoming paper by Atiyah and Hörmander). In some ways this is more direct and natural than verifying (B2″), on the sphere. However, it involves more explicit analysis which we avoid by the topological deformations implied in (4.7).

$$2 \operatorname{ind} j_!(1) = 2$$
$$(1 - \xi) \operatorname{ind} j_!(1) = 1 - \xi .$$

Now the annihilator of $1 - \xi$ in

$$R(O(1)) = \mathbf{Z}[\xi]/(1 - \xi^2)$$

consists of integral multiples of $(1 + \xi)$. Hence we deduce

$$\operatorname{ind} j_!(1) = 1 \in RSO(2)$$
$$\operatorname{ind} j_!(1) = 1 + a(1 + \xi) \in R(O(1))$$

for some integer a. Restricting the second equation to the identity of $O(1)$, and applying (iii) of (B2''), we deduce

$$1 = 1 + 2a \quad \text{i.e., } a = 0 .$$

Thus $\operatorname{ind} j_!(1) = 1$ both for $SO(2)$ and for $O(1)$. This is precisely Axiom (B2'), bearing in mind that ind on $K(T\mathbf{R}^n)$ is defined *via* the compactification $\mathbf{R}^n \to S^n$ (cf. footnote[18]).

To conclude this section, we might point out that the topological index of § 3 clearly satisfies axioms (A1), (A2), and also (B2). It is not quite so obvious that it satisfies (B1) and (B3) but, with a little work, these can be established directly. On the other hand, once we have shown that the analytical index (to be introduced in § 6) satisfies (B1), (B2), and (B3), the uniqueness theorem (4.6) will imply that t-ind = a-ind, and so t-ind will also satisfy these axioms.

5. Pseudo-differential operators

In this section, we shall review the basic analytical facts concerning pseudo-differential operators. Their application to the index of elliptic operators will be treated in § 6. For proofs of the results stated here, we refer to Kohn-Nirenberg [11], Seeley [15], Hörmander [9][10], and Palais [14]. We shall however attempt to present the material so that only standard results are given without proof.

The term "pseudo-differential" is applied, in different places, to slightly different classes of operators. For our purposes any one of these classes would be equally good. In fact, we shall eventually form a closure of this class and, by that stage, any differences would disappear. Perhaps the largest and most natural class is that given in Hörmander [10], and we begin therefore by recalling his definition.[20]

[20] Actually [10] is concerned primarily with classes $L^m_{\rho,\delta}$, and we only use the case $\rho = 1, \delta = 0$. The other values of ρ, δ are used in [10] to treat certain classes of hypoelliptic operators. For these the index problem can be solved by reducing it to the standard elliptic case discussed here. This is proved in a forthcoming paper of Atiyah and Hörmander.

Consider first an open set U of $\mathbf{R}^n$, and let $x = (x_1, \cdots, x_n)$ be the standard coordinates. For any integer m, we denote by $S^m(U)$ the set of all smooth functions $p(x, \xi)$ on $U \times \mathbf{R}^n$ such that, for every compact $K \subset U$, and all multi-indices α, β, we have

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|}, \qquad x \in K, \xi \in \mathbf{R}^n .$$

Here D_ξ^α stands for the partial derivative

$$(-i\partial/\partial\xi_1)^{\alpha_1}(-i\partial/\partial\xi_2)^{\alpha_2} \cdots (-i\partial/\partial\xi_n)^{\alpha_n},$$

$|\alpha| = \sum \alpha_i$ and $C_{\alpha,\beta,K}$ is a constant depending on α, β, K, and p. For any such p, we define a linear operator

$$P: \mathfrak{D}(U) \longrightarrow \mathfrak{E}(U)$$

by the formula

$$Pu = (2\pi)^{-n}\int p(x, \xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi .$$

Here $\mathfrak{E}$ denotes the smooth functions on U, $\mathfrak{D}$ those with compact support, and $\hat{u}$ is the Fourier transform of u. If p is a polynomial in ξ of degree m with smooth coefficients, then $p \in S^m(U)$, and P is the differential operator associated to it in the usual way. For this reason, when we want to show the dependence of P on p, in the general case, we write

$$P = p(x, D),$$

where D stands formally for the vector with components $-i(\partial/\partial x_j)$.

A *pseudo-differential operator* is one which is locally of the above type. Precisely, we denote by $L^m(U)$ the set of all mappings $P: \mathfrak{D}(U) \to \mathfrak{E}(U)$ such that, for all $f \in \mathfrak{D}(U)$, there exists some $p_f \in S^m(U)$ with $P(fu) = p_f(x, D)u$ for all $u \in \mathfrak{D}(U)$. An equivalent definition [10; § 2] is that P is continuous, and that the commutator

$$p_f(x, \xi) = e^{-i\langle x,\xi\rangle}P(fe^{i\langle x,\xi\rangle})$$

belongs to $S^m(U)$ for all $f \in \mathfrak{D}(U)$.

In this paper we shall introduce a sub-class $\mathcal{P}^m \subset L^m$ consisting of operators P for which all the functions $p_f(x, \xi)$ above lie in a certain subspace $S_0^m(U \times \mathbf{R}^n)$ of $S^m(U \times \mathbf{R}^n)$. A function $p \in S_0^m$ if, for $\xi \neq 0$, the limit

$$\sigma(p)(x, \xi) = \lim_{\lambda\to\infty} \frac{p(x, \lambda\xi)}{\lambda^m}$$

exists. Then $\sigma(p)$ is a C^∞ function on $U \times (\mathbf{R}^n - 0)$, and it is homogeneous of degree m in ξ. For an operator $P \in \mathcal{P}^m$, we define the symbol $\sigma(P)$ by

$$\sigma(P)(x, \xi) = \sigma(p_f)(x, \xi),$$

where f is any function equal to 1 near x. It does not depend on the choice of f. The symbol is a C^∞ function of (x, ξ) outside $\xi = 0$, and it is homogeneous of degree m. It follows from results of [10] that L^m and $\mathcal{P}^m$ are invariant under diffeomorphisms of U, and hence corresponding classes of operators can be defined globally for smooth vector bundles over smooth (paracompact) manifolds (for the details of this, see [10] or [4]). If E and F are smooth vector bundles over a smooth manifold X, we shall denote by $\mathcal{P}^m(X; E, F)$ the space of pseudo-differential operators[21]

$$P: \mathcal{D}(X; E) \longrightarrow \mathcal{E}(X; F)$$

of type $\mathcal{P}^m$. Locally, with respect to a coordinate patch on X and bases in E, F, such an operator P is given by a matrix $p_{ij}(x, D)$ of operators in euclidean space as above.

The symbol $\sigma(P)$ of $\mathcal{P}^m(X; E, F)$ is globally well-defined as a smooth homomorphism

$$\sigma(P): \pi^*E \longrightarrow \pi^*F$$

of vector bundles on the cotangent space TX (with the zero-section removed). Here π denotes the projection $\pi: TX \to X$. On each fibre of TX, $\sigma(P)$ is (positively) homogeneous of degree m. We denote by $\mathrm{Symb}^m(X; E, F)$ the space of all homomorphisms $\pi^*E \to \pi^*F$ which are defined and smooth outside the zero-section and, on each fibre of TX, are homogeneous of degree m. If we choose the unit sphere bundle $S(X)$ of TX defined by some smooth riemannian metric, we can clearly identify $\mathrm{Symb}^m(X; E, F)$ with the space of smooth homomorphisms $\pi_S^*E \to \pi_S^*F$ where $\pi_S: S(X) \to X$ is the projection.

We recall briefly a few important properties of pseudo-differential operators. First of all they compose well; that is,

$$P \in \mathcal{P}^m, Q \in \mathcal{P}^q, f \in \mathcal{D}(X) \Longrightarrow PfQ \in \mathcal{P}^{m+q},$$

and

$$\sigma(PfQ) = \sigma(P)f\sigma(Q).$$

If X is a compact manifold then the function f may be omitted. Secondly they transpose well; that is,

$$P \in \mathcal{P}^m(X; E, F) \Longrightarrow P^t \in \mathcal{P}^m(X; F', E'),$$

where $E' = \mathrm{Hom}(E, \Omega)$ (Ω denoting the volume bundle of X as in [4; § 5]), and P^t is the transpose of P (so that the distributional extension of P^t coincides with the dual of P). Moreover $\sigma(P^t) = \sigma(P)'$ where $\sigma \mapsto \sigma'$ is the

[21] We write $\mathcal{E}(X; E)$ for the spaces of smooth sections of E over X and $\mathcal{D}(X; E)$ for those with compact support. When X is clearly understood, we shall omit it.

map of symbols induced by the isomorphism $\mathrm{Hom}(E, F) \to \mathrm{Hom}(F', E')$ of bundles. If we choose a smooth hermitian metric for E, and a smooth positive measure on X, we get an anti-linear isomorphism $E \to E'$. If we also have a metric for F, then the transpose $P^t \in \mathcal{P}^m(X; F', E')$ can be replaced by the "formal adjoint" $P^* \in \mathcal{P}^m(X; F, E)$. For the symbols, we then have $\sigma(P^*) = \sigma(P)^*$.

We come now to the Sobolev spaces. If E is a smooth vector bundle over a smooth manifold X, and if s is any non-negative integer[22], we denote by $H_s^{\mathrm{loc}}(X; E)$ the space of those distributional sections u of E for which $Du \in L_2^{\mathrm{loc}}$ for all differential operators

$$D: \mathfrak{D}(X; E) \longrightarrow \mathfrak{D}(X; 1)$$

with smooth coefficients, and of order $\leqq s$. If, for local coordinates (x_j), and a local base (e_i) of E, we write $u = \sum u_i(x) e_i$, then (in the coordinate patch)

$$u \in H_s^{\mathrm{loc}} \Longleftrightarrow \left(\frac{\partial}{\partial x}\right)^{\alpha} u_i \in L_2^{\mathrm{loc}} \qquad \text{for all } \alpha \text{ with } |\alpha| \leqq s\,.$$

The space $H_s^{\mathrm{loc}}(X; E)$ has a natural topology given by a countable[23] set of semi-norms. It is a Fréchet space. We also consider

$$H_s^{\mathrm{comp}}(X; E) \subset H_s^{\mathrm{loc}}(X; E)$$

consisting of sections with compact support. It has its own natural topology as a direct limit (over compact $K \subset X$) of Hilbert spaces (this is not of course the induced topology from H_s^{loc}). We define $H_{-s}^{\mathrm{loc}}(X; E)$ as the dual of $H_s^{\mathrm{comp}}(X; E')$, and $H_{-s}^{\mathrm{comp}}(X; E)$ as the dual of $H_s^{\mathrm{loc}}(X; E')$ where $E' = \mathrm{Hom}(E, \Omega)$ as above.

If X is compact, then $H_s^{\mathrm{comp}} = H_s^{\mathrm{loc}}$, and we write it simply as H_s. An explicit Hilbert space norm for H_s can be defined in terms of a hermitian metric and connection for E, and a smooth positive measure on X as follows. For $s = 0$ we have the usual L_2-norm

$$\|u\| = \left(\int_X \langle u, u\rangle\right)^{1/2}.$$

For $s > 0$, we first introduce the positive definite operator $\Delta = 1 + D^*D$, where $D: \mathfrak{D}(E) \to \mathfrak{D}(E \otimes T)$ is the covariant derivative given by the connection, and then put

$$\|u\|_s = \left(\int_X \langle \Delta^s u, u\rangle\right)^{1/2}.$$

---

[22] These spaces are also defined for real s [8] but the integer case is enough for our purposes. The invariance properties for integral s are then rather trivial.

[23] All our manifolds are assumed paracompact so that we can take a countable set of coordinate patches.

For $s < 0$, we take the dual norm. Note that, if a compact group G acts differentiably on X and E, then an invariant measure on X, and an invariant connection will lead to Hilbert space norms on $H_s(X; E)$ invariant by G.

Pseudo-differential operators behave well with respect to the H_s-spaces, namely we have

(5.1) *A pseudo-differential operator*

$$P: \mathcal{D}(X; E) \longrightarrow \mathcal{E}(X; F)$$

in $\mathcal{P}^m(X; E, F)$ *extends, for integer* s, *to a continuous linear operator*

$$P_s: H_s^{\text{comp}}(X; E) \longrightarrow H_{s-m}^{\text{loc}}(X; F) .$$

Let $Op_s^m = Op_s^m(X; E, F)$ denote the space of all continuous linear maps $H_s^{\text{comp}}(X; E) \to H_{s-m}^{\text{loc}}(X; F)$ with the topology of bounded convergence. Then $P \mapsto P_s$ defines a map $\mathcal{P}^m \to Op_s^m$ whose image we denote by $\mathcal{P}_s^m$. If X is compact, Op_s^m is a Banach space (with norm denoted by $\| \ \|_s^m$) and the closure $\bar{\mathcal{P}}_s^m$ has a rather simple structure as described in the following result.

(5.2) *Let* X *be compact, then the symbol*

$$\sigma: \mathcal{P}_s^m(X; E, F) \longrightarrow \text{Symb}^m(X; E, F)$$

is continuous for the sup *norm topology on the unit sphere bundle of* TX; *it extends by continuity to a map*

$$\sigma_s: \bar{\mathcal{P}}_s^m(X; E, F) \longrightarrow \overline{\text{Symb}^m}(X, E, F)$$

which is surjective, and has the compact operators $H_s \to H_{s-m}$ *as kernel.*

Remark. We recall that $\text{Symb}^m(X; E, F)$ is isomorphic (by restriction to the unit sphere bundle $S(X)$ in TX) to the space of smooth homomorphisms $\pi_S^* E \to \pi_S^* F$ where $\pi_S: S(X) \to X$ is the projection. The closure $\overline{\text{Symb}^m}(X; E, F)$ can thus be identified with the space of continuous homomorphisms $\pi_S^* E \to \pi_S^* F$.

To obtain C^∞ results from the H_s-spaces, it is convenient to consider all s simultaneously. For this purpose, we introduce (for any X, not necessarily compact) the space $Op^m(X; E, F)$ of all linear operators $\mathcal{D}(X; E) \to \mathcal{D}'(X; F)$ which extend by continuity to operators in $Op_s^m(X; E, F)$ for all s. Now the Sobolev lemma implies that

$$\mathcal{D}(X; E) = \bigcap_s H_s^{\text{comp}}(X; E) , \qquad \mathcal{E}(X; F) = \bigcap_s H_s^{\text{loc}}(X; F)$$

(with the inverse limit topology), and so an operator in Op^m actually maps $\mathcal{D}(X; E) \to \mathcal{E}(X; F)$ (continuously). For each s we have an embedding of Op^m in Op_s^m, and hence of Op^m in $\prod_s Op_s^m$: its image is closed, and we give Op^m the induced topology making it a Fréchet space.

The space Op^m is a *local* space of operators in the sense that the Schwartz kernels on $X \times X$ are a local space of distributions. This means that an operator P is in Op^m if and only if, for every pair of C^∞ functions φ, ψ on X with compact support, $\varphi P \psi \in Op^m$. Reasoning as in [3; Appendix] it follows that, if $\{U_i\}$ is a coordinate covering of X so that $\{U_i \times U_i\}$ covers $X \times X$,

$$P \in Op^m(X) \Longleftrightarrow P \mid U_i \in Op^m(U_i) \qquad \text{for all } i .$$

Moreover the semi-norms in the U_i determine the semi-norms in X.

By (5.1) $\mathcal{P}^m \subset Op^m$, and we now form the closure $\bar{\mathcal{P}}^m$. Since $\mathcal{P}^m$ is by definition a local space of operators, and since Op^m is local, it follows that $\bar{\mathcal{P}}^m$ is local. Since $\mathcal{P}^m \longrightarrow \mathcal{P}^m_s$ is, by definition, continuous, (5.2) implies that the symbol extends by continuity to give a diagram

$$\begin{array}{ccc} \bar{\mathcal{P}}^m & \longrightarrow & \bar{\mathcal{P}}^m_s \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma_s} & \\ & \overline{\mathrm{Symb}^m} & \end{array} .$$

Note that, in this diagram, while σ_s is surjective, σ is not. In fact, the image of σ is rather difficult to describe, and we shall not go into the question. An operator $P \in \bar{\mathcal{P}}^m_s$ will be called elliptic if $\sigma_s(P)$ is invertible. An operator $P \in \bar{\mathcal{P}}^m$ will be called elliptic if $\sigma(P)$ is invertible (in the space Symb^m of continuous symbols); this means that

$$P \text{ elliptic} \Longleftrightarrow P_s \text{ elliptic for some } s \Longleftrightarrow P_s \text{ elliptic for all } s.$$

The most important reason for introducing the closure $\bar{\mathcal{P}}^m$ lies in the behavior of pseudo-differential operators for products of manifolds as we shall now explain. Thus let E, F be smooth vector bundles over X, and G a smooth vector bundle over Y. If P is a continuous linear map $\mathcal{D}(X; E) \to \mathcal{E}(X; F)$, we denote by $\tilde{P}$ the "lifted operator" from $\mathcal{D}(X \times Y; E \boxtimes G) \to \mathcal{E}(X \times Y; F \boxtimes G)$, that is the unique continuous linear map such that

$$\tilde{P}(u \otimes v) = Pu \otimes v \qquad u \in \mathcal{D}(X; E), v \in \mathcal{D}(Y; G) .$$

If $m \geqq 0$, this lifting operation behaves well with respect to the Op^m spaces, that is $P \mapsto \tilde{P}$ defines a continuous map

$$Op^m(X; E, F) \longrightarrow Op^m(X \times Y; E \boxtimes G, F \boxtimes G) . \tag{5.3}$$

To verify this, it is enough to check the case when X and Y are domains in euclidean space, and all bundles are trivial of dimension one. Suppose then that $P \in Op^m(X)$. Then for $f \in \mathcal{D}(X \times Y)$, compact sets $K \subset X$ $L \subset Y$ and $|\alpha| + |\beta| = s - m \geqq 0$, we have

$$\int_{K\times L} |D_x^\alpha D_y^\beta \tilde{P} f|^2 dx\, dy = \int_L dy \int_K |D_x^\alpha \tilde{P} D_y^\beta f|^2 dx$$
$$\leqq C \int_L dy \int_K \sum_{|\gamma|\leqq|\alpha|+m} |D_x^\gamma D_y^\beta f|^2 dx$$
$$\leqq C \int_{K\times L} \sum_{|\beta|+|\gamma|\leqq s} |D_x^\gamma D_y^\beta f|^2 dx dy\ .$$

This shows that (for $s \geqq m$) $\tilde{P}$ is continuous $H_s^{\text{comp}}(X \times Y) \to H_{s-m}^{\text{loc}}(X \times Y)$, and that $P \mapsto \tilde{P}$ is continuous from $Op^m(X)$ to $Op_s^m(X \times Y)$. Passing to adjoints gives the corresponding result for $s \leqq 0$. Hence for $m = 0$ or 1, we are through and obtain the continuity of (5.3). For $m > 1$, another small argument is needed, but since $m = 1$ suffices for applications, we shall omit the proof.

Unfortunately it is *not* true in general that $P \in \mathcal{P}^m \Rightarrow \tilde{P} \in \mathcal{P}^m$. However for the closure $\bar{\mathcal{P}}^m$, we do have

(5.4) *For $m > 0$, if $P \in \bar{\mathcal{P}}^m(X; E, F)$, then the lifted operator*

$$\tilde{P} \in \bar{\mathcal{P}}^m(X \times Y; E \boxtimes G, F \boxtimes G)\ .$$

Moreover $\sigma(\tilde{P}) = \tilde{\sigma}(P)$ where $\tilde{\sigma}$ is the lift of σ defined by

$$\tilde{\sigma}_{(\xi,\eta)}(e \otimes g) = \sigma_\xi(e) \otimes g, \qquad \xi \in TX, \eta \in TY, e \in E, g \in G.$$

This is essentially property (S6) of [14; Ch. XI], but we shall recall the proof which is quite elementary.

Since $\bar{\mathcal{P}}^m$ is a local class of operators, it will be sufficient to deal with the case when $X = U \subset \mathbf{R}^n$ and $Y = V \subset \mathbf{R}^q$ are domains in euclidean space, and all bundles are trivial of dimension one. Moreover by the continuity of (5.3) it will be sufficient to prove that

$$P \in \mathcal{P}^m(U) \Longrightarrow \tilde{P} \in \bar{\mathcal{P}}^m(U \times V)\ .$$

Again, since $\bar{\mathcal{P}}^m$ is a local class, it will be sufficient to show that, for all $\varphi, \psi \in \mathcal{D}(U)$ and $\varphi_1, \psi_1 \in \mathcal{D}(V)$, $Q = \varphi\varphi_1\tilde{P}\psi\psi_1 \in \bar{\mathcal{P}}^m(U \times V)$. To do this we shall construct a family $R^t \in \mathcal{P}^0(U \times V)$, defined for $t > 0$, and such that

(i) $Q \circ R^t \in \mathcal{P}^m(U \times V)$

(ii) $Q \circ R^t \to Q$ in $Op^m(U \times V)$ as $t \to 0$.

First of all we choose a family of functions $\sigma^t(\xi, \eta)$, defined for $t > 0$, taking values in the unit interval and such that

(a) σ^t is homogeneous of degree zero and C^∞ outside the origin,

(b) $\sigma^t = 1$ for $|\xi| < t|\eta|$
$= 0$ for $|\xi| > 2t|\eta|$.

Also let $\varphi(\lambda)$ be a C^∞ function of $\lambda \in \mathbf{R}$ such that

$$\varphi(\lambda) = 0 \qquad \text{for } |\lambda| \leqq 1$$
$$= 1 \qquad \text{for } |\lambda| \geqq 2\ .$$

Then put

$$\rho^t(\xi, \eta) = 1 - \varphi\big(t(|\xi|^2 + |\eta|^2)^{1/2}\big)\sigma^t(\xi, \eta) .$$

The appearance of this function is indicated below: $\rho^t = 1$ in the horizontally shaded region, and $\rho^t = 0$ in the vertically shaded region

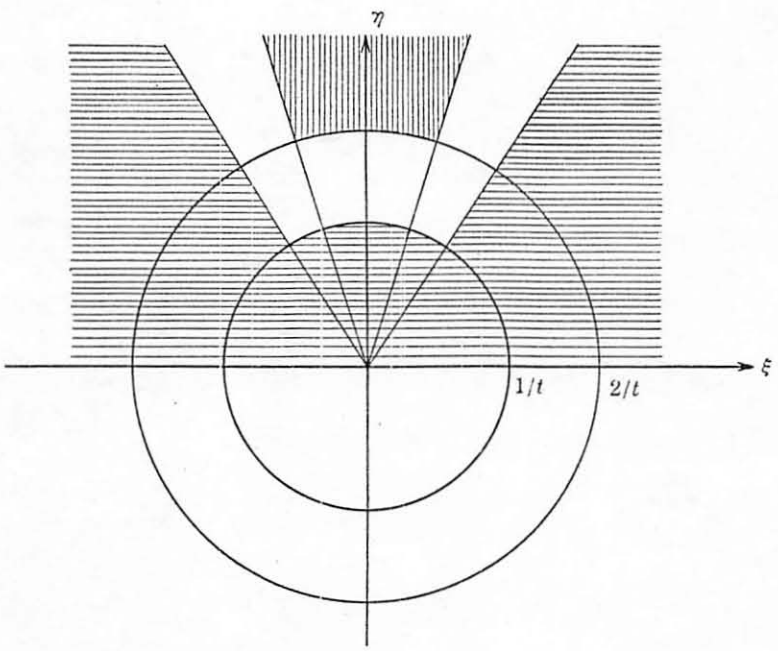


FIG. 2

Finally we define R^t to be convolution by the inverse Fourier transform of ρ^t so that

$$R^t u(x, y) = (2\pi)^{-n-q}\int \rho^t(\xi, \eta)\hat{u}(\xi, \eta)e^{i\langle x,\xi\rangle + i\langle y,\eta\rangle}d\xi d\eta .$$

Then the operator $Q^t = Q \circ R^t$ is given by the integral formula

$$(Q^t u)(x, y) = (2\pi)^{-n-q}\int \varphi_1(y)\varphi(x)p_\psi(x, \xi)\rho^t(\xi, \eta)\widehat{\psi_1 u}(\xi, \eta)d\xi d\eta .$$

The properties of the function ρ^t show that $p_\psi \rho^t \in S_0^m(U \times V \times \mathbf{R}^{n+q})$ so that $Q^t \in \mathcal{P}^m(U \times V)$. Now, as remarked in [11], the H_s estimates for pseudo-differential operators do not require regularity in (ξ, η). The fact that, for $m \geqq 0$, $D_x^\beta p_\psi(x, \xi)/(1 + |\xi| + |\eta|)^m$ is bounded then implies that $Q \in Op^m$ (which we already know). Similarly for $m > 0$, the inequalities

$$\left| \frac{D_x^\beta\{p_\psi(x, \xi)(\rho^t(\xi, \eta) - 1)\}}{(1 + |\xi| + |\eta|)^m} \right| < C_\beta t^m ,$$

(which are easy consequences of the properties of ρ^t) show that, as $t \to 0$, $Q^t - Q \to 0$ in Op^m. As for the symbols we have

$$\sigma(Q^t) = \varphi_1\varphi\sigma(p_\psi)(1 - \sigma^t)\psi_1 = \varphi_1\varphi\tilde{\sigma}(P)(1 - \sigma^t)\psi\psi_1 ,$$

and, as $t \to 0$, this converges to $\varphi_1\varphi\tilde{\sigma}(P)\psi\psi_1$. Since the symbol is also a local object, this implies that $\sigma(\tilde{P}) = \tilde{\sigma}(P)$ as required.

The final property of pseudo-differential operators we need concerns the situation of a group action. Thus let X be a compact manifold, and G a compact Lie group acting smoothly on X and on vector bundles E, F over X. Then G acts on the space $\bar{\mathcal{P}}^m(X; E, F)$ (because of the invariance of $\bar{\mathcal{P}}^m$ under diffeomorphism). If $g \in G$ and $P \in \bar{\mathcal{P}}^m$, we denote the action simply by $g(P)$. Note that, if u is a section of E, and $u \mapsto gu$ denotes the action of G on sections, then

$$g(P)u = gPg^{-1}u .$$

Then we shall need the following continuity property.[24]

(5.5) *For fixed $P \in \bar{\mathcal{P}}^m(X; E, F)$, the map $G \to \bar{\mathcal{P}}^m \subset Op^m$ given by $g \mapsto g(P)$ is continuous.*

Since G can be assumed to act unitarily on all H_s spaces, and since $\bar{\mathcal{P}}^m$ is the uniform closure of $\mathcal{P}^m$ it will be sufficient to prove (5.5) for $P \in \mathcal{P}^m$. Let A be an element of the Lie algebra of G, and let A_E, A_F denote the first order differential operators defined by the action of A on E, F, respectively. The symbol $\sigma(A_E)$ is given by

$$\sigma(A_E)\xi = A(\xi)I_E ,$$

where I_E is the identity of E, and $A(\xi)$ is inner product of the cotangent vector $\xi \in (TX)_x$ with the tangent vector A_x defined by the action of A on X. Hence

$$\sigma(P)\sigma(A_E) = \sigma(A_F)\sigma(P) ,$$

and so, by a basic property of pseudo-differential operators,

$$PA_E - A_FP \in \mathcal{P}^m .$$

For A in a bounded neighborhood of zero in the Lie algebra, it follows that we can find constants C_s so that

$$\| PA_E - A_FP \|_s^m < C_s .$$

Now, for $g_t = \exp tA$ and $u \in \mathcal{D}(E)$, we put

[24] The proof of (5.5) that follows we owe to L. Hörmander. In fact, the proof in [10] of the invariance of pseudo-differential operators under diffeomorphism also gives a certain uniformity which essentially includes (5.5) as a special case.

$$f_t = g_t(P)u = \exp tA_F \circ P \circ \exp(-tA_E)u \ .$$

Then

$$\left\| \frac{df_t}{dt} \right\|_{m-s} = \| \exp tA_F \circ (PA_E - A_F P) \circ \exp(-tA_E)u \|$$
$$< C_s \| u \|_s \ ,$$

since G can be assumed to act unitarily on all H_s spaces. Hence

$$\| f_t - f_0 \|_{m-s} \leq C_s t \| u \|_s$$

establishing the continuity of $P \mapsto g(P)$ at the identity element of G. Since each g acts continuously on $\bar{\mathcal{P}}^m$, the continuity everywhere follows.

In view of (5.5), if $P \in \bar{\mathcal{P}}^m$, and if dg denotes normalized Haar measure on G $\left(\text{so that } \int_G dg = 1\right)$, we can form the average

$$\text{Av}(P) = \int_G g(P)dg$$

and this will still be in[25] $\bar{\mathcal{P}}^m$.

Since $\sigma: \bar{\mathcal{P}}^m \to \overline{\text{Symb}^m}$ is continuous, it follows at once that we have

$$\sigma \, \text{Av}(P) = \text{Av}(\sigma P) \ . \tag{5.6}$$

Similar results hold for the closure $\bar{\mathcal{P}}^m_s$ in each H_s-space.

6. The index of elliptic operators

Let X be a *compact* manifold, E and F smooth vector bundles over X. We recall that an operator $P \in \mathcal{P}^m(X; E, F)$ is said to be *elliptic* of order m if $\sigma(P)$ is invertible. One can then construct $Q \in \mathcal{P}^{-m}(X; F, E)$ such that $PQ - 1$ and $QP - 1$ both have smooth kernels (and in particular are compact). This then leads at once to the basic results on elliptic operators.

(6.1) *P has closed range,* Ker P *and* Coker P *are finite-dimensional and all distributional solutions of P and its adjoint are C^∞,* i.e. Ker P = Ker P_s *and* Coker $P \cong$ Coker $P_s \cong$ Ker P_s^* *for all s.*

The index of P is defined to be

$$\text{index } P = \dim \text{Ker } P - \dim \text{Coker } P \ .$$

In view of the last part of (6.1) we also have

$$\text{index } P = \text{index } P_s \qquad \text{for all } s \ .$$

For operators P in $\bar{\mathcal{P}}^m_s$ with invertible symbols, we can, using (5.2), find $Q_{s-m} \in \bar{\mathcal{P}}^{-m}_{s-m}(X; F, E)$ such that $QP - 1$ and $PQ - 1$ are both compact. This

[25] Actually it is true that $P \in \mathcal{P}^m \Rightarrow \text{Av}(P) \in \mathcal{P}^m$, but this requires a little more work and is not necessary for our purposes.

implies that P is a Fredholm operator (from H_s to H_{s-m}), i.e., that it has closed range and Ker P, Coker P both have finite dimension. Index P is therefore defined for all elliptic $P \in \bar{\mathcal{P}}^m_s$. By standard properties of Fredholm operators in Hilbert space the index is a continuous (and so locally constant) function, and it is unchanged by addition of compact operators. Together with (5.2), this implies

(6.2) *The index depends only on the homotopy class of the symbol in the space of continuous invertible symbols of given order.*

In fact the order of an operator is not significant for purposes of the index as is shown by

(6.3) *If P, Q are elliptic operators in $\mathcal{P}^m, \mathcal{P}^n$ such that $\sigma(P)$ and $\sigma(Q)$ coincide on the unit sphere bundle of TX (for some metric), then* index P = index Q.

To prove (6.3), we observe that $\sigma(P)/\sigma(Q)$ is self-adjoint, and so we can find a self-adjoint elliptic R so that

$$\sigma(P) = \sigma(Q)\sigma(R) .$$

This implies that index P = index Q + index R = index Q, since index $R = 0$ for a self-adjoint operator.

Finally let us add two trivial formal properties of the index

(6.4) index $P \oplus Q$ = index P + index Q

(6.5) index $P = 0$ if $P: \mathcal{D}(X; E) \to \mathcal{D}(X; F)$ is induced by a bundle isomorphism $E \to F$ over X.

If $P \in \bar{\mathcal{P}}^m_s(X, E, F)$ is elliptic, then $\sigma(P)$ defines, as in § 2, an element of $K(TX)$. Moreover our description of $K(TX)$ in § 2 by homogeneous complexes (of length one) and the properties (6.2) – (6.5) of the index above show that index P *depends only on the class of* $\sigma(P)$ in $K(TX)$, and that $P \mapsto$ index P induces a homomorphism

$$K(TX) \longrightarrow \mathbf{Z} .$$

This will be called the *analytical index* and denoted by a-ind. Note that it does not depend on the integers m, s chosen above. The independence of m follows from (6.3) and the independence of s from the fact that $\mathcal{P}^m$ is dense in $\bar{\mathcal{P}}^m_s$, that we have regularity (6.1) for elliptic operators in $\mathcal{P}^m$, and that the index is continuous.

Let us now return and consider an elliptic operator $P \in \bar{\mathcal{P}}^m$, so that P_s is a Fredholm operator for all s. Since $\sigma(P_s)$ is independent of s, it follows from what has been said above, that index P_s is independent of s. But

$$\text{index } P_s = \dim \text{Ker } P_s - \dim \text{Ker } P^*_{m-s} .$$

Since $H_s \supset H_{s+1}$ and $H_{m-s} \subset H_{m-s-1}$, it follows that index P_s is a monotone decreasing function of s, and so can be constant only if Ker P_s and Ker P^*_{m-s} are independent of s. Hence we have established

(6.6) *Let $P \in \bar{\mathcal{P}}^m$ be elliptic, then all distributional solutions of $Pu = 0$ are C^∞ and the same holds for the adjoint P^*.*

This result, together with the fact that each P_s has closed range, easily implies that P has closed range; if f is in the closure of the range of P, it is in the closure of the range of P_s, and so $f = P_s g_s$ for $g_s \in H_s$. But (6.6) then implies $g_s - g_t \in C^\infty$ for all s, t, and so $g_s \in C^\infty$. Thus we can again compute index P_s on C^∞ sections,

$$\text{index } P_s = \text{index } P = \dim \text{Ker } P - \dim \text{Coker } P .$$

Finally we consider the situation of a group action. If the compact Lie group G acts smoothly on X and on vector bundles over X, and if $P \in \bar{\mathcal{P}}^m$, then we saw in § 5 that we can average over G to obtain $\text{Av}(P) \in \bar{\mathcal{P}}^m$, and that averaging commutes with taking symbols. In particular if $\sigma(P)$ is invariant, then

$$\sigma(\text{Av}\, P) = \sigma(P) .$$

Similar results hold for $P \in \bar{\mathcal{P}}^m_s$.

It remains now to show that a G-invariant Fredholm operator $P: H \to H'$, where H and H' are Hilbert spaces acted on by G, has an index in $R(G)$ with the usual properties. The definition of index P is clear, we put

$$\text{index } P = [\text{Ker } P] - [\text{Coker } P] \in R(G) ,$$

which makes sense because Ker P and Coker P are finite-dimensional G-modules. To prove that this is locally constant for the norm topology of P, we proceed as follows.[20] Choose $V \subset H$ to be any G-invariant closed subspace of finite co-dimension with $V \cap \text{Ker } P = 0$ (for example $V = (\text{Ker } P)^\perp$), and let $\tilde{P}: H/V \to H'/P(V)$ be induced by P. From the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & H & \longrightarrow & H/V & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow P & & \downarrow \tilde{P} & & \\ 0 & \longrightarrow & P(V) & \longrightarrow & H' & \longrightarrow & H'/P(V) & \longrightarrow & 0 , \end{array}$$

we deduce G-isomorphisms Ker $P \cong$ Ker $\tilde{P}$, Coker $P \cong$ Coker $\tilde{P}$. Hence

$$\begin{aligned} \text{index } P &= [\text{Ker } P] - [\text{Coker } P] = [\text{Ker } \tilde{P}] - [\text{Coker } \tilde{P}] \\ &= [H/V] - [H'/P(V)] \end{aligned}$$

[20] We are simply giving here a proof of the invariance of the usual index which extends naturally to group actions.

by a simple property of the index for finite-dimensional spaces. Now the map $(PV)^\perp \oplus V \to H'$ given by $x \oplus y \mapsto x + Qy$ is an isomorphism for $Q = P$, and so also an isomorphism for $\| P - Q \|$ small. Hence, for such Q, we have $V \cap \operatorname{Ker} Q = 0$ and $(PV)^\perp \cong H'/Q(V)$. Hence introducing $\tilde{Q}: H/V \to H'/Q(V)$ we find as before

$$\begin{aligned} \text{index } Q = \text{index } \tilde{Q} &= [H/V] - [H'/Q(V)] \\ &= [H/V] - [(PV)^\perp] \\ &= \text{index } P \,. \end{aligned}$$

Thus index $P \in R(G)$ is locally constant.[27] If K is compact and G-invariant, the homotopy $P + tK$ with $0 \leqq t \leqq 1$ then shows that index P = index $(P + K)$.

As has already been indicated in the introduction, our main theorem will be

THEOREM (6.7). *The analytical index and the topological index coicide as homomorphisms* $K_G(TX) \to R(G)$.

This theorem gives in principle a complete topological answer to the problem of computing the index of G-invariant elliptic operators. Alternative and more explicit methods of computing the topological index will be derived in papers II and III.

In view of the axiomatic characterizations of the topological index given in § 4 we have only to show that the analytical index verifies the appropriate axioms. The fact that

$$\text{a-ind: } K_G(TX) \longrightarrow R(G)$$

is functorial for G-diffeomorphisms of X, and for homomorphisms of groups $G \to G'$, is immediate from the naturality of the construction.

Axiom (A1) is quite trivial for the analytical index. In fact an elliptic operator P on a point is just a G-linear map $P: V \to W$ of finite-dimensional G-modules, and we have

$$[\sigma(P)] = [V] - [W] = \text{index } P \in R(G) \,.$$

The verification of the remaining axioms will be carried out in § 8 and § 9.

7. Elliptic complexes

In this section we digress briefly to discuss the notion of an *elliptic complex*. Let X be a compact manifold, E^i a sequence of smooth vector bundles over X, and

[27] Note that we did not really use the compactness of G. Everything holds for *unitary* representations of any group G provided we interpret $R(G)$ as the group generated by characters of finite-dimensional representations of G.

$$d_i: \mathfrak{D}(X; E^i) \longrightarrow \mathfrak{D}(X; E^{i+1})$$

a pseudo-differential operator of order m with symbol σ_i. The sequence

$$0 \longrightarrow \mathfrak{D}(E^0) \xrightarrow{d_0} \mathfrak{D}(E^1) \longrightarrow \cdots \xrightarrow{d_{n-1}} \mathfrak{D}(E^n) \longrightarrow 0$$

is called an *elliptic complex* (of order m) if

(i) $d_{i+1}d_i = 0$,

(ii) the symbol complex

$$0 \longrightarrow \pi^*E^0 \xrightarrow{\sigma_0} \pi^*E \longrightarrow \cdots \xrightarrow{\sigma_{n-1}} \pi^*E^n \longrightarrow 0$$

over the cotangent bundle TX is exact outside the zero-section.

In [4] it is shown that[28] the homology groups $H^i(E) = \operatorname{Ker} d_i/\operatorname{Im} d_{i-1}$ are finite-dimensional, and so the Euler characteristic

$$\chi(E) = \sum (-1)^i \dim H^i(E)$$

is defined. If a compact group G acts on E (commuting with the d_i), then $H^i(E)$ is a G-module, and we can define $\chi(E)$ as an element of $R(G)$.

If the complex is of length one, then we have just one elliptic operator d_0 and $\chi(E)$ becomes just the index of d_0. Thus an elliptic complex is a natural generalization of a single elliptic operator. Moreover elliptic complexes occur naturally in differential geometry; the two significant examples being the de Rham complex and its complex analogue, the Dolbeault complex.

The symbol complex $\sigma(E)$ of E is, in the terminology of § 2, a compactly supported homogeneous complex over TX, and so it defines an element of $K(TX)$ or of $K_G(TX)$ in the group situation. The problem of computing $\chi(E)$ in terms of the class of $\sigma(E)$ in $K_G(TX)$ can be reduced to the problem of the index of a single operator by the following simple device.

Introduce G-invariant metrics in all the bundles and on X so that we obtain adjoints d_i^* for the d_i. Now consider the single operator

$$D: \mathfrak{D}(\textstyle\bigoplus_i E^{2i}) \longrightarrow \mathfrak{D}(\bigoplus_i E^{2i+1})$$

defined by $D = d + d^*$. More precisely

$$D(u_0, u_2, \cdots) = (d_0u_0 + d_1^*u_2, d_2u_2 + d_3^*u_4, \cdots) .$$

Since $d^2 = 0$, we have $(d^*)^2 = 0$, and so

$$D^*D = \textstyle\bigoplus_i \Delta_{2i} , \qquad DD^* = \bigoplus_i \Delta_{2i-1} ,$$

where Δ_i denotes the "laplacian" on $\mathfrak{D}(E_i)$ given by

$$\Delta_i = d_{i-1}d_{i-1}^* + d_i^*d_i .$$

[28] Actually in [4] the degrees of the d_i are not assumed to be the same. Here for simplicity we have made this assumption.

Now the exactness of the symbol complex $\sigma(E)$ (off the zero-section) implies that

$$\sigma(\Delta_i) = \sigma_{i-1}\sigma_{i-1}^* + \sigma_i^*\sigma_i \in \mathrm{Hom}\,(\pi^*E^i, \pi^*E^i)$$

is an isomorphism (off the zero-section). Hence Δ_i, and so also D, is elliptic. Using the regularity properties of elliptic operators, it then follows, as in the usual Hodge theory, that

$$\mathrm{Ker}\, D = \bigoplus_i H^{2i} \cong \bigoplus_i H^{2i}(E)$$
$$\mathrm{Coker}\, D = \bigoplus_i H^{2i+1} \cong \bigoplus H^{2i+1}(E)$$

where H^i stands for the space of "harmonic" sections of E^i, namely $\mathrm{Ker}\,\Delta_i$. Hence

$$\mathrm{index}\, D = \chi(E)\ .$$

Moreover $\sigma(D)$ and $\sigma(E)$ represent the same class in $K_G(TX)$ [1; 2.6.10]. Thus we are reduced to the case of a single operator, and our main theorem (6.7) yields immediately

THEOREM (7.1). *Let E be a G-invariant elliptic complex over the compact G-manifold X, and let $[\sigma(E)] \in K_G(TX)$ be the class of the symbol sequence of E. Then the Euler characteristic*

$$\chi(E) = \sum (-1)^i H^i(E) \in R(G)$$

is given by

$$\chi(E) = \text{t-ind}\,[\sigma(E)]\ .$$

Remark. [3] was concerned with a general endomorphism T of an elliptic complex (not necessarily arising from a compact group), and the introduction of metrics invariant under T is not in general possible. For this reason complexes are needed essentially in [3] but not here.

8. The excision and normalization axioms

In this section, we shall show that the analytical index of § 6 satisfies the excision axiom (B1) and the normalization axiom (B2″) of § 4.

Consider first the excision (B1). Thus we suppose that U is an open G-invariant subset of the compact G-manifold X, and we denote by $j\colon U \to X$ the inclusion. Then j induces

$$j_*\colon K_G(TU) \longrightarrow K_G(TX)\ .$$

Shortly we shall show that any element $a \in K_G(TU)$ is the symbol class of an elliptic G-invariant operator P which is "equal to the identity" outside a compact set. More precisely, this means that $P \in \bar{\mathcal{P}}^0(U; E, F)$ is G-invariant with $[\sigma(P)] = a$,

$$\alpha: E \mid U - L \longrightarrow (U - L) \times \mathbf{C}^n$$
$$\beta: F \mid U - L \longrightarrow (U - L) \times \mathbf{C}^n$$

are G-bundle isomorphisms outside some compact set L, and for $u \in \mathfrak{D}(U-L;E)$, we have

$$Pu = \beta^{-1}\alpha u \,. \tag{8.1}$$

For the moment assume we have such an operator P. Then it extends in an obvious fashion to a G-invariant operator j_*P on X; we extend E, F trivially to bundles j_*E, j_*F on X using α and β, and then extend P outside U by (8.1). Clearly we have

$$[\sigma j_*(P)] = j_*[\sigma(P)] = j_*(a) \in K_c(TX) \,.$$

On the other hand, if $u \in \mathfrak{D}(X; j_*E)$, (8.1) shows that

$$(j_*P)\, u = 0 \Longrightarrow \operatorname{supp} u \subset U \quad \text{and} \quad Pu = 0 \,.$$

Thus $\operatorname{Ker} P \cong \operatorname{Ker} j_*P$, and similarly for the adjoint P^*. Hence

$$\operatorname{index}^X j_*P = [\operatorname{Ker} P] - [\operatorname{Ker} P^*] \in R(G) \,.$$

This shows that $\operatorname{index}^X j_*P$ can be computed from the operator P on U, and so does not depend on X. This is axiom (B1).

It remains now to show how to construct the operator P. As shown in § 2, we can represent $a \in K_c(TU)$ as the class of a complex over TU

$$0 \longrightarrow \pi^*E \xrightarrow{\sigma} \pi^*F \longrightarrow 0 \,,$$

where E, F are vector bundles on U, σ is homogeneous of degree zero and, outside a compact set L_1 of U, we have isomorphisms

$$\alpha: E \mid U - L_1 \longrightarrow (U - L_1) \times \mathbf{C}^n$$
$$\beta: F \mid U - L_1 \longrightarrow (U - L_1) \times \mathbf{C}^n \,,$$

such that $\sigma = \pi^*(\beta^{-1}\alpha)$. Moreover, everything is G-invariant, and we can assume σ is smooth. Since the construction of a pseudo-differential operator with given symbol is done locally and then globalized by partitions of unity, it is clear that we can find $P_1 \in \mathcal{P}^0(U; E, F)$ with $\sigma(P_1) = \sigma$, and such that P_1 is equal to the identity (or precisely is induced by $\beta^{-1}\alpha$) outside some (G-invariant) compact $L \supset L_1$. The required operator P is then the average $\operatorname{Av}(P_1)$. Strictly speaking we have only shown that averaging preserves the class $\bar{\mathcal{P}}^0$ on *compact* manifolds. Since U is non-compact, we should either extend the proof to non-compact manifolds (which is not hard), or we can argue as follows. We can certainly form $\operatorname{Av}(P_1)$ in some weak sense (e.g., using the distribution topology of kernels). On the other hand, because P_1 is the identity outside a compact set, it is clear that averaging over G will

commute with extension from X to U, that is

$$\mathrm{Av}(j_*P_1) = j_*\mathrm{Av}(P_1) .$$

Since X is compact, $\mathrm{Av}(j_*P_1) \in \bar{\mathcal{P}}^0(X; j_*E, j_*F)$. Hence its restriction to U, namely $\mathrm{Av}(P_1)$, belongs to $\bar{\mathcal{P}}^0(U; E, F)$.

We now verify (B2″). For parts (i) and (ii) we have to consider the de Rham complex of exterior differential forms on the n-sphere for $n = 1, 2$; the symbol class of this is just the element ρ_{S^n} of Axiom (B2″). If we are prepared to use the de Rham theorems, which assert that the cohomology of this complex is naturally isomorphic to the ordinary cohomology groups $H^q(S^n, \mathbf{C})$, then (i) and (ii) of (B2″) become obvious. In fact we have

$$H^q(S^n, \mathbf{C}) = 0 \qquad \text{for } 0 < q < n$$
$$\dim H^0(S^n, \mathbf{C}) = \dim H^n(S^n, \mathbf{C}) = 1 .$$

The connected group $SO(2)$ acts trivially on $H^q(S^2, \mathbf{C})$, and so

$$\text{a-ind } \rho_{S^2} = 2 \in R(SO(2)) ,$$

proving (i). On the other hand, the generator of $O(1)$ changes the orientation of S^1, and so acts as -1 on $H^1(S^1, \mathbf{C})$ (but trivially on $H^0(S^1, \mathbf{C})$). Thus

$$\text{a-ind } \rho_{S^1} = 1 - \xi \in R(O(1))$$

proving (ii).

A direct proof without appealing to the de Rham theorems (or to the cohomology of spheres!) is however quite easy. For the circle, the de Rham complex is just $f \mapsto df = (df/dx)\cdot dx$ where $x \bmod 1$ is a parameter for the circle. Hence $\mathrm{Ker}\, d$ consists of the constant functions, and $\mathrm{Coker}\, d$ is generated by dx. The generator of $O(1)$ is the map $x \mapsto -x$ which induces $dx \mapsto -dx$ on $\mathrm{Coker}\, d$ and, of course, the identity on $\mathrm{Ker}\, d$. Thus

$$\text{a-ind } \rho_{S^1} = 1 - \xi .$$

For S^2, the de Rham complex has three terms:

$$0 \longrightarrow \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \longrightarrow 0 .$$

Again $\mathrm{Ker}\, d^0$ consists of constants. Taking adjoints, we see that $\mathrm{Coker}\, d^1$ is generated by the volume form in Ω^2. Since $SO(2)$ acts trivially on both these spaces, it remains to show that there is no contribution from Ω^1, i.e., that

$$dw = 0 \Longrightarrow w = df \qquad w \in \Omega^1, f \in \Omega^0 .$$

Now[29] outside the north pole, the Poincaré lemma shows that $w = df_0$, and similarly $w = df_\infty$ outside the south pole. Then $d(f_0 - f_\infty) = 0$, and so

[29] The proof that follows is essentially a specially simple version of the proof of the de Rham theorems.

$f_0 - f_\infty =$ constant. Thus f_0 is actually defined everywhere, i.e., $f_0 \in \Omega^0$.

We come now to part (iii) of Axiom (B2''). Here the operator whose index we have to compute is not a differential operator, and the whole problem in this case is a matter of sign conventions.[30] In fact, if we make use of the multiplicative Axiom (B3'') we can compute $(\text{a-ind } j_!(1))^2$ on $S^1 \times S^1$ (or better on S^2). Using part (i) of (B2'') which we have just verified, one finds

$$(\text{a-ind } j_!(1))^2 = 1 ,$$

so that a-ind $j_!(1) = \pm 1$, and the whole problem is to show that, *with our choice of sign conventions*, we get the plus sign. This we now proceed to do.

Consider first the operator P on the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ defined by

$$\begin{aligned} Pe^{inx} &= e^{inx} && \text{for } n \geqq 0 \\ &= 0 && \text{for } n < 0 . \end{aligned}$$

We shall show that $P \in \mathcal{P}^0$. Let $f \in \mathcal{D}(\mathbf{R})$ have support in an interval of length $< 2\pi$ (which we identify with its image in S^1). The Fourier coefficients of $f(x)e^{ix\xi}$ are $\hat{f}(n - \xi)$. Hence, in supp f,

$$\begin{aligned} p_f(x, \xi) = e^{-ix\xi} P\big(f(x)e^{ix\xi}\big) &= \textstyle\sum_0^\infty \hat{f}(n - \xi)e^{ix(n-\xi)} \\ &= f(x) - \textstyle\sum_{-1}^{-\infty} \hat{f}(n - \xi)e^{ix(n-\xi)} . \end{aligned}$$

As $\xi \longrightarrow -\infty$, the sum $\sum_0^\infty$ converges to 0 faster than any power of $|\xi|$, and so do all its derivatives with respect to x and ξ. When $\xi \longrightarrow +\infty$, the sum $\sum_{-1}^{-\infty}$ has the same properties. Thus P is pseudo-differential of order zero. Since

$$\begin{aligned} p_f(x, \xi) &\longrightarrow f(x) && \text{as } \xi \longrightarrow +\infty \\ &\longrightarrow 0 && \text{as } \xi \longrightarrow -\infty \end{aligned}$$

it follows that $P \in \mathcal{P}^0$, and that its symbol σ_P is given by

$$\begin{aligned} \sigma_P(x, \xi) &= 1 && \text{for } \xi > 0 \\ &= 0 && \text{for } \xi < 0 . \end{aligned}$$

Now define $A = e^{ix}P + (1 - P)$. This is then an operator in $\mathcal{P}^0$, and its symbol is given by

$$\begin{aligned} \sigma_A(x, \xi) &= e^{ix} && \text{for } \xi > 0 \\ &= 1 && \text{for } \xi < 0 . \end{aligned} \tag{8.2}$$

A is therefore elliptic. On the other hand, by definition

$$\begin{aligned} Ae^{inx} &= e^{i(n+1)x} && \text{for } n \geqq 0 \\ &= e^{inx} && \text{for } n < 0 , \end{aligned}$$

[30] See [14, p. 281] for a discussion of the various places where a convention of sign is involved. Because we have no cohomology so far, our sign problem is a little less acute than in [14].

and so Ker $A = 0$ and Coker A is generated by the constant functions. Thus index $A = -1$.

To establish axiom (B2″) of the analytical index, it remains therefore to show that the natural homomorphism

$$K(T\mathbf{R}^1) \longrightarrow K(TS^1) \tag{8.3}$$

takes the element $-j_1(1)$ into the class of σ_A in $K(TS^1)$. To do this, we define a continuous symbol σ on R^1 by equation (8.2) for $0 \leqq x \leqq 2\pi$, and by $\sigma = 1$ for $x < 0$ or $x > 2\pi$. It is then clear that $[\sigma] \to [\sigma_A]$ in the homomorphism (8.3). Thus we have finally to check that

$$[\sigma] = -j_1(1) \in K(T\mathbf{R}^1) .$$

Now both these elements are defined by maps

$$T\mathbf{R}^1 - (\text{compact set}) \longrightarrow \mathbf{C} - (0) .$$

For σ, the compact set is the rectangle $0 \leqq x \leqq \pi$, $|\xi| \leqq 1$, while for $j_1(1)$, it is the disc $|x + i\xi| \leqq 1$.

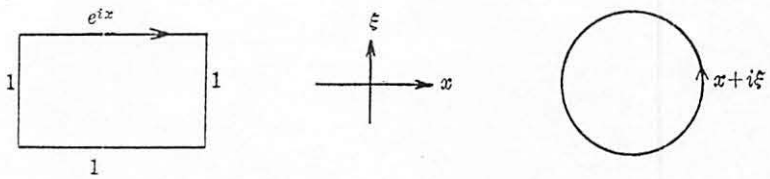


FIG. 3

On the rectangle, σ is e^{ix} on the top and 1 elsewhere, while $j_1(1)$ is $x + i\xi$ on the unit circle. An elementary homotopy then deforms σ into $-j_1(1)$ and the verification of Axiom (B2″) is then complete.

9. The multiplicative axiom

In this section, we show that the analytical index satisfies the multiplicative axiom (B3) (and hence also the weaker axiom (B3′)). Once this has been done we shall have established all the axioms necessary to apply Theorem (4.6), and deduce that the analytical and topological indices coincide. Thus our main theorem (6.7) will be proved.

We start by recalling the context of axiom (B3). Thus let $Y = P \times_H F$ be a fibre bundle over a compact manifold X where $P \to X$ is a principal bundle with compact Lie group H, and F is a compact manifold on which H acts on the left. In addition a second compact Lie group G acts on both P and F commuting with the action of H. Hence G acts on $X = P/H$ and $Y = P \times_H F$. In what follows, metrics are chosen on P, X, F, Y, and assorted

vector bundles over these spaces so as to be invariant under G and/or H.

Axiom (B3) is concerned with two elements

$$a \in K_G(TX) , \qquad b \in K_{G\times H}(TF)$$

and their product, in a suitable sense, which is an element

$$ab \in K_G(TY) .$$

What we have to do is to calculate the analytic index of ab in terms of the analytical indices of a and b. For this purpose we must first represent a, b, and ab, by elliptic operators. This we proceed to do rather carefully.

Consider first $a \in K_G(TX)$. We choose a representative smooth symbol α of order[31] 1 (and G-invariant). Let $A_1 \in \mathcal{P}^1$ have symbol α, but for the moment we do not insist that A_1 be G-invariant. Now take a covering $\{U_j\}$ of X with trivializations of P, and hence of Y, over each U_j. Let $\{\varphi_j^2\}$ be a smooth partition of unity on X for this covering, and consider the operator $A_1^j = \varphi_j A_1 \varphi_j$ on U_j. Let $Y_j = p^{-1}(U_j) \cong U_j \times F$, where p: $Y \to X$ is the projection, then the lifted operator $\tilde{A}_1^j$ on Y_j belongs by (5.4) to the class $\bar{\mathcal{P}}^1(Y_j)$. Because of the φ_j it can also be regarded as an operator on Y, and so it belongs to $\bar{\mathcal{P}}^1(Y)$. Now form the average over G

$$\tilde{A} = \mathrm{Av}(\sum \tilde{A}_1^j) \in \bar{\mathcal{P}}^1(Y) .$$

Since $\sigma(\sum A_1^j) = \sum \varphi_j^2 \sigma(A_1) = \alpha$, and since taking symbols commutes both with lifting (5.4) and with averaging (5.6), it follows that

$$\sigma(\tilde{A}) = \tilde{\alpha} .$$

Note that because we have chosen invariant metrics everywhere, the lifting $\tilde{\alpha}$ of the symbol α is globally well-defined: if we split the cotangent space of Y into horizontal components ξ and vertical components η, then $\tilde{\alpha}(\xi, \eta) = \alpha(\xi)$ (where we identify ξ with the corresponding vector on the base X).

If we restrict $\tilde{A}_1^j$ to sections coming from the base, i.e., constant on the fibres of Y, we recover the original operator A_1^j. Hence the restriction of $\tilde{A}$ to such sections is just the G-invariant elliptic operator

$$A = \mathrm{Av}(\sum A_1^j) \in \bar{\mathcal{P}}^1(X)$$

with symbol α.

We turn next to the element $b \in K_{G\times H}(TF)$. We choose a $G \times H$-invariant operator $B \in \bar{\mathcal{P}}^1(F)$ with $\beta = \sigma(B)$ in the class b. Let $\tilde{B}_1$ be the operator on $P \times F$ obtained by lifting B. Since it is $G \times H$-invariant, it induces a G-invariant operator $\tilde{B}$ on $Y = P \times_H F$. We simply restrict $\tilde{B}_1$ to sections constant along the fibres of $P \times F \to Y$. Since P is locally a product, it

[31] As promised in § 5 we are only going to apply (5.4) for $m = 1$.

follows that, over the open sets $Y_j = p^{-1}(U_j) \cong U_j \times F$, the restriction $\tilde{B}_j$ of $\tilde{B}$ is just the lifting of B. It then follows from (5.4) that $\tilde{B}_j \in \bar{\mathcal{P}}^1(Y_j)$, and so $\tilde{B} \in \bar{\mathcal{P}}^1(Y)$. Let $\tilde{\beta} = \sigma(\tilde{B})$. It is invariantly defined by β, and in terms of horizontal (ξ) and vertical (η) components, we have $\tilde{\beta}(\xi, \eta) = \beta(\eta)$.

Finally we come to the element ab. For this we take the operator

$$D = \begin{pmatrix} \tilde{A} & -\tilde{B}^* \\ \tilde{B} & \tilde{A}^* \end{pmatrix} \in \bar{\mathcal{P}}^1(Y) .$$

It is G-invariant (since $\tilde{A}$ and $\tilde{B}$ are), and we have

$$\sigma(D) = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta}^* \\ \tilde{\beta} & \tilde{\alpha}^* \end{pmatrix} .$$

Remark. If $A \in \bar{\mathcal{P}}^1(X; E^0, E^1)$ and $B \in \bar{\mathcal{P}}^1(F; G^0, G^1)$, then

$$D \in \bar{\mathcal{P}}^1(Y; E^0 \boxtimes G^0 \oplus E^1 \boxtimes G^1, E^1 \boxtimes G^0 \oplus E^0 \boxtimes G^1) ,$$

and the two liftings of A, which occur in D, are liftings to different bundles. Thus the $\tilde{A}$ in the top-left is the lifting to G^0, while $\tilde{A}^*$ in the bottom right is (adjoint of) the lifting to G^1. However the position in the matrix makes clear which bundles are involved, and to simplify the notation we shall omit all reference to the actual bundles involved.

As explained in § 2, $\sigma(D)$ is a representative for the class ab. What we have to do now is to compute the index of the elliptic operator D.

We first show that $\tilde{A}$ and $\tilde{B}$ commute. From the local product representation $Y_j \cong U_j \times F$, it follows that $\tilde{B}_j$ commutes with $\tilde{A}_1^j$. Thus $\tilde{B}$ commutes with $\sum \tilde{A}_1^j$ and, because $\tilde{B}$ is G-invariant, it also commutes with $\tilde{A} = \mathrm{Av}(\sum \tilde{A}_1^j)$. Similar remarks apply with $\tilde{B}^*$ for $\tilde{B}$, and so the off-diagonal terms in D commute with the diagonal terms. Hence

$$D^*D = \begin{pmatrix} \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}\tilde{A}^* + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} P_0 & 0 \\ 0 & Q_0 \end{pmatrix} \quad \text{say}$$

$$DD^* = \begin{pmatrix} \tilde{A}\tilde{A}^* + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}^*\tilde{A} + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & Q_1 \end{pmatrix}$$

Computing the kernels (on smooth sections), it follows that

$$\operatorname{Ker} D = \operatorname{Ker} D^*D = \operatorname{Ker} P_0 \oplus \operatorname{Ker} Q_0$$
$$\operatorname{Ker} D^* = \operatorname{Ker} DD^* = \operatorname{Ker} P_1 \oplus \operatorname{Ker} Q_1 .$$

But by (6.6) (regularity for elliptic operators in $\bar{\mathcal{P}}^m$), it follows that we have $\operatorname{Ker} D^* \cong \operatorname{Coker} D$ and so

$$\operatorname{index}^V D = (\operatorname{Ker} P_0 - \operatorname{Ker} P_1) + (\operatorname{Ker} Q_0 - \operatorname{Ker} Q_1) \in R(G) .$$

Consider now the operator $P_0 = \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B}$ (on smooth sections). For any smooth section u we have

$$\langle P_0 u, u\rangle = \langle \tilde{A}u, \tilde{A}u\rangle + \langle \tilde{B}u, \tilde{B}u\rangle ,$$

and so $\operatorname{Ker} P_0 = \operatorname{Ker}\tilde{A} \cap \operatorname{Ker}\tilde{B}$. Since $\tilde{B}$ is the natural extension of the operator B on the fibres, it follows that $\operatorname{Ker}\tilde{B}$ consists of those smooth sections which, on each fibre Y_x, lie in $\operatorname{Ker} B_x$ (where B_x denotes the operator on Y_x corresponding to B on the standard fibre F). But this is just another way of saying that $\operatorname{Ker}\tilde{B}$ is the space of smooth sections of the vector bundle $K_B = P \times_H \operatorname{Ker} B$ over X. Since $\tilde{A}$ and $\tilde{B}$ commute, it follows that $\tilde{A}$ induces an operator C on the sections of K_B. Since $\tilde{A} = \operatorname{Av}(\sum \varphi_j \tilde{A}_1^j \varphi_j)$, we have $C = \operatorname{Av}(\sum \varphi_j C_j \varphi_j)$ where C_j is the operator induced on $K_B | U_j$ by $\tilde{A}_1^j$. But, by definition of $\tilde{A}_1^j$, this means that $C_j = A_1 \otimes \operatorname{Id}(K_B)$. Hence $C_j \in \mathcal{P}^1$, $C \in \bar{\mathcal{P}}^1$ and

$$\sigma(C) = \alpha \otimes \operatorname{Id}(K_B) .$$

Thus C is a G-invariant elliptic operator over X, and the class of $\sigma(C)$ in $K_G(TX)$ is the product $a[K_B]$ of the class $a \in K_B(TX)$ of the symbol α and the class $[K_B] \in K_G(X)$ of the vector bundle K_B. Since $\tilde{A}^* = (A^*)^{\sim}$, replacing $\tilde{A}$ by $\tilde{A}^*$ replaces P_0 by P_1 and C by C^*. Hence

$$\operatorname{Ker} P_0 - \operatorname{Ker} P_1 = \operatorname{Ker} C - \operatorname{Ker} C^* = \text{a-ind}^X a[K_B] \in R(G) .$$

Replacing B by B^*, we similarly get

$$\operatorname{Ker} Q_1 - \operatorname{Ker} Q_0 = \text{a-ind}^X a[L_B] \in R(G) ,$$

where $L_B = P \times_H \operatorname{Coker} B$. Hence finally we obtain

$$\operatorname{index}^Y D = \text{a-ind}^X \left(a([K_B] - [L_B])\right) = \text{a-ind}^X \left(a \cdot \mu_P(\text{a-ind}_{G\times H^F} b)\right) \in R(G) .$$

Since the class of $\sigma(D)$ is the product ab, we have therefore established Axiom (B3) for the analytical index. The proof of Theorem (6.7) is thus complete.

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The index of elliptic operators: II

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Introduction

The purpose of this paper is to show how the index theorem of [5] can be reformulated as a general "Lefschetz fixed-point theorem" on the lines of [2]. In this way we shall obtain the main theorem of [2], generalized to deal with arbitrary fixed-point sets, but only for transformations belonging to a compact group.

The content of this paper is essentially topological, and it should be viewed as a paper on the equivariant K-theory of manifolds. The analysis has all been done in [5], and what we do here is simply to express the topological index in terms of fixed-point sets. This is quite independent of the main theorem of [5] asserting the equality of the topological and analytical indices.

As in [5], we avoid cohomology and use only K-theory. In paper III of this series, we shall pass over to cohomology obtaining explicit formulas in terms of characteristic classes.

The basic result in K_G-theory which leads to the fixed-point formula is what we call the Localization Theorem. We review this in § 1. In § 2, this is then applied to the topological index. Some cases of special interest are then discussed in § 3.

1. The localization theorem

In [5], we reviewed some of the basic facts about the functor $K_G(X)$, defined for G a compact Lie group, and X a locally compact G-space. We did not however introduce the group $K_G^1(X)$ and the exact sequence for K_G-theory which we shall need to use here. Let us recall then [1] that one defines

$$K_G^{-n}(X) = K_G(\mathbf{R}^n \times X)$$

(where G acts trivially on $\mathbf{R}^n$), and that the periodicity theory gives natural isomorphisms

$$K_G^{-n} \cong K_G^{-n-2} .$$

Considering n as an integer mod 2, we then introduce

$$K_G^* = K_G^0 \oplus K_G^1 \qquad (K_G^0 = K_G) .$$

For a compact pair (X, Y), one then has an exact triangle

$$\begin{array}{ccc} & K_G^*(X) & \\ \nearrow & & \searrow \\ K_G^*(X, Y) & \xleftarrow{\delta} & K_G^*(Y) \end{array},$$

where δ interchanges K_G^0 and K_G^1. More generally, for locally compact spaces with Y closed in X, we have an exact triangle

$$\begin{array}{ccc} & K_G^*(X) & \\ \nearrow & & \searrow \\ K_G^*(X - Y) & \xleftarrow{\delta} & K_G^*(Y) \end{array}.$$

This follows at once from the compact case on replacing X, Y by their one point compactifications X^+, Y^+, and observing that

$$K_G^*(X^+, Y^+) = K_G^*(X^+ - Y^+) = K_G^*(X - Y).$$

We recall next that the tensor product induces a pairing

$$K_G(X) \otimes K_G(Y) \longrightarrow K_G(X \times Y)$$

for any two locally compact G-spaces X, Y. In particular, taking $Y =$ point, we see that $K_G(X)$ is a module over $K_G(\text{point}) = R(G)$, the character ring of G. Replacing X by $\mathbf{R}^1 \times X$, the same is true for $K_G^1(X)$, and hence for $K_G^*(X)$. It is therefore possible to study $K_G^*(X)$ from the point of view of commutative algebra by analyzing it with respect to the prime ideals of $R(G)$. It is the purpose of this section to review the main results in this direction.

Let γ be a conjugacy class in G. Then it defines a prime ideal in $R(G)$, namely all characters which vanish on γ. For any $R(G)$-module M, we denote by M_γ the module obtained from M by *localizing* at this prime ideal. Thus M_γ is a module over the local ring $R(G)_\gamma$. An element of $R(G)_\gamma$ is a "fraction" u/s with $u, s \in R(G)$ and $s(\gamma) \neq 0$, but two fractions u/s and u'/s' represent the same element of $R(G)_\gamma$ if $\exists t \in R(G)$ with $t(\gamma) \neq 0$ and $tus' = tu's$. Elements of M_γ are "fractions" m/s $(m \in M, s \in R(G), s(\gamma) \neq 0)$ with a similar equivalence relation.

On the other hand if X is a G-space, we can consider the subspace

$$X^\gamma = \bigcup_{g \in \gamma} X^g$$

where X^g denotes the fixed point set of g acting on X. Then X^γ is a *closed*[1] G-subspace of X. The main result is then the following.

LOCALIZATION THEOREM (1.1). *Let γ be a conjugacy class in G, $i: X^\gamma \to X$ the inclusion. Then*

$$i^*: K_G(X) \longrightarrow K_G(X^\gamma)$$

[1] γ is compact and X^γ is the image in X of a closed subspace of $\gamma \times X$ under the projection $\gamma \times X \to X$.

becomes an isomorphism

$$i_\gamma^*: K_G(X)_\gamma \longrightarrow K_G(X^\gamma)_\gamma$$

when localized at the prime ideal of $R(G)$ defined by γ.

The proof of this theorem is given in [7]. However, as it plays a key role in this paper, we shall review the proof. The first step is the following lemma about characters.

LEMMA (1.2). *Let H be a closed subgroup of a compact Lie group G, and let γ be a conjugacy class of G not meeting H. Then there exists $\chi \in R(G)$ such that*

(i) $\chi(\gamma) \neq 0$,

(ii) $\chi(h) = 0$ *for all* $h \in H$.

For general G, the proof of (1.2) is surprisingly difficult. However for the purposes of this paper, it is sufficient to consider *abelian* groups. In this case, (1.2) is a trivial consequence of the fact that characters separate points in the quotient group G/H.

When we localize at the prime ideal of $R(G)$ defined by γ, the element χ of (1.2) becomes (by (i)) a unit of $R(G)_\gamma$. By (ii), this unit annihilates $R(H)_\gamma$; this being viewed as an $R(G)_\gamma$-module in the obvious way. Thus we obtain

COROLLARY (1.3). *In the notation of* (1.2), *we have* $R(H)_\gamma = 0$.

Observe now that, for any conjugacy class γ of G, $R(H)_\gamma$ is itself a ring of fractions of $R(H)$. Also if M is an $R(H)$-module, and so an $R(G)$-module, M_γ is an $R(H)_\gamma$-module. Since our modules are always "unitary" (i.e. the identity of the ring acts as the identity on the module), it follows that $M_\gamma = 0$ whenever $R(H)_\gamma = 0$. This situation arises if γ satisfies the hypothesis of (1.2), and we take $M = K_G^*(X)$ where X is a compact G-space admitting a G-map onto G/H. The factorization $X \to G/H \to$ point gives rise to a factorization

$$\begin{array}{ccccc} K_G^*(X) & \longleftarrow & K_G^*(G/H) & \longleftarrow & K_G^*(\text{point}) \\ & & \uparrow & & \uparrow \\ & & R(H) & \longleftarrow & R(G)\,, \end{array}$$

so that $K_G^*(X)$ is in fact an $R(H)$-module. Thus we deduce

COROLLARY (1.4). *Let γ, G, H be as in* (1, 2), *and let X be a compact G-space admitting a G-map onto G/H. Then*

$$K_G^*(X)_\gamma = 0\,.$$

Remark. If Y is any closed G-subspace of X, then it also admits a G-map $Y \to G/H$, and so $K_G^*(Y)_\gamma = 0$. The exact triangle of (X, Y), together with

the exactness of localization, then shows that $K_G^*(X, Y)_\gamma = 0$.

Suppose now that X is any locally compact G-space, and let $Y \subset X$ be an orbit with isotropy group H. Then *we can always find a closed G-neighborhood V of Y in X with a G-retraction onto Y*. This follows from the existence of a "slice" (cf. [7]); but for our present purposes, we only need to apply this when X is a differentiable G-manifold. In this case Y is a G-submanifold, and we can take V to be a closed tubular neighbourhood defined by a G-invariant riemannian metric. In any case, given that such neighbourhoods V of orbits exist, we can cover any compact G-subspace L of X by a finite number of sets $L_i = V_i \cap L$. Let H_i be the isotropy group connected with V_i, then we have G-maps $L_i \to G/H_i$. Assume now that γ is a conjugacy class of elements of G having *no fixed points* in X, then $\gamma \cap H_i = \varnothing$ for all i, and so $K_G^*(L_i)_\gamma = 0$ by (1.4). A simple induction on the number of L_i using exact sequences, and the remark following (1.5), then shows that $K_G^*(L)_\gamma = 0$. Replacing L by any compact G-subspace L', and then using the exact triangle for (L, L'), we deduce that $K_G^*(L, L')_\gamma = 0$. In particular, if U is an open relatively compact G-subspace of X, we have

$$K_G^*(U)_\gamma = K_G^*(\bar{U}, \partial\bar{U})_\gamma = 0 .$$

Since $K_G^*(X)$ is the direct limit of these $K_G^*(U)$, and since localization commutes with taking direct limits, we deduce

PROPOSITION (1.5). *Let X be a locally compact G-space, γ a conjugacy class of G having no fixed points in X. Then $K_G^*(X)_\gamma = 0$.*

Proposition (1.5) is the special case of (1.2) in which $X^\gamma = \varnothing$. To prove the localization theorem in general, we consider the exact triangle

$$\begin{array}{ccc} & K_G^*(X) & \\ \nearrow & & \searrow^{i^*} \\ K_G^*(X - X^\gamma) & \longleftarrow & K_G^*(X^\gamma) . \end{array}$$

We now localize this at the prime ideal defined by γ, and recall again that localization preserves exactness. Applying (1.5) to the space $X - X^\gamma$ (where γ has no fixed points), we obtain the exact triangle

$$\begin{array}{ccc} & K_G^*(X)_\gamma & \\ \nearrow & & \searrow^{i_\gamma^*} \\ 0 & \longleftarrow & K_G^*(X^\gamma)_\gamma \end{array}$$

which establishes (1.1).

Remark. The proof of (1.1) which we have just given is complete for G *abelian* and X *a differentiable G-manifold* (note that we were careful to

avoid taking the one-point compactification of X). Only this case will be used in the rest of the paper.

2. The Lefschetz formula

Let G be a compact Lie group, X a compact differentiable G-manifold. The tangent bundle TX is then a differentiable G-manifold and we can consider the group $K_G(TX)$. In [5] we defined two $R(G)$-homomorphisms

$$\text{a-ind}: K_G(TX) \longrightarrow R(G)$$
$$\text{t-ind}: K_G(TX) \longrightarrow R(G) .$$

The main theorem of [5] asserts that these two homomorphisms coincide. In this section we shall show how the topological index t-ind can be computed in terms of fixed-point sets. Combined with the main theorem of [5], this will then give a "Lefschetz fixed-point formula" for elliptic operators. For simplicity of notation (and since in any case a-ind = t-ind), we shall omit the prefix and write ind for t-ind. When it is necessary to prevent confusion, we shall write $\text{ind}_G{}^X$ to indicate the space and group involved.

We recall first that in [5] we defined a functorial homomorphism

$$i_!: K_G(TX) \longrightarrow K_G(TY)$$

for any G-embedding $i: X \to Y$. In terms of this the topological index was defined as follows. Let $i: X \to E$ be a G-embedding of X in a real representation space of G, and let $j: P \to E$ be the inclusion of the origin. Then,

$$j_!: K_G(TP) \longrightarrow K_G(TE)$$

is an isomorphism and the topological index

$$\text{ind}: K_G(TX) \longrightarrow K_G(TP) = R(G)$$

is defined by

$$\text{ind} = (j_!)^{-1} \circ i_! .$$

It follows at once from this definition (and the functoriality of $i_!$) that, if $i: Z \to X$ is a G-embedding of the closed submanifold Z, the following diagram commutes

(2.1)
$$\begin{array}{ccc} K_G(TZ) & \xrightarrow{i_!} & K_G(TX) \\ & \searrow^{\text{ind}^Z} \quad \downarrow{\scriptstyle \text{ind}^X} & \\ & R(G) . & \end{array}$$

In addition to the covariant map

$$i_!: K_G(TZ) \longrightarrow K_G(TX) ,$$

there is of course the usual contravariant restriction homomorphism

$$i^*: K_G(TX) \longrightarrow K_G(TZ) .$$

The relation between these is given by the following lemma [5; (3.1)].

LEMMA (2.2). *If $i: Z \to X$ is a G-embedding with normal bundle N, then*

$$i^* i_!: K_G(TZ) \longrightarrow K_G(TZ)$$

is multiplication by

$$\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C}) = \sum (-1)^i \lambda^i (N \otimes_{\mathbf{R}} \mathbf{C}) \in K_G(Z)$$

where the λ^i are the exterior powers, and $K_G(TZ)$ is regarded as a $K_G(Z)$-module in the usual way.

Suppose now that G is a topologically cyclic group, i.e., that it possesses an element g whose powers are dense in G. Then the fixed point set of g in X is fixed by the whole group, i.e.,

$$X^g = X^G .$$

Hence we have[2] (cf. [5; § 2]):

$$K_G(X^g) \cong K(X^g) \otimes R(G) \tag{2.3}$$

and hence, localizing at the prime ideal of $R(G)$ defined by the conjugacy class $\{g\}$,

$$K_G(X^g)_g \cong K(X^g) \otimes R(G)_g . \tag{2.4}$$

The following lemma (valid for any group G) characterizes the units in this ring.

LEMMA (2.5). *Let Y be a compact space on which G acts trivially, p any prime ideal of $R(G)$. Then an element $u \in K_G(Y)_p$ is a unit if and only if its restriction to each point $P \in Y$ is a unit in $K_G(P)_p = R(G)_p$.*

PROOF. Let[3] $H^0(Y; \mathbf{Z})$ denote the group of continuous maps $Y \to \mathbf{Z}$. Then by assigning to each vector bundle on Y its dimension (or rank), we obtain a homomorphism

$$\mathrm{rk}: K(Y) \longrightarrow H^0(Y; \mathbf{Z}) .$$

This splits so, if $K_1(Y) = \mathrm{Ker}\,(\mathrm{rk})$, we have a decomposition

$$K(Y) = K_1(Y) \oplus H^0(Y; \mathbf{Z}) .$$

In [1, (3.1.6)] it is shown that every element of $K_1(Y)$ is nilpotent. Hence an element of

[2] Tensor products are taken over the integers $\mathbf{Z}$ unless othewise indicated.

[3] For our purposes $Y = \sum_{i=1}^{n} Y_i$ will be a finite sum of connected spaces Y_i so that $H^0(Y; \mathbf{Z})$ is a free abelian group with one generator for each component Y_i.

$$K_G(Y)_p \cong K(Y) \otimes R(G)_p \cong \left(K_1(Y) \otimes R(G)_p\right) \oplus \left(H^0(Y; \mathbf{Z}) \otimes R(G)_p\right)$$

is a unit if and only if its image in $H^0(Y; \mathbf{Z}) \otimes R(G)_p$ is a unit. But $H^0(Y; \mathbf{Z}) \otimes R(G)_p$ can be identified with the ring of continuous functions $Y \to R(G)_p$, and an element of this ring is a unit if and only if its value at every point $P \in Y$ is a unit.

Before proceeding further, we must make some remarks about the fixed point set X^g of an element $g \in G$. Since G is compact we can, by averaging over G, assume g is an isometry for some riemannian metric. Suppose $P \in X^g$, and let T_P denote the tangent space to X at P. Then g induces a linear transformation $g \mid T_P$ on T_P. If $\xi \in T_P$ is fixed by $g \mid T_P$, then so is the geodesic in the direction ξ. It follows easily from this that, in a neighbourhood of P, X^g is just the image under the exponential map[4] of the $(+1)$-eigenspace of $g \mid T_P$. Thus X^g is a submanifold of X, and the linear transformation $g \mid N_P$, induced by g on the normal N_P to X^g at P, has no eigenvalue $+1$. In other words

$$\det(1 - g \mid N_P) \neq 0 . \tag{2.6}$$

We shall now apply Lemma (2.5) to deduce:

LEMMA (2.7). *Let G be topologically cyclic generated by g, and let X be a compact G-manifold. Let N^g denote the normal bundle[5] of X^g in X. Then*

$$\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C}) \in K_G(X^g)$$

becomes a unit in $K_G(X^g)_g$.

PROOF. By Lemma (2.5), it is sufficient to restrict to each point $P \in X^g$. Now an element χ of $R(G)$, i.e. a character, becomes a unit in $R(G)_g$ if and only if $\chi(g) \neq 0$. Taking χ to be the restriction of $\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})$ to a point $P \in X^g$ we find[6]

$$\begin{aligned} \chi(g) &= \sum (-1)^i \operatorname{Trace} \lambda^i(g \mid N_P \otimes_{\mathbf{R}} \mathbf{C}) \\ &= \det_{\mathbf{C}}(1 - g \mid N_P \otimes_{\mathbf{R}} \mathbf{C}) \\ &= \det_{\mathbf{R}}(1 - g \mid N_P) \\ &\neq 0 \qquad \text{by (2.6).} \end{aligned}$$

This completes the proof.

We are now ready to prove our key result.

PROPOSITION (2.8). *Let G be topologically cyclic generated by g and let X be a compact G-manifold. Then*

[4] defined by the riemannian metric.

[5] X^g may have components of different dimensions so that N^g will be a vector bundle with different dimensions over the different components of X^g.

[6] We write $\det_{\mathbf{R}}$ and $\det_{\mathbf{C}}$ when necessary to distinguish between determinants of real and complex linear transformations.

$$i_!: K_G(TX^g) \longrightarrow K_G(TX) ,$$

when localized at the prime ideal of $R(G)$ *defined by* g, *becomes an isomorphism*

$$(i_!)_g: K_G(TX^g)_g \longrightarrow K_G(TX)_g .$$

Its inverse is

$$\frac{i_g^*}{\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})}$$

where i_g^* *is obtained by localizing the restriction homomorphism*

$$i^*: K_G(TX) \longrightarrow K_G(TX^g) .$$

PROOF. By the Localization Theorem (2.1), i_g^* is an isomorphism. We need only observe that by (2.6)

$$TX^g = (TX)^g .$$

In other words, the tangent vectors of X fixed by g are precisely the tangent vectors of X^g. By (2.2) the composition $i^* i_!$ is multiplication by $\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})$. Since G acts trivially on TX^g, it follows that

$$K_G(TX^g) \cong K(TX^g) \otimes R(G)$$

and so (if 1 denotes the group with one element),

$$\mathrm{ind}_G^{X^g} \cong \mathrm{ind}_1^{X^g} \otimes \mathrm{Id} .$$

Thus the dependence of $\mathrm{ind}_G^{X^g}$ on G is rather trivial. On the other hand the dependence of ind_G^X on G is not so trivial. However, by localizing (2.1), we get the commutative diagram

$$\begin{array}{ccc} K_G(TX^g)_g & \xrightarrow{(i_!)_g} & K_G(TX)_g \\ & \searrow^{(\mathrm{ind}_G^{X^g})} & \downarrow{\scriptstyle (\mathrm{ind}_G^X)_g} \\ & & R(G)_g \end{array} \tag{2.9}$$

and we know, by (2.8), that $(i_!)_g$ is an isomorphism. This means that ind_G^X, when localized at g, can be computed in terms of $\mathrm{ind}_1^{X^g}$. In fact from (2.8) and (2.9), we get the following precise formula.

PROPOSITION (2.10). *Let* G, g, X *be as in* (2.8), *and let* $u \in K_G(TX)$. *Then we have*[7]

$$(\mathrm{ind}_G^X u)_g = (\mathrm{ind}_G^{X^g})_g \left[\frac{i^* u}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})} \right] .$$

[7] The expression in square brackets is of course to be understood as an element in the localized ring $K_G(TX^g)_g$.

This formula computes the image of $\mathrm{ind}_G^X u$ in the localized ring $R(G)_g$. It is more usual to compute an element of $R(G)$ by regarding it as a function on G and evaluating it at all elements of G. To evaluate it on the generator g we observe that the evaluation map

$$R(G) \longrightarrow \mathbf{C}$$

given by $\chi \mapsto \chi(g)$ factors through the local ring $R(G)_g$: the evaluation map

$$R(G)_g \longrightarrow \mathbf{C}\ ,$$

given by $\chi/\psi \mapsto \chi(g)/\psi(g)$ being well-defined since $\psi(g) \neq 0$. Thus (2.10) will yield a formula for $(\mathrm{ind}_G^X u)(g)$. To write this in a convenient form, we shall, for any trivial G-space Y, introduce the evaluation maps

$$K_G(Y) \cong K(Y) \otimes R(G) \longrightarrow K(Y) \otimes \mathbf{C}$$
$$K_G(Y)_g \cong K(Y) \otimes R(G)_g \longrightarrow K(Y) \otimes \mathbf{C}$$

given respectively by

$$u \otimes \chi \longmapsto u \otimes \chi(g)$$
$$u \otimes \chi/\psi \longmapsto u \otimes \chi(g)/\psi(g)$$

for $u \in K(Y)$. Taking $Y = TX^g$ we can evaluate (2.10) on the element g, and we obtain the formula:

$$(\mathrm{ind}_G^X u)(g) = (\mathrm{ind}_1^{X^g} \otimes \mathrm{Id})\left\{\frac{i^* u(g)}{\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)}\right\}\ , \tag{2.11}$$

where

$$\mathrm{ind}_1^{X^g} \otimes \mathrm{Id}\colon K(TX^g) \otimes \mathbf{C} \longrightarrow \mathbf{Z} \otimes \mathbf{C} = \mathbf{C}$$

is the natural extension of $\mathrm{ind}_1^{X^g}$ obtained by tensoring with $\mathbf{C}$.

In (2.11) we have assumed that G is topologically cyclic and generated by g. Now for any group G, and for any $g \in G$, let H be the closed subgroup generated by g. If $u \in K_G(TX)$, let u_H be the element of $K_H(TX)$ induced by u. By naturality of the topological index, we have

$$\mathrm{ind}_G^X u(g) = \mathrm{ind}_H^X u_H(g)\ .$$

Applying (2.11) with H intead of G, we then obtain an explicit formula for the topological character-index ind_G^X in terms of ordinary topological (integer) indices on various fixed point sets. In other words, the group G has been eliminated from the problem.

If we now combine this formula for the topological index with the main theorem of [5] we will obtain a general "Lefschetz formula". Thus let E be an elliptic complex on X invariant under G, and let $\sigma(E)$ be its symbol sequence. Then this defines an element

$$u = [\sigma(E)] \in K_G(TX) ,$$

and the analytical index of u evaluated at g is just the Lefschetz number

$$L(g, E) = \sum (-1)^i \operatorname{Trace}\left(g \mid H^i(E)\right)$$

where $H^i(E)$ are the homology groups of the elliptic complex.

Formula (2.11) then leads to the following general Lefschetz fixed-point theorem for G-invariant elliptic complexes.

THEOREM (2.12). *Let G be topologically cyclic generated by g, X a compact G-manifold, E an elliptic complex on X on which G acts. Let X^g denote the fixed-point set of g, N^g the normal bundle of X^g in X. Finally let $u = [\sigma(E)] \in K_G(TX)$ denote the class of the symbol of E, $i^*u \in K_G(TX^g)$ its restriction to X^g. Applying the evaluation map*

$$K_G(TX^g) \cong K(TX^g) \otimes R(G) \longrightarrow K(TX^g) \otimes \mathbf{C} ,$$

given by $a \otimes \chi \mapsto a \otimes \chi(g)$, we can then form $i^*u(g)$ and $\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g)$ in $K(TX^g) \otimes \mathbf{C}$. The latter is invertible, and so

$$\frac{i^*u(g)}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g)} \in K(TX^g) \otimes \mathbf{C}$$

is well-defined. Then the Lefschetz number $L(g, E)$ is given by

$$L(g, E) = \operatorname{index}\left\{\frac{i^*u(g)}{\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)}\right\} ,$$

where index: $K(TX^g) \otimes \mathbf{C} \to \mathbf{C}$ *denotes the natural extension of the topological index $K(TX^g) \to \mathbf{Z}$.*

Remarks. 1. Theorem (2.12) reduces the problem of calculating an index in $R(G)$, or a Lefschetz number, to that of ordinary indices in $\mathbf{Z}$. In principle we could have used this to deduce the equality a-ind = t-ind for groups G from the corresponding equality without groups. However this would have been rather artificial, because the main step in either case is the commutative diagram (2.1) (for both a-ind and t-ind).

2. Applying the explicit cohomological formula for the index given in [4] to (2.12) we will of course obtain a corresponding expression for $L(g, E)$. This will be done in detail in paper III.

3. When the fixed point set X^g is finite, the topological index $K(TX^g) \to \mathbf{Z}$ is rather trivial, and (2.12) leads immediately to the explicit Lefschetz formula of [2]. This will be developed in the next section, but we should emphasize at this stage the exact relation of our Theorem (2.12) to the main result of [2]. In [2] general maps with simple fixed points are considered. These maps need not be invertible and, even if they are, they may not lie in

any compact group of automorphisms of the elliptic complex. The deformation methods which are available for compact groups (because of the discreteness of the characters) do not therefore apply, and the proof of [2] is by direct analysis. The case of higher dimensional fixed-point sets, however, requires more subtle analysis for direct treatment, and our present deformation method is therefore very advantageous, when applicable, i.e., where we have a compact group.

4. The right hand side of the formula in (2.12) is *a priori* only a complex number. The Lefschetz number $L(g, E)$, on the other hand, is the value on g of a character of G. For example, if G is finite these values must be algebraic integers. In this way (2.12) leads to "integrality theorems" for actions of groups on manifolds. These will be illustrated in the next section. It is perhaps worth pointing out that, for cyclic groups, these "integrality theorems" do not depend on the analysis in [5], since we can define the Lefschetz number topologically.

3. Special cases and applications

First, as indicated in Remark (3) above, we shall consider the case when the fixed point set X^g consists of a finite set of points. In this case, we have

$$K(TX^g) = \prod_P K(P) ,$$

where P runs over the fixed points of g. The topological index coincides on each factor $K(P)$ with the natural isomorphism $K(P) \cong \mathbf{Z}$. The normal bundle N^g at P is just the tangent space T_P to X at P, and

$$\lambda_{-1}(T_P \otimes_{\mathbf{R}} \mathbf{C})(g) = \det(1 - g \mid T_P) .$$

Finally if E_i are the bundles of the elliptic complex E, then the component $(i^*u)_P$ is just $\sum (-1)^i E_i$, and evaluated on g, this gives

$$\sum (-1)^i \operatorname{Trace}(g \mid E_i) .$$

Thus, as a special case of (2.12), we obtain

THEOREM (3.1). *Let G be a compact Lie group, X a compact G-manifold, E an elliptic complex on X on which G acts. Let $g \in G$ have a finite set of fixed points. Then the Lefschetz number $L(g, E)$ is given by*

$$L(g, E) = \sum_P \nu(P)$$

where the summation is taken over the fixed points of g, and

$$\nu(P) = \frac{\sum_i (-1)^i \operatorname{Trace}(g \mid E_{i,P})}{\det(1 - g \mid T_P)} .$$

We shall now show that this formula is essentially identical with the one

given in [2]. For this, we take $f: X \longrightarrow X$ to be given by $f(x) = g^{-1}x$, and the maps

$$\varphi_{i,x}: E_{i,f(x)} \longrightarrow E_{i,x}$$

to be given by the action of g. The map T_i on sections of E_i induced by (f, φ_i) then coincides with the natural action of g: if $s \in \Gamma(E_i)$, we have

$$(gs)(gx) = g\big(s(x)\big)$$

or

$$(gs)(y) = g\big(s(g^{-1}y)\big) = \varphi_{i,x}s\big(f(y)\big) = (T_i s)(y) \ .$$

Thus $L(g; E) = L(T)$ is the same Lefschetz number as in [2]. As for the terms $\nu(P)$ in [2], they are defined as

$$\frac{\sum (-1)^i \operatorname{Trace} \varphi_{i,P}}{|\det (1 - df_p)|} \ .$$

Now $\varphi_{i,P} = g \mid E_{i,P}$ and $df_P = g^{-1} \mid T_P$. Since $g \mid T_P$ is orthogonal, it follows that

$$\det (1 - g^{-1} \mid T_P) = \det (1 - g \mid T_P) > 0 \ .$$

Thus we have

$$\det (1 - g^{-1} \mid T_P) = |\det (1 - df_P)| \ ,$$

so that the terms $\nu(P)$ occurring in (3.1) are the same as those given in [2].

Theorem (3.1) has a number of interesting applications but, as these are dealt with fully in [3], we shall not pursue them here.

Another case of special interest is to take X a compact complex manifold, and G a finite group of complex analytic automorphisms of X. Suppose moreover that V is a holomorphic G-vector bundle over X. Then the Dolbeault complex $A(V)$

$$\longrightarrow A^{0,p}(V) \xrightarrow{\bar{\partial}} A^{0,p+1}(V) \longrightarrow$$

(where $A^{0,p}(V)$ denotes the differential forms on X of type $(0, p)$ with coefficients in V) is an elliptic complex acted on by G. The homology groups of this complex may be identified with the sheaf cohomology groups $H^p(X, \mathcal{O}(V))$ where $\mathcal{O}(V)$ denotes the sheaf of germs of holomorphic sections of V. The Lefschetz number is thus

$$L(g, A(V)) = \textstyle\sum_p (-1)^p \operatorname{Trace} \big(g \mid H^p(X, \mathcal{O}(V))\big) \ .$$

Consider now the fixed-point set X^g of g. It is a complex submanifold of X. If $a(X, V) \in K_G(TX)$ denotes the symbol class of the Dolbeault complex $A(V)$, then its restriction to $K_H(TX^g)$ (H the subgroup generated by g_1) is given by

$$i^*a(X, V) = a(X^g, V \mid X^g)\lambda_{-1}(\bar{N}^*) ,$$

where N is the (complex) normal bundle of X^g in X. This follows from the usual decomposition of the exterior algebra of a direct sum. Now we have isomorphisms

$$N \otimes_{\mathbf{R}} \mathbf{C} \cong N \oplus \bar{N} \qquad \bar{N} \cong N^*$$

and so

$$\frac{\lambda_{-1}(\bar{N}^*)}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})} = \frac{\lambda_{-1}(N)}{\lambda_{-1}(N)\cdot\lambda_{-1}(N^*)} = \frac{1}{\lambda_{-1}(N^*)} .$$

Thus we get

$$\begin{aligned} \frac{i^*a(X, V)}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})} &= \frac{a(X^g, V \mid X^g)}{\lambda_{-1}(N^*)} \\ &= \frac{a(X^g)[V \mid X^g]}{\lambda_{-1}(N^*)} , \end{aligned} \tag{3.2}$$

where $a(X^g)$ is the symbol class of the Dolbeault complex of X^g.

If U is a holomorphic vector bundle over a compact complex manifold Y, one usually denotes by $\chi(Y, U)$ the Euler characteristic

$$\sum (-1)^p \dim H^p(Y, \mathcal{O}(U)) ,$$

i.e., the index of the Dolbeault complex $A(Y, U)$. We extend this notation to any $u \in K(Y) \otimes \mathbf{C}$. Thus

$$\chi(Y, u) = \text{index}\,(a(Y)\cdot u) \in \mathbf{C} .$$

In this notation (3.2) yields

$$\text{index}\left\{\frac{i^*a(X, V)(g)}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g)}\right\} = \chi\left(X^g, \frac{[V \mid X^g](g)}{\lambda_{-1}(N^*)(g)}\right) .$$

Putting this into Theorem (2.12), we have therefore established

THEOREM (3.3). *Let X be a compact complex manifold, V a holomorphic vector bundle over X, G a finite group of automorphisms of the pair (X, V). For any $g \in G$, let X^g denote the fixed-point set in X of g, and let N^g denote the (complex) normal bundle of X^g in X. Then we have*

$$\sum (-1)^p \,\text{Trace}\,(g \mid H^p(X, \mathcal{O}(V))) = \chi\left(X^g, \frac{[V \mid X^g](g)}{\lambda_{-1}((N^g)^*)(g)}\right) .$$

Remark. In the case of a general elliptic complex E with symbol class u, we can restrict u to $K_G(TX^g)$ but not the actual complex. In the holomorphic case however there is, in some sense, a natural restriction, and so Theorem (3.3) can be expressed without reference to symbols.

We shall now show that Theorem (3.3) implies in effect a Riemann-Roch

theorem for the complex space X/G. For this we need the following lemma.

LEMMA (3.4). *Let X, V, G be as in* (3.3), *and let $f: X \to X/G$ be the projection of X onto the analytic space X/G. Then we have natural isomorphisms*

$$(H^p(X, \mathcal{O}(V)))^G \cong H^p(X/G, (f_*\mathcal{O}(V))^G)$$

where $(\ )^G$ denotes the invariant part under the action of G.

PROOF. We recall that the structure sheaf of X/G is by definition $f_*(\mathcal{O}_X)^G$. Now the higher direct images $R^q f_*(\mathcal{O}(V))$ $(q \geqq 1)$ are all zero because any $y \in X/G$ has a base of neighbourhoods U with $f^{-1}(U)$ a disjoint finite union of complex balls. The Leray spectral sequence

$$H^p(X/G, R^q f_*(\mathcal{O}(V))) \Longrightarrow H^{p+q}(X, \mathcal{O}(V))$$

therefore collapses and gives rise to isomorphisms

$$H^p(X/G, f_*(\mathcal{O}(V))) \cong H^p(X, \mathcal{O}(V)) .$$

Taking invariant parts on both sides, and observing that, (since $(\ )^G$ is an exact functor for vector spaces over C),

$$H^p(X/G, (f_*(\mathcal{O}(V)))^G) \cong H^p(X/G, f_*(\mathcal{O}(V)))^G ,$$

the required result follows.

If W is a holomorphic vector bundle on $Y = X/G$, then $V = f^*W$ is a holomorphic G-vector bundle on X, and $(f_*(\mathcal{O}(V)))^G \cong \mathcal{O}(W)$. The Euler characteristic

$$\chi(Y, W) = \sum (-1)^p \dim H^p(Y, \mathcal{O}(W))$$

can then be computed by the following theorem.

THEOREM (3.5). *Let G be a finite group of automorphisms of a compact complex manifold X, and let W be a holomorphic vector bundle over the complex space $Y = X/G$. Then we have*

$$\chi(Y, W) = \frac{1}{|G|} \sum_{g \in G} \mu(g)$$

where

$$\mu(g) = \chi\left(X^g, \frac{[f^*W \mid X^g](g)}{\lambda_{-1}((N^g)^*)(g)}\right)$$

N^g is the normal bundle to X^g in X, and $|G|$ denotes the order of G.

PROOF. By (3.4) we have

$$\chi(Y, W) = \sum (-1)^p \dim H^p(X, \mathcal{O}(f^*W))^G .$$

The theorem follows by combining this with Theorem (3.3) and, recalling that

for any G-module M, we have

$$\dim M^G = \frac{1}{|G|} \cdot \sum_{g \in G} \text{Trace}\,(g \mid M)\ .$$

Remarks. 1. (3.5) reduces the Riemann-Roch theorem for the *singular space* X/G to the Riemann-Roch theorem for the *manifolds* X^g. Thus, if the terms $\mu(g)$ are expressed in explicit cohomological form, we shall obtain an explicit formula for $\chi(Y, W)$. This is of particular interest in the case of automorphic forms (cf. [6]).

2. It is probable that (3.3) and (3.5) continue to hold in abstract algebraic geometry provided $|G|$ is prime to the characteristic of the ground field.

Another interesting case of Theorem (2.12) is given by the Dirac operator of a spin-manifold, but we shall defer this example until paper III, (for isolated fixed points see [3; § 8]).

Theorem (2.12) becomes especially simple if the normal bundle N^g of X^g is G-trivial (i.e., isomorphic to $X^g \times M$ for some G-module M), or at least if it is G-trivial over each connected component. In this case, over each component, $\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})$ is just an element of $R(G)$ and evaluated at g it gives

$$\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g) = \det(1 - g \mid N_P)\ ,$$

where P is any point of the component. Thus (2.12) becomes

$$L(g, E) = \sum_j \sigma_j\ ,$$

where the summation is over the connected components X_j^g of X^g,

$$\sigma_j = \frac{1}{\det(1 - g \mid N_j)}\ \text{index}\ i_j^! u(g)\ ,$$

$i^* u_j \in K_G(TX_j^g)$, and N_j is the fibre of N at some point of X_j^g. Theorem (3.1), dealing with finite X^g, is of course a special case in which the normal bundle is G-trivial over each component.

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The index of elliptic operators: III

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Introduction

In [7], paper I of this series, the index of an elliptic operator was computed in terms of K-theory. In this paper, we carry out what is essentially a routine exercise by passing from K-theory to cohomology. In this way, we end up with (2.12) the explicit cohomological formula for the index announced in [6].

In [7] we also considered elliptic operators (or complexes) compatible with a compact group G of transformations. The index in this case is a character of G, and the main theorem of [7] gave a construction for this in K_G-theory. In [5], paper II of this series, the value of this index-character at an element $g \in G$ was expressed as the index of a new "virtual operator" on the fixed-point set of g. This was referred to as a Lefschetz fixed-point formula. By combining this formula with the cohomological formula for the index, we obtain finally an explicit cohomological formula (3.9) for the index-character. We shall describe this formula in detail for a number of important operators. In particular we draw attention to the "integrality theorems" obtained in this way for actions of finite groups on manifolds. Most of these do not depend on the analysis in [7], but are a consequence of combining the purely topological results of [5] and the present paper.

We begin in § 1 with a brief review of the theory of characteristic classes and of the relation between cohomology and K-theory. The index formula (2.12) is then derived in § 2, and the more general index-character or Lefschetz formula (3.9) is given in § 3. The proofs are simple formal consequences of the results of papers I and II. We also give in (2.17) a rather more explicit form of the index theorem for the kind of operator which arises naturally from differential-geometric structures.

The bulk of the paper is actually devoted to examples and applications of the main theorem. The explicit formula (3.9) is of a rather formidable kind, and it seemed worthwhile to show what it reduces to in various important special cases. Thus in § 4 we discuss the Riemann-Roch theorem (4.3) for compact complex manifolds, and also the corresponding Lefschetz formula (4.6) for finite groups of automorphisms. In § 5 we examine briefly the Dirac operator of a Spin manifold. In some ways the most interesting elliptic oper-

ator is the one, discussed in § 6, the index of which is the Hirzebruch signature of a manifold. Because of its significance for differential topology, we examine this case in considerable detail. Besides obtaining the original theorem (6.6) of Hirzebruch, we also derive the corresponding Lefschetz formula (6.12). One noteworthy feature here is that this formula is of interest for all even-dimensional (oriented) manifolds, not just those with dimension divisible by 4, as is the case with (6.6).

In § 7 we show how the Lefschetz formula of the preceding sections can be used to define invariants for free G-actions (on odd-dimensional manifolds). Some of these are cobordism-type invariants, but the operator of § 6 gives rise to a more refined invariant, and we devote most of § 7 to this case.

In § 8 we consider the infinitesimal case of our general Lefschetz formula, and we derive results similar to those of Bott [9] [10] connecting characteristic numbers with zeros of vector fields.

Finally in § 9 we make some simple deductions from the general index and Lefschetz formulas. Under various special assumptions, these formulas simplify considerably, and, in many situations, one gets a zero index. Most of these special cases are already mentioned in [6].

As far as the ordinary index theorem goes, the particular cases of §§ 4, 5, 6, are quite adequately described in [19]. Our reason for reproducing this material again here is partly for the sake of completeness, and partly because we want to go on to discuss Lefschetz numbers. Our treatment however is somewhat briefer than that in [19], and the reader can refer to it for more detail.

1. Cohomology and characteristic classes

In this section we shall, for the benefit of the reader, give a brief summary of the theory of characteristic classes. For further details we refer to [8].

For the category of compact spaces, the most natural cohomology theory is Čech cohomology. This is also the one that fits in with vector bundles, K-theory, and general fibre bundle theory. For locally compact spaces, it is then convenient to take cohomology with compact supports, which amounts to taking the reduced cohomology of the one-point compactification. For differentiable manifolds, where our interest lies, the Čech cohomology groups (with real coefficients) can, by sheaf theory, be identified with the de Rham groups; i.e., the cohomology of the complex of exterior differential forms.

Let G be a compact Lie group. A *characteristic class* of G can be defined as a functor which assigns to every (compact) principal G-bundle P a coho-

mology class[1] of $X = P/G$ (with coefficients $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$ as the case may be). The set of all characteristic classes forms a ring which we denote by[2] H_G^*, or $H_G^*(A)$ when we want to specify the coefficient ring A. The following facts about H_G^* are known.

(1.1) If $G = T$ a torus, $H_T^2(Z)$ is naturally isomorphic to the character group (or Pontrjagin dual) $\hat{T}$ of T and $H_T^*(A)$ is the (completed) symmetric algebra of $\hat{T}$ over A; thus, if $x_1, \cdots, x_n$ are a basis for $\hat{T}$,

$$H_T^*(A) = A[[x_1, \cdots, x_n]]$$

is the ring of formal power series in $x_1, \cdots, x_n$.

(1.2) If G has maximal torus T and Weyl group W, the natural homomorphism

$$H_G^*(A) \longrightarrow H_T^*(A) \qquad \text{for } A = \mathbf{Q}, \mathbf{R}, \text{ or } \mathbf{C}$$

identifies $H_G^*(A)$ with the invariants of W acting on $H_T^*(A)$.

(1.3) If $G = U(n)$ is the unitary group, then the conclusion of (1.2) also holds for $A = \mathbf{Z}$. Thus, for any A, if $x_1, \cdots, x_n$ denote the standard characters of the maximal torus T of $U(n)$ (so that W is the permutation group S_n) we have

$$\begin{aligned} H_{U(n)}^*(A) &\cong A[[x_1, \cdots, x_n]]^{S_n} \\ &\cong A[[c_1, \cdots, c_n]] , \end{aligned}$$

where c_i is the i^{th} elementary symmetric function of $x_1, \cdots, x_n$.

The characteristic classes c_i of $U(n)$ are called *Chern classes*. Since a principal $U(n)$-bundle defines and (up to isomorphism) is defined by a complex vector bundle of dimension n, we can regard the Chern classes as functors from vector bundles to cohomology. Thus if E is a complex vector bundle over X, we have

$$c_i(E) \in H^{2i}(X; \mathbf{Z}) \qquad i = 1, \cdots, n = \dim E .$$

Introducing an indeterminate t, we define the Chern polynomial

$$c(E) = \sum c_i(E) t^i \qquad (c_0 = 1) .$$

Then one has

$$c(E \oplus F) = c(E) \cdot c(F) . \tag{1.4}$$

For a trivial bundle E, one has $c(E) = 1$. Together with (1.4), this shows that the Chern classes can be defined on $K(X)$.

---

[1] We allow a cohomology class to be inhomogeneous, i.e., to be an element of the infinite product $\prod_q H^q(X)$.

[2] The usual notation is $H^{**}(B_G)$ where B_G is the classifying space. For brevity we shall avoid introducing B_G.

The power series

$$\sum_{i=1}^{n} e^{x_i} = n + \sum x_i + \frac{\sum x_i^2}{2!} + \cdots$$

can be expressed uniquely in terms of the elementary symmetric functions c_i. This defines a characteristic class (over $\mathbf{Q}$) called the *Chern character* and denoted by ch. It has the formal properties

$$\operatorname{ch}(E \oplus F) = \operatorname{ch} E + \operatorname{ch} F$$
$$\operatorname{ch}(E \otimes F) = (\operatorname{ch} E)(\operatorname{ch} F) .$$

It extends to a ring homomorphism

$$\operatorname{ch}: K(X) \longrightarrow H^*(X; \mathbf{Q}) .$$

Suppose now that G has maximal torus S, and that $\rho: G \to U(n)$ is a representation of G with $\rho(S) \subset T$ the standard maximal torus of $U(n)$. Then ρ induces a homomorphism

$$\hat{\rho}: \hat{T} \longrightarrow \hat{S}$$

and the elements $y_i = \hat{\rho}(x_i) \in \hat{S}$ are called the weights of ρ. The naturality of the isomorphism (1.2) implies

(1.5) If $\rho^*: H^*_{U(n)} \to H^*_G$ is the homomorphism, induced by ρ, then

$$\rho^* c = \prod (1 + y_i t)$$
$$\rho^* \operatorname{ch} = \sum e^{y_i} ,$$

where $c = \sum c_i t^i$, and t is an indeterminate. If M is the G-module defined by ρ (i.e., $M = \mathbf{C}^n$ with action $g(x) = \rho(g)x$), we shall also write $\operatorname{ch} M$ for the characteristic class

$$\rho^* \operatorname{ch} = \sum e^{y_i} .$$

Then $M \mapsto \operatorname{ch} M$ defines a ring homomorphism

$$\operatorname{ch}: R(G) \longrightarrow H^*_G(\mathbf{Q}) ,$$

the "universal" Chern character. If T is the maximal torus of G, we have a commutative diagram

$$\begin{array}{ccc} R(G) & \xrightarrow{\operatorname{ch}} & H^*_G(\mathbf{Q}) \\ \downarrow & & \downarrow \\ R(T) & \xrightarrow{\operatorname{ch}} & H^*_T(\mathbf{Q}) . \end{array}$$

For the torus T, we have $R(T) = \mathbf{Z}[\hat{T}]$ (the integral group ring of $\hat{T}$), and the Chern character is the ring homomorphism defined by

$$x_i \longmapsto e^{x_i} ,$$

where $x_1, \cdots, x_m$ are a basis for T.

If E is a real vector bundle over X, the *Pontrjagin classes*

$$p_i(E) \in H^{4i}(X; \mathbf{Z})$$

are defined by

$$p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbf{R}} \mathbf{C}) .$$

Using (1.3) and (1.5), one finds

$$H^*_{O(n)}(A) = A[[p_1, \cdots, p_m]] \qquad m = \left[\frac{n}{2}\right], \ A = \mathbf{Q}, \mathbf{R}, \text{ or } \mathbf{C} .$$

If $x_1, \cdots, x_m$ are the basic characters of the standard maximal torus of $O(n)$, the p_i are given by

$$\begin{aligned} \sum (-1)^i p_i t^{2i} &= \prod_1^m (1 + x_i t)(1 - x_i t) \\ &= \prod (1 - x_i^2 t^2) , \end{aligned}$$

so that the p_i are the elementary symmetric functions of $x_1^2, \cdots, x_m^2$. For the special orthogonal group $SO(2m)$ there is, in addition, a further invariant of the Weyl group, namely

$$e = \prod_1^m x_i . \tag{1.6}$$

Like the p_i, this is in fact the image of an integral characteristic class called the *Euler class*. This may be defined as follows.

Let E be an oriented real vector bundle of dimension n over X. Then we have a Thom isomorphism

$$\psi: H^*(X; \mathbf{Z}) \longrightarrow H^*(E; \mathbf{Z}) , \tag{1.7}$$

given by $\psi(u) = u \cdot \psi(1)$. Thus $H^*(E; \mathbf{Z})$ is a free $H^*(X; \mathbf{Z})$-module on the one generator $\psi(1)$. If $n = 2m$, then the Euler class $e(E)$ is defined by

$$e(E) = i^* \psi(1) \qquad \in H^n(X; \mathbf{Z}) , \tag{1.8}$$

where $i^*: H^*(E; \mathbf{Z}) \to H^*(X; \mathbf{Z})$ is induced by the zero-section $i: X \to E$.

If V is a complex vector bundle of dimension m over X, and E is the underlying real oriented vector bundle, then one has

$$c_m(V) = e(E) . \tag{1.9}$$

In rational cohomology, this follows at once from (1.3), (1.6), and the fact that $U(m)$ and $SO(2m)$ have the same maximal torus. The result for integer cohomology then follows from the fact (1.3) that

$$H^*_{U(n)}(\mathbf{Z}) \longrightarrow H^*_{U(n)}(\mathbf{Q})$$

is injective.

We conclude this survey of characteristic classes with a quick look at

the role of curvature. We suppose now that we are in the differentiable framework, i.e., all spaces are differentiable manifolds and all bundles are differentiable. Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra, T a maximal torus of G, $\mathfrak{t}$ its Lie algebra. Then the inclusion $\mathfrak{t} \to \mathfrak{g}$ induces an isomorphism

$$S(\mathfrak{g}^*)^G \longrightarrow S(\mathfrak{t}^*)^W \tag{1.10}$$

from the polynomials on $\mathfrak{g}$ invariant under the adjoint action of G, onto the polynomials on $\mathfrak{t}$ invariant under the Weyl group W. $S(\mathfrak{t}^*)$ is just the polynomial algebra over $\mathbf{R}$ generated by $\hat{T}$, and so by (1.2)

$$S(\mathfrak{t}^*)^W \cong H_G^*(\mathbf{R}) . \tag{1.11}$$

Thus (1.10) and (1.11) imply that the real characteristic ring of G may be identified with the invariant polynomials on its Lie algebra.

Suppose now that P is a principal G-bundle over a compact manifold X. Let α be a connection on P. Then one can define the curvature $\theta(\alpha)$ of α. This is an exterior differential 2-form on X with coefficients in the Lie algebra bundle. More precisely, let $\mathfrak{P} = P \times_G \mathfrak{g}$ be the vector bundle associated with P by the adjoint action of G on $\mathfrak{g}$. Then, for any x,

$$\theta(\alpha)_x \in \Omega_x^2 \otimes \mathfrak{P}_x ,$$

where Ω^2 is the bundle of 2-forms on X.

If $f \in S(\mathfrak{g}^*)^G$ is an invariant polynomial of degree k, then $f(\theta(\alpha))$ is a well-defined exterior differential form on X of degree $2k$. It is closed, and its (de Rham) cohomology class is independent of the choice of the connection α. Thus we have a differentiable characteristic class

$$[f(\theta(\alpha))] \in H^{2k}(X; \mathbf{R})$$

for each $f \in S(\mathfrak{g}^*)^G$. Except for a numerical factor (involving π), this is precisely the characteristic class corresponding to f by the isomorphism

$$S(\mathfrak{g}^*)^G \cong H_G^*(\mathbf{R}) .$$

Thus, in the differentiable situation, characteristic classes can be represented by explicit differential forms constructed out of curvature.

For manifolds, the Thom isomorphism can be re-interpreted in a somewhat simpler manner. Assume that X is an oriented manifold of dimension N, and let E be an oriented real vector bundle over X of dimension n. Then we have the Poincaré duality isomorphisms

$$H^q(X; \mathbf{Z}) \cong H_{N-q}(X; \mathbf{Z})$$
$$H^p(E; \mathbf{Z}) \cong H_{N+n-p}(E; \mathbf{Z}) ,$$

where the homology is singular homology. The Thom isomorphism then transforms into the homology isomorphism

$$H_{N-q}(X; \mathbf{Z}) \longrightarrow H_{N-q}(E; \mathbf{Z})$$

induced by the zero-section inclusion $X \to E$ (which is, of course, a homotopy equivalence).

Note that the singular homology of X is homology with compact support, but there is also homology with arbitrary support, and the fundamental class $[X]$ belongs to this latter group. If $u \in H^N(X; \mathbf{Z})$, we can evaluate u on $[X]$. For real coefficients if u is represented by a closed N-form with compact support, this value is simply the integral

$$u[X] = \int_X u \ .$$

If $Du \in H_0(X; \mathbf{Z})$ is the dual homology class, then one has

$$u[X] = \varepsilon(Du) \ ,$$

where the augmentation $\varepsilon: H_0(X; \mathbf{Z}) \to \mathbf{Z}$ is induced by the projection $X \to$ point. The interpretation of the Thom isomorphism *via* Poincaré duality then implies

$$\psi(u)[E] = u[X] \qquad u \in H^n(X; \mathbf{Z}) \ . \tag{1.12}$$

2. Cohomological form of the index theorem[3]

We begin by comparing the Thom isomorphisms

$$\varphi_V: K(X) \longrightarrow K(V)$$
$$\psi_V: H^*(X; \mathbf{Q}) \longrightarrow H^*(V; \mathbf{Q})$$

of K-theory and cohomology, where V denotes a complex vector bundle over X. We suppose first that X is compact so that $K(X)$ has an identity element 1. We can then consider the cohomology class

$$\mu(V) = \psi_V^{-1} \operatorname{ch} \varphi_V(1) \qquad \in H^*(X; \mathbf{Q}) \ .$$

Because φ and ψ are both natural,

$$V \longmapsto \mu(V)$$

is a functor from complex vector bundles to rational cohomology. It is therefore given by an element μ of the characteristic ring (cf. (1.3))

$$H^*_{U(n)}(\mathbf{Q}) = \mathbf{Q}[[c_1, \cdots, c_n]] = \mathbf{Q}[[x_1, \cdots, x_n]]^{S_n} \ .$$

Now if $i^*: H^*(E; \mathbf{Q}) \to H^*(X; \mathbf{Q})$ is the restriction homomorphism, then for any $u \in H^*(X; \mathbf{Q})$, we have by (1.8)

$$i^*\psi_V(u) = i^*\big(u \cdot \psi_V(1)\big) = ui^*\big(\psi_V(1)\big) = u \cdot e(V) \ .$$

[3] The calculations in this section are essentially the same as those in [4] and are, by now, fairly routine.

Taking $u = \mu(V)$, and recalling that the fundamental class

$$\lambda_V = \varphi(1) \in K(E)$$

restricts to

$$\lambda_{-1}(V) = \sum (-1)^i \lambda^i(V) \qquad \in K(X) ,$$

we get

$$\begin{aligned} \mu(V) \cdot e(V) &= i^* \psi_V \cdot \psi_V^{-1} \operatorname{ch} \varphi_V(1) \\ &= i^* \operatorname{ch} \lambda_V \\ &= \operatorname{ch} \lambda_{-1}(V) . \end{aligned}$$

Since this holds for all V, we have an identity

$$\mu \cdot e = \operatorname{ch} \lambda_{-1} \in H^*_{U(n)}(\mathbf{Q}) . \tag{2.1}$$

By (1.9) we have $e = c_n$, and by (1.5) we have

$$\operatorname{ch} \lambda_{-1} = \prod (1 - e^{x_i}) ,$$

Since $c_n = \prod x_i$ is not a zero divisor in $H^*_{U(n)}(\mathbf{Q})$, we deduce

$$\mu = \prod \left(\frac{1 - e^{x_i}}{x_i} \right) . \tag{2.2}$$

If we expand (2.2) in terms of the elementary symmetric functions c_i of $x_1, \cdots, x_n$, we find $\mu = \sum \mu_k$ where the μ_k are explicit polynomials in the c_i. For any V, we then have $\mu(V) = \sum \mu_k(c_1(V), \cdots, c_k(V))$. Note that $\mu_0(V) = (-1)^n$; and so if V is trivial, $\mu(V) = (-1)^n$.

Since φ, ψ, and ch are all module homomorphisms, it follows that, for any $u \in K(X)$,

$$\psi_V^{-1} \operatorname{ch} \varphi_V u = \operatorname{ch} u \cdot \mu(V) . \tag{2.3}$$

More generally the same proof shows that (2.3) also holds for $u \in K(X, Y)$ where Y is a closed subspace of X. Suppose now that X, Y are differentiable manifolds with X compact, and let $i: X \to Y$ be an embedding with normal bundle N. Then, as in [7; § 3] we can identify TN with the complex vector bundle $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$ over TX, where $\pi: TX \to X$ is the projection. The homomorphism $i_!: K(TX) \to K(TY)$ of [7; § 3] is defined as the composition

$$K(TX) \xrightarrow{\varphi} K(TN) \xrightarrow{k_*} K(TY) ,$$

where φ is the Thom homomorphism, and k_* is the natural map induced by the open inclusion $k: TN \to TY$ (N being identified with a tubular neibourhood of X in Y). Applying the relative form of (2.3) to an element

$$u \in K\big(B(X), S(X)\big) = K(TX)$$

($B(X)$ and $S(X)$ denoting the unit ball and unit sphere bundles of TX), and with $V = \pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$, we get

$$\operatorname{ch} \varphi_V(u) = \psi_V(\operatorname{ch} u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})) . \tag{2.4}$$

Here we regard $H^*(TX)$ as a module over $H^*(X)$ in the usual way. For any manifold X we can regard TX as an almost complex manifold, on the same lines as in [7; § 3]. Locally we have

$$X \cong \mathbf{R}^n, \qquad TX \cong T\mathbf{R}^n \cong \mathbf{C}^n$$

the last isomorphism being given by $(x, \xi) \mapsto x + i\xi$ where $x \in \mathbf{R}^n$ and $\xi \in T_x$. In particular this gives TX a definite orientation and so defines a fundamental class. Note that, if $X \subset Y$, the orientation of the normal bundle of TX in TY induced by our orientation of TX and TY coincides with the orientation of the complex vector bundle $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$. If we now evaluate (the top-dimensional component of) (2.4) on the fundamental class of TN and use (1.12), we obtain

$$\operatorname{ch} \varphi_V(u)[TN] = \operatorname{ch} u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})[TX] . \tag{2.5}$$

By naturality, on the other hand, we have

$$\begin{aligned} \operatorname{ch} \varphi_V(u)[TN] &= \operatorname{ch} k_* \varphi_V(u)[TY] \\ &= \operatorname{ch} i_!(u)[TY] . \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) therefore we obtain

$$\operatorname{ch} i_!(u)[TY] = \operatorname{ch} u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})[TX] . \tag{2.7}$$

We now consider the two inclusions $i: X \to E$, $j: P \to E$ where E is a euclidean space. Let $\dim X = n$, $\dim E = n + q$, then the trivial case of (2.7) with i replaced by j gives

$$\begin{aligned} \operatorname{ch} j_!(v)[TE] &= (-1)^{n+q} \operatorname{ch} v[TP] \\ &= (-1)^{n+q} v \qquad \text{for } v \in K(P) \cong \mathbf{Z} . \end{aligned}$$

In other words, the inverse of the isomorphism $j_!$ is given by

$$j_!^{-1}(w) = (-1)^{n+q} \operatorname{ch} w[TE] \qquad w \in K(TE) . \tag{2.8}$$

Recalling that the *topological index*

$$\text{t-ind}: K(TX) \longrightarrow \mathbf{Z}$$

was defined [7; § 3] by t-ind $= j_!^{-1} \circ i_!$, (2.7) and (2.8) yield the formula

$$\text{t-ind } u = (-1)^{n+q} \operatorname{ch} u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})[TX] . \tag{2.9}$$

It is convenient to express (2.9) in terms of the tangent bundle T rather than the normal bundle N of X. For this, we observe that, as a consequence of (2.2), we have

$$\mu(E \oplus F) = \mu(E)\mu(F)$$

for any two complex vector bundles. Moreover

$$\mu(E) = \pm 1 + \text{higher terms} \qquad \in H^*(X; \mathbf{Q}),$$

and so is invertible. Moreover for a trivial bundle W, we have $\mu(W) = (-1)^{\dim W}$. Thus

$$\mu(N \otimes_{\mathbf{R}} \mathbf{C}) = \mu(T \otimes_{\mathbf{R}} \mathbf{C})^{-1} = \mu^{-1}(T \otimes_{\mathbf{R}} \mathbf{C})$$

where μ^{-1} is the characteristic class of $U(n)$ given by

$$\mu^{-1} = \prod_1^n \frac{x_i}{1 - e^{x_i}} .$$

Let us now write

$$\mathcal{T}^* = (-1)^n \mu^{-1} = \prod_1^n \frac{-x_i}{1 - e^{x_i}}$$

$$\mathcal{T} = \prod \frac{x_i}{1 - e^{-x_i}} .$$

The class $\mathcal{T}$ is called the *Todd class*, and $\mathcal{T}^*$ may be called the dual Todd class. In fact, if V^* is the dual of V, then we have

$$\mathcal{T}^*(V) = \mathcal{T}(V^*) .$$

In particular, when $V = E \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of a real bundle, then $V \cong V^*$ and so $\mathcal{T}(V) = \mathcal{T}^*(V)$.

The functor $E \mapsto \mathcal{T}(E \otimes_{\mathbf{R}} \mathbf{C})$ defines a characteristic class of $O(n)$, the image of $\mathcal{T}$ in the homomorphism

$$H^*_{U(n)}(\mathbf{Q}) \longrightarrow H^*_{O(n)}(\mathbf{Q}) .$$

We shall call this class the *Index class* and write it as $\mathcal{I}$. Thus

$$\mathcal{I}(E) = \mathcal{T}(E \otimes_{\mathbf{R}} \mathbf{C}) .$$

If $y_1, \cdots, y_m$ are the usual basic characters for the maximal torus of $O(n)$ (where $m = [n/2]$) then (1.5) shows that

$$\mathcal{I} = \prod \frac{-y_i}{1 - e^{y_i}} \prod \frac{y_i}{1 - e^{-y_i}} . \tag{2.10}$$

Expressing this in terms of the elementary symmetric functions p_i of $y_1^2, \cdots, y_m^2$, we obtain an expansion

$$\mathcal{I} = \sum \mathcal{I}_{k,m}(p_1, \cdots, p_k)$$

where the $\mathcal{I}_{k,m}$ are explicit polynomials of weight k. Moreover $\mathcal{I}_{k,m}$ is independent of m provided $k \leqq m$. If we write $\mathcal{I}_k$ for this common value, then we always have

$$\mathcal{J} = \sum \mathcal{J}_k(p_1, \cdots, p_k) = 1 - \frac{p_1}{12} \cdots$$

provided we agree to put $p_k = 0$ for $k > m$. This is standard in this sort of formalism.

Finally, we follow the usual convention, and write

$$p_i(X) = p_i(TX) ,$$

calling these the Pontrjagin classes of X. Correspondingly we define the Index class of X by

$$\mathcal{J}(X) = \mathcal{J}(TX) = \sum_k \mathcal{J}_k\big(p_1(X), \cdots, p_k(X)\big) .$$

With this notation, (2.9) gives the following cohomological formula for the topological index.

PROPOSITION (2.11). *Let X be a compact differentiable manifold of dimension n and let the Index class*

$$\mathcal{J}(X) = \sum_k \mathcal{J}_k\big(p_1(X), \cdots, p_k(X)\big) \qquad \in H^*(X; \mathbf{Q})$$

be defined as the Todd class of the complexification of the tangent bundle. Let

$$\text{t-ind}: K(TX) \longrightarrow \mathbf{Z}$$

be the topological index as defined in [7; § 3]. *Then for any $u \in K(TX)$, we have*

$$\text{t-ind}\, u = (-1)^n\{\text{ch}\, u \cdot \mathcal{J}(X)\}[TX]$$

where, on the right, we evaluate the top-dimentional component of $\text{ch}\, u \cdot \mathcal{J}(X)$ *on the fundamental homology class of TX, and TX is oriented as an almost complex manifold with "horizontal part real and vertical part imaginary".*

Now the main theorem (6.7) of [7], for the special case when the group G is reduced to one element, asserts that

$$\text{a-ind} = \text{t-ind} .$$

Combined with (2.11), this then gives the cohomological form of the index theorem.

INDEX THEOREM (2.12). *Let P be an elliptic operator over a compact manifold X, and let $u \in K(TX)$ be the symbol class of P. Then the index of P is given by*

$$\text{index}\, P = (-1)^n\{\text{ch}\, u \cdot \mathcal{J}(X)\}[TX]$$

where $\mathcal{J}(X)$ is the Index class of X, TX is oriented as in (2.11) *and $n = \dim X$.*

Remarks. 1. The factor $(-1)^n$ could have been eliminated by giving TX

the dual almost complex structure, as in [19]. There are reasons however to prefer the convention we have adopted here.

2. As explained in [7; § 7], we can in (2.12) replace the single operator P by a complex E and the index of P by the Euler characteristic of E. We shall do this in future.

If X is an oriented manifold, then the evaluation on TX in (2.12) can be replaced by an evaluation on X by using (1.12). We have to watch sign conventions however because the orientation of TX induced from that of X differs from our orientation. In fact, to pass from one fundamental class to the other, we need a sign $(-1)^{n(n-1)/2}$. Thus if $\psi: H^*(X) \to H^*(TX)$ is the Thom isomorphism[4] (2.12) can be replaced by

$$\text{index } P = (-1)^{n(n+1)/2}\{(\psi^{-1} \operatorname{ch} u)\cdot \mathcal{J}(X)\}[X] \,. \tag{2.13}$$

This is now the form of the index theorem announced in [6], except for notation and for conventions which were not made explicit in [6].

If X is not orientable, then a formula like (2.13) can also be written down, but we must use twisted coefficients, both for the fundamental class of X and for $\psi^{-1} \operatorname{ch} u$.

In the remaining sections we shall apply (2.13) to a number of particularly interesting elliptic complexes associated with some geometrical structure on X. All these are in fact examples of operators associated to an H-structure and, for these, (2.13) can be made even more explicit. We proceed to describe this class of operators.

Let H be a compact Lie group, V a fixed real oriented H-module. Then an *H-structure* on X will mean a principal H-bundle P over X together with an isomorphism (of oriented bundles)

$$P \times_H V \cong T(X)$$

where $P \times_H V$ denotes, as usual, the vector bundle over X associated to P by the H-module V. We then have a natural homomorphism

$$\alpha_P: K_H(V) \to K_H(P \times V) = K(TX) \,.$$

If $v \in K_H(V)$, the image

$$u = \alpha_P(v) \in K(TX)$$

may be called an elliptic symbol class associated to the H-structure. The element v, which may be called a "universal" elliptic symbol class for H-structures, usually arises in the following way. Let $M^0, \cdots, M^n$ be a sequence of complex H-modules, and let

[4] For n odd, it is important to specify whether the Thom isomorphism is $u \mapsto u\psi(1)$ or $u \mapsto \psi(1)u$; we have chosen the first alternative, hence the sign $(-1)^{n(n-1)/2}$.

$$\varphi_i\colon V \longrightarrow \mathrm{Hom}\,(M^i, M^{i+1})$$

be an H-map[5] so that for $\xi \in V$, $\xi \neq 0$,

$$0 \longrightarrow M^0 \xrightarrow{\varphi_0(\xi)} M^1 \longrightarrow \cdots \xrightarrow{\varphi_{n-1}(\xi)} M^n \longrightarrow 0$$

is exact. This sequence may be viewed as a complex of H-vector bundles on V with compact support [7; § 2], and so represents an element $v \in K_H(V)$.

We shall now show how to compute $\mathrm{ch}\, u$ when u is associated to an H-structure. To do this we consider any principal H-bundle P and put $T = P \times_H V$. Then we observe that

$$P \longmapsto \psi_T^{-1}\,\mathrm{ch}\,\alpha_P(v) \tag{2.14}$$

is a functor from principal H-bundles to rational cohomology, and so is given by an element of $H_H^*(\mathbf{Q})$. Now let $\dim V = 2l$, and let $\rho^*(e) \in H_H^*(\mathbf{Q})$ be the image of the Euler class e under the homomorphism

$$\rho^*\colon H^*_{\mathrm{SO}(2l)}(\mathbf{Q}) \longrightarrow H_H^*(\mathbf{Q})$$

induced by $\rho\colon H \to \mathrm{SO}(2l)$. Using (1.8), we then deduce

$$\left[\psi_T^{-1}\,\mathrm{ch}\,\alpha_P(v)\right]\cdot\rho^*(e)(P) = \mathrm{ch}\,\alpha_P(i^*v) \tag{2.15}$$

where $i^*\colon K_H(V) \to K_H(\text{point}) = R(H)$ is restriction to the origin, and α_P is also used for the homomorphism

$$\alpha_P\colon R(H) \longrightarrow K(X)$$

induced by $M \mapsto P \times_H M$ (for H-modules M). If $\rho^*(e) \neq 0$ in $H_H^*(\mathbf{Q})$, then (2.15) can be used to express the characteristic class (2.14) in terms of $i^*(v)$. If v is defined by a sequence of H-modules M^i and homomorphisms as described above, then

$$i^*(v) = \sum (-1)^i M^i \in R(H)\,.$$

By (1.5) the functor $P \mapsto \mathrm{ch}\,\alpha_P M$ gives the characteristic class

$$\mathrm{ch}\, M = \sum e^{y_i} \in H_H^*(\mathbf{Q})$$

where the y_i are the weights of M. Hence

$$\mathrm{ch}\,\alpha_P i^*(v) = \sum (-1)^i\,\mathrm{ch}\, M^i\,.$$

Using (2.15) and assuming $\rho^*(e) \neq 0$, we then get

$$\psi_T^{-1}\,\mathrm{ch}\,\alpha_P(v) = \frac{\sum (-1)^i\,\mathrm{ch}\, M^i}{\rho^*(e)}(P)\,. \tag{2.16}$$

To examine the condition $\rho^*(e) \neq 0$, we choose a maximal torus S of H map-

[5] φ_i is not necessarily linear although, as we shall see, the most interesting examples occur with φ_i linear.

ping into the standard maximal torus T of $SO(2l)$. If $x_1, \cdots, x_l \in \hat{T}$ are the basic characters, we have $e = \prod_1^l x_i$ and so $\rho^*(e) = \prod_1^l y_i$ where $y_i \in \hat{S}$ is the character of S induced by x_i. Thus $\rho^*(e) = 0$ is equivalent to the vanishing of one of the y_i or, in other words, to the existence of a fixed non-zero vector for S in R^{2l}.

Inserting (2.16) into the index formula (2.13), we then deduce

PROPOSITION (2.17). *Let $\rho: H \to SO(2l)$ be a homomorphism such that the maximal torus of H has no fixed non-zero vector in R^{2l}. Then $\rho^*(e) \neq 0$, where $e \in H^*_{SO(2l)}(\mathbf{Q})$ is the Euler class, and*

$$\rho^*: H^*_{SO(2l)}(\mathbf{Q}) \longrightarrow H^*_H(\mathbf{Q})$$

is induced by ρ. Now let X be a compact oriented manifold of dimension $2l$ with an H-structure, i.e., *we have a principal H-bundle P over X, and TX is associated to P via ρ. Let $M^0, \cdots, M^N$ be complex H-modules, $E^0, \cdots, E^N$ the associated vector bundles over X, and suppose*

$$0 \longrightarrow \mathcal{D}(E^0) \longrightarrow \cdots \longrightarrow \mathcal{D}(E^N) \longrightarrow 0$$

is an elliptic complex whose symbol class in $K(TX)$ is associated to the H-structure. Then the index of this complex is given by

$$(-1)^l \left\{ \frac{\sum (-1)^i \operatorname{ch} M^i}{\rho^*(e)}(P)\mathcal{J}(X) \right\}[X] .$$

Remarks. 1. This formula shows that, for H-structures (under the assumptions of (2.17)), the index depends only on the bundles E^i, or rather on the H-modules M^i. In other words, two complexes where the M^i are the same, but the operators are different, have the same index, provided always that the symbol class is associated to the H-structure.

2. The group H in (2.17) should not be confused with the group G in [7; (6.7)]. In (2.17) we are concerned with the ordinary integer-index not the more general character-index. Of course one could envisage an H-structure as in (2.17) invariant under a further group G. In particular, suppose G acts trivially on X, then it follows that the action of G on the principal H-bundle P is given by some homomorphism $\rho: G \to \text{Centre}\,(H)$. Let χ^i be the character of G induced by ρ from the character of the H-module M^i. Then the Lefschetz number $L(g, E)$ is given by the formula in (2.17) with $\operatorname{ch} M^i$ replaced by $\chi^i(g) \cdot \operatorname{ch} M^i$. The proof is essentially the same as that of (2.17).

3. Lefschetz fixed-point formula

In Theorem (2.12) of [5] we gave a "Lefschetz fixed-point formula" which computed Lefschetz numbers of automorphisms of elliptic complexes in terms

of the index of associated elliptic symbol classes on fixed-point sets. Simply by inserting the cohomological formula for the index given in the preceding section into Theorem (2.12) of [5], we shall then obtain a cohomological form of the Lefschetz fixed-point formula for elliptic complexes.

It will be convenient to adopt the following convention. Let X be a trivial G-space so that

$$K_G(X) \cong K(X) \otimes R(G) ,$$

and let $g \in G$.

The homomorphism

$$K_G(X) \longrightarrow H^*(X; \mathbf{C})$$

given by

$$u = \sum a_i \otimes \chi_i \longmapsto \sum \chi_i(g) \cdot \operatorname{ch} a_i$$

will be denoted by $\operatorname{ch} u(g)$. With this notation, Theorem (2.12) of [5] and (2.11) of this paper, give the formula

$$L(g, E) = (-1)^n \left\{ \frac{\operatorname{ch} i^* u(g)}{\operatorname{ch} \lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)} \mathcal{J}(X^g) \right\} [TX^g] . \tag{3.1}$$

Here E is an elliptic complex acted on by a topologically cyclic group G, with symbol class $u \in K_G(TX)$, X^g is the fixed point set of a generator g of G, i^* is the restriction to $K_G(TX^g)$, N^g is the normal bundle of X^g in X, and $n = \dim X^g$.

The denominator in (3.1) can be made more explicit as we shall now see. We first examine the action of G on the normal bundle N^g. For each $x \in X^g$, the fibre N^g_x of N^g at x is a real G-module. Since G is cyclic, its irreducible real representations are of two types

(i) one-dimensional with $g \mapsto \pm 1$,

(ii) two-dimensional with

$$g \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

In (ii) the representations given by θ and $-\theta$ are equivalent, and we may therefore restrict to the case $0 < \theta < \pi$. Such a two-dimensional real G-module has then a canonical complex structure in which g acts as the complex scalar $e^{i\theta}$. The real G-module N^g_x can therefore be written canonically as a direct sum

$$N^g_x = N^g_x(-1) \oplus \textstyle\sum_{0<\theta<\pi} N^g_x(\theta) . \tag{3.2}$$

The eigenvalue $+1$ does not occur, because N^g is normal to the fixed-point set X^g, and of course only a finite number of values of θ can occur (with non-zero space). Each space $N^g_x(\theta)$ has a natural complex structure in which g acts

as $e^{i\theta}$. Since the decomposition can be defined by projection operators, (3.2) defines a direct sum decomposition of the vector bundle N^q (cf. [1; (1.6.2)] for the analogous proof in the complex case)

$$N^q = N^q(-1) \oplus \textstyle\sum_{0<\theta<\pi} N^q(\theta) \,. \tag{3.3}$$

The bundle $N^q(-1)$ is real, and each $N^q(\theta)$ is complex. *Thus $N^q(-1)$ has Pontrjagin classes, and each $N^q(\theta)$ has Chern classes.*

We propose now to express the denominator of formula (3.1) in terms of these characteristic classes. From (3.3), we get

$$N^q \otimes_{\mathrm{R}} \mathrm{C} = (N^q(-1) \otimes_{\mathrm{R}} \mathrm{C}) \oplus \textstyle\sum_\theta (N^q(\theta) \oplus N^q(\theta)^*)$$
$$\lambda_{-1}(N^q \otimes_{\mathrm{R}} \mathrm{C}) = \lambda_{-1}(N^q(-1) \otimes_{\mathrm{R}} \mathrm{C}) \cdot \textstyle\prod_\theta \lambda_{-1}(N^q(\theta)) \cdot \prod_\theta \lambda_{-1}(N^q(\theta))^* \,.$$

For the real bundle $N^q(-1)$, we have

$$\mathrm{ch}\, \lambda_{-1}(N^q(-1) \otimes_{\mathrm{R}} \mathrm{C})(g) = \varepsilon\{\textstyle\prod_{j=1}^{r} (1 + e^{x_j})(1 + e^{-x_j})\} N^q(-1) \,, \tag{3.4}$$

where $r = [s(-1)/2]$, $s(-1) = \dim N^q(-1)$, $\varepsilon = 1$ or 2 according as $s(-1)$ is even or odd, and

$$\textstyle\prod_j (1 + e^{x_j})(1 + e^{-x_j}) \in H^*_{O(s(-1))}(\mathrm{C}) \,.$$

For the complex vector bundle $N^q(\theta)$, we have

$$\mathrm{ch}\, \lambda_{-1}(N^q(\theta))(g) = \{\textstyle\prod_{j=1}^{s(\theta)} (1 - e^{y_j + i\theta})\} N^q(\theta) \tag{3.5}$$

where $s(\theta) = \dim_{\mathrm{C}} N^q(\theta)$ and

$$\textstyle\prod_{j=1}^{s(\theta)} (1 - e^{y_j + i\theta}) \in H^*_{U(s(\theta))}(\mathrm{C}) \,.$$

Finally, for the dual $N^q(\theta)^*$, we have

$$\mathrm{ch}\, \lambda_{-1}(N^q(\theta))^*(g) = \{\textstyle\prod_{j=1}^{s(\theta)} (1 - e^{-y_j - i\theta})\} N^q(\theta) \,. \tag{3.6}$$

In order to express these formulas in terms of Pontrjagin and Chern classes, we shall introduce the sequences of polynomials

$$\mathcal{R}_r(p_1, \cdots, p_r) \qquad \mathcal{S}^\theta_r(c_1, \cdots, c_r)$$

defined by the formal identities

$$\left\{\prod_j \left(\frac{1 + e^{x_j}}{2}\right)\left(\frac{1 + e^{-x_j}}{2}\right)\right\}^{-1} = \sum \mathcal{R}_r(p_1, \cdots, p_r)$$
$$\left\{\prod_j \left(\frac{1 - e^{y_j + i\theta}}{1 - e^{i\theta}}\right)\left(\frac{1 - e^{-y_j - i\theta}}{1 - e^{-i\theta}}\right)\right\}^{-1} = \sum \mathcal{S}^\theta_r(c_1, \cdots, c_r) \,, \tag{3.7}$$

where the p_i are the elementary symmetric functions of the x_j^2, the c_i are the elementary symmetric functions of the y_j, and to define the r^{th} term of the sum, we take $\prod_j$ to be a product of r terms. More precisely, we put

$$\textstyle\prod_{j=1}^{N} = \sum \mathcal{R}_{r,N} \,,$$

and observe that $\mathcal{R}_{r,N}$ is independent of N for $r \leqq N$ and similarly for $\sum \mathbb{S}^\theta_r$.

We put

$$\mathcal{R} = \sum \mathcal{R}_r , \qquad \mathbb{S}^\theta = \sum \mathbb{S}^\theta_r .$$

By (1.3) we can regard these as characteristic classes (over C) of $O(n)$ and $U(n)$ respectively, for any n. For a real vector bundle E over a space Y, we have

$$\mathcal{R}(E) = \mathcal{R}\big(p_1(E), \cdots, p_n(E)\big) \qquad \in H^*(Y; \mathrm{C}) ,$$

and for a complex vector bundle F over Y,

$$\mathbb{S}^\vartheta(F) = \mathbb{S}^\theta\big(c_1(F), \cdots, c_n(F)\big) \qquad \in H^*(Y; \mathrm{C}) .$$

Note that $\mathcal{R}$ and $\mathbb{S}^\theta$ depend only on the stable classes of E, F, and not on the augmentation. Thus

$$\mathcal{R}(E \oplus 1_{\mathrm{R}}) = \mathcal{R}(E) \qquad \mathbb{S}^\theta(F \oplus 1_{\mathrm{C}}) = \mathbb{S}^\theta(F)$$

where 1_{R} and 1_{C} denote trivial line-bundles (real and complex respectively).

If we compare the formulas (3.7), we see that they differ from those in (3.4), (3.5), and (3.6) by a total factor

$$2^{s(-1)} \prod_\theta (1 - e^{i\theta})(1 - e^{-i\theta}) .$$

For any $x \in X^g$, this is just the value of

$$\det (1 - g \mid N^g_x) .$$

It is constant on each component of X^g, and can be regarded as an element of $H^0(X^g; \mathrm{C})$, we shall write this element as $\det (1 - g \mid N^g)$.

Combining (3.4), (3.5), and (3.6), we therefore get

$$\{\mathrm{ch}\, \lambda_{-1}(N^g \otimes_{\mathrm{R}} \mathrm{C})(g)\}^{-1} = \frac{\mathcal{R}(N^g(-1)) \prod_\theta \mathbb{S}^\theta(N^g(\theta))}{\det (1 - g \mid N^g)} . \tag{3.8}$$

Substituting this in (3.1), we finally obtain

LEFSCHETZ THEOREM (3.9). *Let g be a generator of the (topologically cyclic) compact Lie group G, X a compact G-manifold, E an elliptic complex on X on which G acts. Let X^g denote the fixed-point set of g, N^g the normal bundle of X^g in X, and*

$$N^g = N^g(-1) \oplus \textstyle\sum_{0<\theta<\pi} N^g(\theta)$$

the decomposition of N^g determined by the action of G. Let $u \in K_G(TX)$ be the symbol class of E, $i^*u \in K_G(TX^g)$ its restriction to X^g. Let $\mathcal{I}(X) \in H^*(X; \mathrm{Q})$ denote the Index class of X where $\mathcal{I}$ is defined by (2.10), *and let $\mathcal{R}$, $\mathbb{S}^\theta$ be the characteristic classes of the orthogonal and unitary groups defined by* (3.7). *Then the Lefschetz number $L(g, E)$ is given by*

$$L(g, E) = (-1)^n\left\{\frac{\operatorname{ch} i^*u(g)\mathcal{R}(N^g(-1))\prod_{0<\theta<\pi}\mathcal{S}^\theta(N^g(\theta))\cdot\mathcal{J}(X^g)}{\det(1-g\,|\,N^g)}\right\}[TX^g]$$

where $\det(1-g\,|\,N^g)\in H^0(X^g;\mathbf{C})$ *assigns to the component of* X^g *containing* x *the value* $\det(1-g\,|\,N_x^g)$, *and* n *assigns to each component its dimension.*

Remark. Theorem (3.9) gives a complete cohomological formula for the index character

$$\text{a-ind: } K_G(TX)\longrightarrow R(G)$$

for any compact Lie group G. To evaluate $\chi=\operatorname{ind} u$ on an element $g\in G$, we simply apply (3.9) to the closed cyclic group H generated by g.

The formula for $L(g, E)$ given in (3.9) involves the symbol u of E and the following cohomological invariants of (X, G):

(i) Pontrjagin classes of X^g,

(ii) Pontrjagin classes of $N^g(-1)$,

(iii) Chern classes of all the $N^g(\theta)$.

If X^g is oriented then, as in (2.13), we can replace the evalution on $[TX^g]$, by an evalution on $[X^g]$. We simply replace $(-1)^n\operatorname{ch} i^*(u)(g)$ in (3.9) by

$$(-1)\frac{n(n+1)}{2}\psi^{-1}(\operatorname{ch} i^*(u)(g))\,, \tag{3.10}$$

where $n=\dim X^g$, and

$$\psi\colon H^*(X^g;\mathbf{C})\longrightarrow H^*(TX^g;\mathbf{C})$$

is the Thom isomorphism. If moreover $i^*(u)$ is associated to an H-structure on X^g, as in (2.17), we can compute (3.10) in terms of characteristic classes of this H-structure. This situation will be illustrated by some of the examples we consider in subsequent sections.

We should perhaps point out that in practice, for the "classical" operators, it is usually simpler to go back to the K-theory formulas of [5] rather than apply the formidable cohomological expression in (3.9).

4. The Riemann-Roch theorem

One of the most important examples of an elliptic complex arises in connection with complex manifolds. The index theorem (2.12) becomes in this case the Hirzebruch form of the celebrated Riemann-Roch theorem [13]. In fact this will be a case of an H-structure, and so we will be able directly to apply Proposition (2.17). We proceed to describe the details.

Let X be a compact complex manifold of dimension n, V a holomorphic vector bundle over X, $\mathcal{O}(V)$ the sheaf of germs of holomorphic sections of V. Moreover let $A(V)$ denote the Dolbeault complex of V

$$0 \longrightarrow A^{0,0}(V) \xrightarrow{\bar\partial} A^{0,1}(V) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} A^{0,n}(V) \longrightarrow 0$$

where $A^{0,p}(V)$ denotes the differential forms of type $(0, p)$ with coefficients in V. The Dolbeault isomorphism [13; § 15] asserts that the cohomology groups of $A(V)$ are isomorphic to the sheaf cohomology groups $H^q(X, \mathcal{O}(V))$, and so $\sum (-1)^q \dim H^q(A(V)) = \sum (-1)^q \dim H^q(X, \mathcal{O}(V))$ is the Euler characteristic $\chi(X, V)$ which one wants to compute in the Riemann-Roch theorem.

The complex $A(V)$ is easily seen to be elliptic. In fact, let TX denote now the *complex* tangent bundle of X and choose a hermitian metric on X, so that we can identify $\bar T$ with the bundle of forms of type $(1, 0)$ and so T with the bundle of forms of type $(0, 1)$. Then we see that the symbol sequence of $A(V)$ is just the complex over TX defined by $V \otimes \Lambda^*(T)$, and hence is exact outside the zero section of TX. Moreover this shows that the symbol class $a(V)$ of $A(V)$ is given by

$$a(V) = [V]\lambda_T \in K(TX)\ , \tag{4.1}$$

where $\lambda_T \in K(TX)$ is the fundamental element, given by $\Lambda^*(T)$, and $K(TX)$ is regarded as a $K(X)$-module. Now λ_T is a symbol associated to the $U(n)$-structure of X (given by the hermitian metric). For the case $V = 1$ therefore, we are precisely in a position to apply (2.17), the M^i being the $U(n)$-modules $\lambda^i(\mathbf{C}^n)$. For the general case we either observe from (4.1) that ch (V) simply enters (2.12) as a multiplier or we formally give X a $U(n) \times U(m)$ structure where $m = \dim V$, and $U(m)$ acts trivially on X. This makes $V \cdot \lambda_T$ an associated symbol-class, and we can then apply (2.17) directly. Either way we obtain the formula

$$\chi(X, V) = (-1)^n \left\{ \mathrm{ch}\ V \frac{\prod (1 - e^{x_i})}{\prod x_i}(T) \cdot \mathcal{J}(X) \right\}[X] \tag{4.2}$$

where

$$\frac{\prod (1 - e^{x_i})}{\prod x_i} \in H^*_{U(n)}(\mathbf{Q})$$

is applied to the complex tangent bundle T, and

$$\begin{aligned} \mathcal{J}(X) &= \mathcal{J}(T \otimes_{\mathbf{R}} \mathbf{C}) = \mathcal{J}(T \oplus T^*) \\ &= \prod \left(\frac{x_i}{1 - e^{-x_i}} \cdot \frac{-x_i}{1 - e^{x_i}} \right)(T)\ . \end{aligned}$$

Thus cancelling the factor $\prod x_i(1 - e^{-x_i})^{-1}$, we end up finally with

RIEMANN-ROCH THEOREM (4.3). *Let X be a compact complex manifold, V a holomorphic vector bundle over X. Let*

$$\mathcal{T}(X) = \prod \frac{x_i}{1 - e^{-x_i}}(T)$$

be the Todd class of X. *Then the Euler characteristic* $\chi(X, V)$ *of the sheaf of germs of holomorphic sections of* V *is given by*

$$\chi(X, V) = \{\mathrm{ch}\, V \cdot \mathcal{T}(X)\}[X] .$$

Remarks. 1. This theorem was previously known only for projective algebraic manifolds [13]. The proof we have given, using general elliptic operators, is the only one known at present for the general case.

2. For a complex manifold X, we know, by the Thom isomorphism in K-theory, that $K(TX)$ is a free $K(X)$-module generated by λ_T. In other words, there is a single elliptic symbol which, in a sense, generates everything. It is thus natural to replace the index homomorphism

$$K(TX) \longrightarrow \mathbf{Z}$$

of the general theory by the homomorphism

$$K(X) \longrightarrow \mathbf{Z}$$

given by $v \mapsto \mathrm{ind}\,(v\lambda_T)$. This is in effect what is usually done in algebraic geometry.

If now G is a finite[6] group of automorphisms of the pair (X, V), we can apply our general Lefschetz theorem (3.9). Alternatively we can return to Theorem (3.3) of [5] and combine that with the Riemann-Roch theorem (4.3). This gives

$$\sum (-1)^p \,\mathrm{Trace}\,(g \mid H^p(X, \mathcal{O}(V))) = \left\{\frac{\mathrm{ch}\,(V \mid X^g)(g)\mathcal{T}(X^g)}{\mathrm{ch}\,\lambda_{-1}((N^g)^*)(g)}\right\}[X^g] . \tag{4.4}$$

Now the complex vector bundle N^g has a decomposion

$$N^g = \sum N^g(\theta) ,$$

where $N^g(\theta)$ is the sub-bundle on which g acts as $e^{i\theta}$. Note that this notation differs slightly from that adopted in § 3 where N^g was a real bundle. Each $N^g(\theta)$ is a complex vector bundle and has Chern classes. Moreover we have

$$\mathrm{ch}\,\lambda_{-1}(N^g(\theta))^* = \prod_j (1 - e^{-x_j - i\theta})(N^g(\theta)) ,$$

where

$$\prod_j (1 - e^{-x_j - i\theta}) \in H^*_{U(m)}(\mathbf{C}) , \qquad m = \dim N^g(\theta) .$$

Let us therefore introduce for $0 < \theta < 2\pi$ the stable characteristic class

$$\mathcal{U}^\theta = \sum \mathcal{U}^\theta_r = \left\{\prod_j \left(\frac{1 - e^{-x_j - i\theta}}{1 - e^{-i\theta}}\right)\right\}^{-1} . \tag{4.5}$$

[6] This is perhaps more natural in the holomorphic case.

Each $\mathfrak{U}_r^q$ is thus a polynomial with complex coefficients in the Chern classes. With this notation we have

$$\{\mathrm{ch}\,\lambda_{-1}(N^g(\theta))^*\}^{-1} = \frac{\mathfrak{U}^\theta(N^g(\theta))}{(1-e^{-i\theta})^m}\,.$$

Thus, taking the product over all θ,

$$\{\mathrm{ch}\,\lambda_{-1}(N^g)^*\}^{-1} = \frac{\prod \mathfrak{U}^\theta(N^g(\theta))}{\det_{\mathbf{C}}(1-g(N^g)^*)}\,,$$

where $\det_{\mathbf{C}}(1-g\,|\,(N^g)^*) \in H^0(X^g;\mathbf{C})$ assigns to the component of $x \in X^g$ the value $\det_{\mathbf{C}}(1-g\,|\,(N_x^g)^*)$. Substituting this in (4.4), we finally obtain.

HOLOMORPHIC LEFSCHETZ THEOREM (4.6). *Let X be a compact complex manifold, V a holomorphic vector bundle over X, and let G be a finite group of automorphisms of the pair (X, V). For any $g \in G$, let X^g denote the fixed point set of g, and let*

$$N^g = \sum N^g(\theta)$$

denote the (complex) normal bundle of X^g decomposed according to the eigenvalues $e^{i\theta}$ of g. Let $\mathfrak{U}^\theta$ denote the characteristic class defined by (4.5). *Then we have*

$$\sum (-1)^p \,\mathrm{Trace}\,(g\,|\,H^p(X, \mathcal{O}(V))) = \left\{\frac{\mathrm{ch}\,(V\,|\,X^g)(g)\cdot\prod_\theta \mathfrak{U}^\theta(N^g(\theta))\cdot\mathcal{T}(X^g)}{\det(1-g\,|\,(N^g)^*)}\right\}[X^g]\,.$$

Using this and Theorem (3.5) of [5], we are then led to

RIEMANN-ROCH THEOREM FOR ORBIT SPACES (4.7). *Let G be a finite group of automorphisms of a compact complex manifold X, and let W be a holomorphic vector bundle over the complex space $Y = X/G$. Then we have*

$$\chi(Y, W) = \frac{1}{|G|}\sum_{g\in G}\mu(g)$$

where

$$\mu(g) = \left\{\frac{\mathrm{ch}\,(f^*W\,|\,X^g)(g)\prod_\theta \mathfrak{U}^\theta(N^g(\theta))\mathcal{T}(X^g)}{\det_{\mathbf{C}}(1-g\,|\,(N^g)^*)}\right\}[X^g]$$

and $N^g(\theta)$, $\mathfrak{U}^\theta$ are as in (4.6).

In [14] Hirzebruch used the Riemann-Roch theorem to compute the dimensions of certain spaces of automorphic forms. The extension (4.7) of the Riemann-Roch theorem to certain singular spaces can be used in the same way and leads to formulas extending those of [14] (cf. Langlands [16]). This will be explained elsewhere (see also [15]).

Theorem (4.6) implies certain "integrality conditions" for the fixed-point sets of G. In fact, the Lefschetz number is an algebraic integer, while the expression on the right-hand side of the formula in (4.6) is *a priori* only an algebraic number. The special case of isolated fixed-points is discussed in [2], and already this shows that these integrality theorems are non-trivial. Of course not every function from G to algebraic integers is a character. Thus the integrality theorems arising from (4.6) for a general finite group G are not just consequences of those for cyclic groups.

As a simple example of (4.6), let us work out the case when $\dim X = 2$, i.e., when X is a complex surface. Assuming that g acts non-trivially, and that X is connected, the fixed point set X^g consists of a finite set of points P_j and curves D_k. We shall, for simplicity, only work out the case when $V = 1$. Thus the Lefschetz number $L(g)$ will be given by

$$L(g) = \sum_j a(P_j) + \sum_k b(D_k) ,$$

where the numbers $a(P)$ and $b(D)$ are given as follows:

$$a(P) = \frac{1}{\det(1 - g \mid T_P)}$$

$$b(D) = \left\{\frac{1 + c/2}{1 - e^{-d-i\theta}}\right\}[D]$$

where c is the first Chern class of D, d is the first Chern class of the normal bundle of D, and $e^{i\theta}$ is the eigenvalue of g on the normal bundle. Evaluating $b(D)$ we get

$$\begin{aligned} b(D) &= \left\{\frac{1 + c/2}{1 - e^{-i\theta}(1 - d)}\right\}[D] \\ &= \left\{(1 - e^{-i\theta})^{-1}\left(1 + \frac{c}{2}\right)\left(1 + \frac{de^{-i\theta}}{1 - e^{-i\theta}}\right)^{-1}\right\}[D] \\ &= (1 - e^{-i\theta})^{-1}\left\{\frac{c[D]}{2} - \frac{e^{-i\theta}}{1 - e^{-i\theta}}d[D]\right\} . \end{aligned}$$

But $c[D] = 2(1 - \pi_D)$, where π_D is the genus of D, and $d[D] = D^2$ is the self-intersection number of D. Thus

$$b(D) = \frac{1 - \pi_D}{1 - e^{-i\theta}} - \frac{e^{-i\theta}}{(1 - e^{-i\theta})^2}D^2 .$$

In particular, if $g^2 = 1$, then $e^{i\theta} = -1$, $a(P) = 1/4$, $b(D) = (1 - \pi_D)/2 + (1/4)D^2$. Thus, as a very simple special case of (4.6), we get

PROPOSITION (4.8). *Let g be a non-trivial involution of a connected complex surface. Let the fixed-point set of g consist of N isolated points and M irreducible curves $\{D_k\}$. Then the holomorphic Lefschetz number $L(g)$ is*

given by

$$L(g) = \frac{N}{4} + \sum_{k=1}^{M} \left(\frac{1 - \pi_k}{2} + \frac{D_k^2}{4} \right),$$

where π_k is the genus of D_k and D_k^2 is its self-intersection number.

Remark. If X is algebraic, then the holomorphic differentials are birational invariants. Thus if we blow up an isolated fixed-point, g will induce $\tilde{g}$ on $\tilde{X}$, and $L(g) = L(\tilde{g})$. This checks with the formula in (4.8) because we lose one point and gain a rational curve with self-intersection -1. Another simple check is to take the involution $(x_0, x_1, x_2) \mapsto (-x_0, x_1, x_2)$ of the projective plane. There is one isolated fixed point, and one fixed line (so $\pi = 0$) with self-intersection $+1$. Thus $L(g) = 1/4 + 1/2 + 1/4 = 1$ which is correct, since there are no holomorphic differentials in positive dimension.

Since the Lefschetz number of an involution is always an integer, (4.8) implies that the number of curves D_k with odd self-intersection number has the same parity as N. This is a simple example of an "integrality theorem" involving higher-dimensional fixed-point sets.

5. The Dirac operator

An interesting elliptic operator called the Dirac[7] operator exists on spin-manifolds. We proceed to discuss this in detail and to examine the general theorems of §§ 2, 3 in this special case.

We recall that $SO(n)$ has a double covering Spin (n) which is its universal covering for $n > 2$. A spin-structure on an oriented compact manifold X of dimension n will mean a Spin (n)-structure in the sense of § 2. Thus we assume given a principal Spin (n)-bundle P over X and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin}(n)} \mathbf{R}^n \cong TX .$$

Equivalently P is a double covering of the princical $SO(n)$-bundle $\mathbf{Q}$ of X (for some riemannian metric) such that P_x is the Spin double covering of Q_x.

The group Spin (n) has a complex representation space Δ of dimension 2^n called the spin-representation[8]. Moreover Δ is a module for the Clifford algebra of $\mathbf{R}^n$ for the negative definite form $-\sum_1^n x_i^2$. In fact Spin (n) is defined as a subgroup of the group of units of the Clifford algebra, and the action of Spin (n) is induced by the Clifford multiplication. If $g \in$ Spin (n),

[7] Of course the original Dirac operator is defined for the indefinite relativistic metric and is not elliptic.

[8] For results about Spin groups and Clifford algebras, the reader may refer to [3] and [8].

$x \in \mathbf{R}^n$, $u \in \Delta$, we have therefore

$$g(xu) = gxg^{-1} \cdot g(u) = \rho(g)x \cdot g(u) ,$$

where ρ: Spin$(n) \to SO(n)$ is the covering map. Thus the Clifford multiplication

$$\mathbf{R}^n \otimes \Delta \longrightarrow \Delta$$

is a homomorphism of Spin (n)-modules. If $n = 2l$ the representation Δ is the direct sum of two irreducible representations $\Delta^{\pm}$ of dimension 2^{n-1}. The decomposition $\Delta = \Delta^+ \oplus \Delta^-$ is interchanged by Clifford multiplication. Thus we have homomorphisms

$$\mathbf{R}^n \otimes \Delta^+ \longrightarrow \Delta^-$$
$$\mathbf{R}^n \otimes \Delta^- \longrightarrow \Delta^+ .$$

If $x_1, \cdots, x_l$ are the basic characters of the maximal torus T_0 of $SO(2l)$, they can also be considered as characters of the maximal torus T of Spin $(2l)$. Since T double covers T_0, the character group $\hat{T}_0$ is of index 2 in $\hat{T}$; an element in the other coset is $\frac{1}{2}(x_1 + \cdots + x_l)$. The weights of Δ^+ are the characters

$$\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_l)$$

with an even number of minus signs. The weights of Δ^- are those with an odd number of minus signs. Thus we have

$$\begin{aligned} \operatorname{ch} \Delta &= \prod_{i=1}^{l} (e^{x/2} + e^{-x/2}) \\ \operatorname{ch} \Delta^+ - \operatorname{ch} \Delta^- &= \prod_{i=1}^{l} (e^{x/2} - e^{-x/2}) . \end{aligned} \tag{5.1}$$

These characteristic classes may be viewed as characteristic classes of either Spin $(2l)$ or of $SO(2l)$, since over $\mathbf{Q}$ the characteristic rings of these two groups coincide.

If X is a Spin $(2l)$-manifold with principal bundle P we can form the associated complex vector bundles

$$E^{\pm} = P \times_{\text{Spin}(2l)} \Delta^{\pm} , \qquad E = E^+ \oplus E^- .$$

The *Dirac operator* is a first order differential operator

$$D: \mathfrak{D}(E) \longrightarrow \mathfrak{D}(E)$$

defined as follows. Recall first that the riemannian metric defines a natural $SO(2l)$-connection, and this lifts to give a connection for P. We may therefore consider the covariant derivative

$$\partial: \mathfrak{D}(E) \longrightarrow \mathfrak{D}(E \otimes T^*) .$$

On the other hand $T \cong T^*$, and we have the bundle homomorphism

$$\mathfrak{D}(E \otimes T) \longrightarrow \mathfrak{D}(E)$$

induced by Clifford multiplication. We define D to be the composition of these two maps. Thus in terms of an orthonormal base e_i of T, we have

$$Ds = \sum e_i(\partial_i s)$$

where $\partial_i s$ is the covariant derivative of s in the direction e_i and $e_i(\ )$ denotes Clifford multiplication. The symbol of D is (up to a factor i) just Clifford multiplication, i.e., its value at $\xi \in T_x$ is the homomorphism

$$\Delta_x \longrightarrow \Delta_x$$

given by Clifford multiplication by ξ. Since $\xi(\xi(u)) = -\|\xi\|^2 \cdot u$ it follows that D is elliptic. Moreover because of the properties of Clifford multiplication D induces two operators

$$D^+: \mathfrak{D}(E^+) \longrightarrow \mathfrak{D}(E^-), \qquad D^-: \mathfrak{D}(E^-) \longrightarrow \mathfrak{D}(E^+)$$

each of which is elliptic. If we introduce the natural inner product on $\mathfrak{D}(E)$ defined by taking the inner product in each fibre, and then integrating over X with respect to the riemannian measure, we find that D is formally self-adjoint, and so D^- is the formal adjoint of D^+.

A solution of $Ds = 0$ will be called a *harmonic spinor*, and the space of harmonic spinors will be denoted by H. It decomposes:

$$H = H^+ \oplus H^-,$$

where $H^+ = \operatorname{Ker} D^+$, $H^- = \operatorname{Ker} D^- \cong \operatorname{Coker} D^+$. Let us put $h^\pm = \dim H^\pm$, so that

$$\operatorname{index} D^+ = h^+ - h^-.$$

We shall refer to this as the *spinor index* of X, and write it as Spin (X).

Finally following Hirzebruch [14], let us define the stable characteristic class $\hat{\mathfrak{A}}$ by

$$\hat{\mathfrak{A}} = \prod_{i=1}^{l} \left(\frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \right) \in H^*_{0(2l)}(\mathbb{Q}). \tag{5.2}$$

For any real vector bundle E,

$$\hat{\mathfrak{A}}(E) = \sum \hat{\mathfrak{A}}_r(p_1(E), \cdots, p_r(E))$$

is an explicit polynomial in the Pontrjagin classes of E.

For a manifold X, we put

$$\hat{\mathfrak{A}}(X) = \hat{\mathfrak{A}}(T(X)).$$

Since the operator D^+ has a symbol-class associated to the Spin-structure, we are in a position to apply the index theorem in the form given by Proposition (2.17). We get

$$\text{index } D^+ = (-1)^l\left\{\left\{\prod\left(\frac{e^{x_i/2}-e^{-x_i/2}}{x_i}\right)\cdot\prod\left(\frac{x_i}{1-e^{x_i}}\right)\prod\left(\frac{-x_i}{1-e^{-x_i}}\right)(X)\right\}\right\}[X]$$
$$= (-1)^l\hat{\mathfrak{A}}(X)[X])\ .$$

Since $\hat{\mathfrak{A}}$ involves only Pontrjagin classes we have

$$\text{index } D^+ = 0 \qquad \text{if } \dim X \not\equiv 0\ (4)$$
$$= \hat{\mathfrak{A}}(X)[X] \qquad \text{if } \dim X \equiv 0\ (4)\ .$$

Thus we have established

THEOREM (5.3). *Let X be a Spin manifold of dimension $4k$, then the spinor index is equal to $\hat{\mathfrak{A}}(X)[X]$ where $\hat{\mathfrak{A}}$ is the characteristic class given by* (5.2).

Remarks. 1. The number

$$\hat{A}(X) = \hat{\mathfrak{A}}(X)[X]$$

is called the $\hat{A}$-genus of X.

2. It is easy to extend (5.3) to spinors with coefficients in a vector bundle V. One has a natural Dirac-type operator whenever V is associated to the tangent bundle. For other V, one has to choose first a specific connection.

3. Lichnerowicz [17] has shown that there are no harmonic spinors if the riemannian metric has strictly positive scalar curvature. In such circumstances (5.3) implies that the $\hat{A}$-genus is zero. The complex projective space $P_{2k}(\mathbf{C})$ is an example of a manifold with positive curvature and $\hat{A}$-genus non-zero; however $P_{2k}(\mathbf{C})$ has no Spin structure.

4. One also has a Dirac operator and a theorem analogous to (5.3) for Spin^c-manifolds where Spin^c is the complex spinor group of [3]. This case provides a natural unification between Spin and almost complex structures. It is in this context also that one has a fundamental element in $K(TX)$ which generates this freely as a $K(X)$-module. In particular for Spin manifolds the symbol class of D^+ is such a generator. This follows from the Thom isomorphism in K-theory for Spin (or Spin^c) bundles [3; 12.3].

Suppose now that X is a Spin $(2l)$-manifold and that the group G acts on the Spin-structure of X. This means not only that G acts on the manifold X and hence on the principal orthogonal bundle Q but also that we are given an action of G on P (the principal Spin-bundle) compatible with its action on Q. In these circumstances we can define the Spinor index of X as a character of G:

$$\text{Spin}\,(G, X) \in R(G)\ .$$

The value of this character at g (i.e., the Lefschetz number $L(g, D^+)$) will be denoted by Spin (g, X). Our general Lefschetz theorem (3.9) will of course

give an explicit formula for this in terms of the fixed point set X^g. We shall compute it in the simple case when g is an *involution* on X. Note that g is not necessarily an involution on the Spin-structure: its action there can have order 4.

For simplicity, let us assume that X^g is orientable, and fix an orientation. Then over X^g we have a diagram of bundles

$$\begin{array}{ccc} P_0 & \subset & P \\ \downarrow & & \downarrow \\ Q_0 & \subset & Q \end{array}$$

where Q_0 is the principal $SO(2k) \times SO(2l-2k)$ bundle given by the tangent and normal bundles of X^g (as usual $2k = \dim X^g$ depends on the component), and P_0 is a principal bundle with structure group

$$H_0 = \text{Spin}\,(2k) \times_{Z_2} \text{Spin}\,(2l-2k)\,.$$

The element g acts on P_0 through the element $(1, \alpha) \in H_0$ where $\alpha \in \text{Spin}\,(2l-2k)$ is one of the two elements lying over $-1 \in SO(2l-2k)$. If $u \in K_G(TX)$ is the symbol class of the Dirac operator D^+ (and G is the group generated by g), then $i^*u \in K_G(TX^g)$ is a symbol class associated to the H_0-structure P_0 of X^g and invariant under G. Using remark 2 after (2.17), and the character formula (5.1), we then find that the general Lefschetz formula (3.9) reduces to

$$\text{Spin}\,(g, X) = \textstyle\sum_j \varepsilon_j i^{k(j)} \dfrac{\hat{A}(X_j^g)}{\text{ch}\,\Delta(N_j^g)}[X_j^g]\,. \tag{5.4}$$

where the summation is over the connected components X_j^g of X^g and $\varepsilon_j = \pm 1$ depends in a rather subtle global manner on the particular component.

For a further more detailed discussion of this sign question, see [2; § 8] where the case of isolated fixed point ($k = 0$) is dealt with, each fixed point then contributes $\pm i^k/2^l$ in (5.4).

6. The signature

For any compact oriented manifold X of dimension $4k$, the cup-product defines a non-degenerate quadratic form on $H^{2k}(X; \mathbf{R})$. Let $p^+(p^-)$ denote the maximal dimension of a subspace on which this form is positive (negative) definite. The difference $p^+ - p^-$ is usually called the index of X and has been extensively studied by Thom and Hirzebruch. An alternative terminology is to call this invariant the *signature* and, to avoid a possible confusion with the index of elliptic operators, we shall adopt this alternative. For brevity we shall write it as Sign (X).

We propose to show that, in fact, Sign (X) *is* the index of a certain elliptic operator on X associated to the $SO(4k)$-structure (i.e., to the orientation and a riemannian metric). The index theorem in the form (2.17) will then give an explicit formula for Sign (X) in terms of Pontrjagin classes. This formula was originally obtained by Hirzebruch using Thom's results on cobordism. Our derivation of it is more direct and is independent of cobordism[9].

We shall also apply the general Lefschetz theorem (3.9) to the elliptic operator referred to above. This will then give a formula for the G-signature of a G-manifold X. This G-signature is a character of G defined by the action of G on $H^{2k}(X; \mathbf{R})$ and our formula will express this in terms of Pontrjagin classes and fixed-point sets. The particular interest of this case is that the result can be formulated purely in terms of differential topology. Our proof however involves analysis in an essential way. Although the original Hirzebruch signature formula was proved by purely topological means, it is not possible to reduce the problem of calculating the G-signature to the special case of the Hirzebruch signature. The reasons for this will become apparent later. Because of this, our formula for the G-signature seems to provide one of the best topological applications of the theory of elliptic operators. In fact a very special case, discussed in detail in [2] because it involves only isolated fixed-points, has already proved its worth in establishing a conjecture of Milnor about lens spaces. It is to be hoped that the more general case developed here will have further interesting applications.

For any compact manifold X of dimension n, we have the de Rham complex Ω of (complex-valued) exterior differential forms

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \longrightarrow 0 .$$

The symbol of this is just the complex defined by the exterior algebra of the cotangent bundle T^*, and so it is certainly elliptic. Its cohomology groups, by the theorems of de Rham, are naturally isomorphic to the ordinary complex cohomology groups $H^q(X; \mathbf{C})$. Its Euler characteristic is therefore the usual Euler characteristic

$$E(X) = \sum (-1)^q \dim H^q(X; \mathbf{C}) .$$

If X is oriented and of even dimension the index theorem in this case merely asserts the well known fact that

$$E(X) = e(X)[X]$$

is obtained by evaluating the Euler class of X on $[X]$. This is not very excit-

[9] This was not true of the first method of proof of the index theorem as given in [19].

ing. Similarly, if a compact group G acts on X, it acts on the de Rham complex, and our Lefschetz theorem reduces to the classical Lefschetz fixed point formula, at least when the fixed-points are isolated. The general case is slightly more interesting, every component of the fixed-point set now contributes its Euler characteristic to the Lefschetz formula. We leave the details to the reader.

To get a really interesting index problem out of the de Rham complex we have to proceed differently. We assume from now on that $\dim X = 2l$, that X is oriented and that we have chosen a riemannian metric on X. This induces metrics on the bundles of i-forms and hence, by integration over X, inner products in the space Ω^i. The formal adjoint of d with respect to these inner products is denoted by d^*. The operator

$$\Delta = dd^* + d^*d$$

is the Laplace operator of the Hodge theory. It is an elliptic operator and preserves the degree of the differential forms, i.e., it maps each Ω^i into itself. The solutions of $\Delta u = 0$ are the *harmonic forms*, and the space H^i of harmonic i-forms is isomorphic to

$$H^i(\Omega) \cong H^i(X; \mathbf{C}) .$$

We consider now the first order operator $D = d + d^*$. It is formally self-adjoint, and we have (since $d^2 = (d^*)^2 = 0$)

$$\Delta = D^*D = D^2 .$$

From this, it follows that the solutions of $Du = 0$ coincide with the solutions of $\Delta u = 0$, i.e., with the harmonic forms. In fact, if $\Delta u = 0$, we have

$$0 = (\Delta u, u) = (Du, Du) ,$$

and so $Du = 0$.

Now the riemannian metric induces a bundle isomorphism

$$*: \lambda^i(T^*) \longrightarrow \lambda^{2l-i}(T^*) .$$

If α, β are real i-forms, their inner product is then given by

$$(\alpha, \beta) = \int \alpha \wedge *\beta .$$

If we extend $*$ linearly to the complexified forms, we then find that, for $\alpha, \beta \in \Omega^i$, their hermitian inner product is given by

$$(\alpha, \beta) = \int \alpha \wedge \overline{*\beta} .$$

Since $*$ arises from the pairing $\alpha \otimes \beta \mapsto \alpha \wedge \beta$, it follows that

(6.1) $$*(*\alpha) = (-1)^i\alpha \qquad \text{when } \alpha \in \Omega^i .$$

From this and the integral formula for the inner product, one then finds

(6.2) $$d^*\alpha = -*d*\alpha .$$

We now introduce a map τ on differential forms defined by

$$\tau(\alpha) = i^{p(p-1)+l}*\alpha , \qquad \alpha \in \Omega^p .$$

Note that τ is real if l is even, and imaginary if l is odd. Then $\tau^2(\alpha) = i^{2l}*^2\alpha$ where

$$\begin{aligned}\varepsilon &= p(p-1) + l + (2l-p)(2l-p-1) + l \\ &= 4l^2 - 4lp + 2p^2 \equiv 2p \mod 4 .\end{aligned}$$

Since $*^2\alpha = (-1)^p\alpha$, it follows that $\tau^2\alpha = \alpha$, and so τ is an involution. We can therefore decompose the space $\Omega = \sum \Omega^i$ into the ± 1-eigenspaces $\Omega^\pm$ of τ. Since τ is a bundle homomorphism, we have a bundle decomposition

$$\Lambda^*(T^* \otimes_{\mathbf{R}} \mathbf{C}) = \Lambda_+ \oplus \Lambda_- ,$$

and $\Omega^\pm$ are the spaces of smooth sections of the bundle $\Lambda_\pm$. Using (6.1) and (6.2), one then verifies that

$$D\tau = -\tau D ,$$

and so D maps Ω^+ into Ω^- and Ω^- into Ω^+. We denote by $D^\pm$ the restriction of D to the appropriate subspaces of Ω. Thus

$$D^+ : \Omega^+ \longrightarrow \Omega^- , \qquad D^- : \Omega^- \longrightarrow \Omega^+ .$$

Each is elliptic, and they are formal adjoints of each other. Moreover the solutions of $D^+u = 0$ are just the harmonic forms in Ω^+, and similarly for Ω^-. We shall denote by H^+ and H^- these subspaces of harmonic forms, and we put

$$h^+ = \dim H^+ , \qquad h^- = \dim H^- .$$

Thus we have

$$\text{index } D^+ = h^+ - h^- .$$

Since $\Delta\tau = \tau\Delta$, it follows that τ induces an involution on the harmonic forms; the ± 1 eigenspaces are precisely H^+ and H^-. Now the space

$$V_k = H^k \oplus H^{2l-k} \qquad 0 \leqq k < l$$

is stable under τ; in fact τ switches the two factors in this decomposition. Thus the dimensions of the $+1$ and -1 eigenspaces of τ in V_k are equal, and so V_k contributes zero to index D^+. Hence

$$\text{index } D^+ = h^l_+ - h^l_-$$

where

$$h^l_\pm = \dim H^l_\pm$$

and $H^l_\pm$ is the ± 1-eigenspace of τ in H^l. We now distinguish two cases according to the parity of l. First suppose l is odd, then $\tau = i\sigma$ with σ real, and $\sigma^2 = -1$. Then the $\pm i$-eigenspaces of σ are conjugate, and so have the same dimension. But these are the ∓ 1-eigenspaces of τ, and so index $D^+ = 0$ in this case. Supose now $l = 2k$ is even, then for $\alpha \in H^{2k}$ we have $\tau\alpha = *\alpha$. Thus for $\alpha \in H^{2k}_+$ real and non-zero we have

$$\int \alpha \wedge \alpha = \int \alpha \wedge *\alpha = (\alpha, \alpha) > 0 ,$$

while if $\alpha \in H^{2k}_-$ is real and non-zero

$$\int \alpha \wedge \alpha = -\int \alpha \wedge *\alpha = -(\alpha, \alpha) < 0 .$$

Since $\alpha \mapsto \int \alpha \wedge \alpha$ is precisely the quadratic form given by the cup-product on $H^{2k}(X; \mathbf{R})$, it follows that

$$h^{2k}_+ = p^+ \qquad h^{2k}_- = p^- ,$$

and so index $D^+ = \mathrm{Sign}\,(X)$ is the Hirzebruch signature of X.

Now it is clear that the symbol of D^+ is associated to the $SO(2l)$-structure of X so that its index can be computed by applying the index theorem in the form of (2.17). It remains only to calculate the characteristic class

$$T \longmapsto \mathrm{ch}\,\Lambda_+(T) - \mathrm{ch}\,\Lambda_-(T) .$$

Now $\Lambda_\pm(R^{2l})$ are $SO(2l)$-modules, defined as above as ∓ 1-eigenspaces of τ. Also τ is multiplicative, i.e.,

$$\tau(u \wedge v) = \tau(u) \wedge \tau(v)$$

for $u \in \Lambda^*(C^{2l})$, $v \in \Lambda^*(C^{2k})$, $u \wedge v \in \Lambda^*(C^{2l+2k})$. This implies that the image of

$$\Lambda_+(R^{2l+2k}) - \Lambda_-(R^{2l+2k}) \qquad \in R\,(SO(2l + 2k))$$

in $R(SO(2l)) \times R(SO(2k))$ is

$$\{\Lambda_+(R^{2l}) - \Lambda_-(R^{2l})\}\{\Lambda_+(R^{2k}) - \Lambda_-(R^{2k})\} .$$

Hence passing from $SO(2l)$ to its maximal torus S, we see that $\Lambda_+(R^{2l}) - \Lambda_-(R^{2l})$ restricts to $\prod_{i=1}^l f(x_i)$ where $x_1, \cdots, x_l$ are the basic characters of S and

$$f(x) = \Lambda_+(R^2) - \Lambda_-(R^2) \qquad \in R(SO(2)) .$$

If e_1, e_2 is the usual basis of R^2, then

$$*e_1 = e_2 , \qquad *e_2 = -e_1 , \qquad *1 = e_1 \wedge e_2 , \qquad *(e_1 \wedge e_2) = 1 .$$

Thus a basis for $\Lambda_+(R^2)$ is $1 + i(e_1 \wedge e_2)$ and $e_1 + ie_2$, while a basis for $\Lambda_-(R^2)$ is

$1 - i(e_1 \wedge e_2)$ and $e_1 - ie_2$. Hence we have

$$\Lambda_+(R^2) = 1 + x^{-1}$$
$$\Lambda_-(R^2) = 1 + x ,$$

where x is the basic character of $SO(2)$. Hence we have

$$\Lambda_+(R^{2l}) - \Lambda_-(R^{2l}) = \prod_{i=1}^{l} (x_i^{-1} - x_i)$$

and so, applying the universal Chern character,

$$\text{ch}\,(\Lambda_+(R^{2l}) - \Lambda_-(R^{2l})) = \prod_{i=1}^{l} (e^{-x_i} - e^{x_i}) . \tag{6.3}$$

Putting this formula into (2.17), we obtain

$$\begin{aligned} \text{index}\, D^+ &= (-1)^l \left\{ \left\{ \prod_{i=1}^{l} \left(\frac{e^{-x_i} - e^{x_i}}{x_i} \right) \prod \frac{x_i}{1 - e^{x_i}} \cdot \frac{-x_i}{1 - e^{-x_i}} \right\}(X) \right\}[X] \\ &= \left\{ \prod_{i=1}^{l} x_i \frac{(e^{x_i} + 1)}{(e^{x_i} - 1)}(X) \right\}[X] \qquad (6.4) \\ &= \left\{ \prod_{i=1}^{l} \frac{x_i}{\tanh x_i/2}(X) \right\}[X] . \end{aligned}$$

Since $\{x/\tanh(x/2)\}$ is an even function of x, this expression involves only Pontrjagin classes of X, and is zero if l is odd. This checks with the analysis. The interesting case is when l is even.

Since $\{x/\tanh(x/2)\} = 2$ when $x = 0$, the characteristic class occurring in (6.4) is not stable. If we want to express our answer in terms of a stable class, we must introduce

$$\mathfrak{L} = \sum \mathfrak{L}_r(p) = \prod \frac{x_i/2}{\tanh x_i/2} . \tag{6.5}$$

Formula (6.4) is then equivalent to

$$\text{index}\, D^+ = 2^l \mathfrak{L}(X)[X] .$$

The expression on the right will be denoted by $L(X)$ and called the L-genus of X. This agrees with the definition of Hirzebruch [13] because

$$2^l \prod \frac{x_i/2}{\tanh x_i/2} \quad \text{and} \quad \prod \frac{x_i}{\tanh x_i}$$

have the same term in degree l.

We have now established, as a special case of our general index theorem:

HIRZEBRUCH SIGNATURE THEOREM (6.6). *Let X be a compact oriented manifold of dimension $4k$. Let* Sign (X) *denote the signature of the quadratic form in $H^{2k}(X; \mathbf{R})$ and let $L(X) = 2^{2k}\mathfrak{L}(X)[X]$ where $\mathfrak{L}$ is the stable characteristic class of the orthogonal group given by* (6.5). *Then*

$$\text{Sign}\,(X) = L(X) .$$

We proceed now to consider the Lefschetz theorem corresponding to (6.6). We suppose therefore that X is oriented of dimension $2l$, and that a compact Lie group G acts differentiably on X *preserving the orientation*. Choosing a G-invariant riemannian metric on X, the operator D^+ will be G-invariant, since it is functorially defined by the metric and orientation.

On the topological side, the bilinear form B on $H^l(X; \mathbf{R})$ given by $B(x, y) = (xy)[X]$ is G-invariant. Note that this form is *symmetric* for l even, but *skew-symmetric* for l odd. In both cases, by Poincaré duality, it is non-degenerate.

Suppose now we choose a positive definite inner product $\langle\ ,\ \rangle$ on H^l, invariant under G, and define the operator A by

$$B(x, y) = \langle x, Ay \rangle .$$

Then A commutes with the action of G and $A^* = (-1)^l A$. Consider first the case when l is *even*, so that A is self-adjoint. Then the positive and negative eigenspaces of A give a decomposition $H^l = H^l_+ \oplus H^l_-$ invariant under G. Thus we have two real representations ρ^+ and ρ^- of G, and, up to isomorphism, they are independent of the choice of inner product. This follows from the following three facts[10]:

(a) the characters of ρ^+ and ρ^- are continuous functions of the inner product,

(b) the space of all (G-invariant) inner products is connected,

(c) the characters of the compact group G are discrete.

As an obvious generalization of the signature, we now define the G-signature of the G-manifold X to be

$$\text{Sign}\,(G, X) = \rho^+ - \rho^- \in RO(G) \subset R(G); \tag{6.7}$$

it is an element of the real representation ring $RO(G)$.

By evaluating the character of $\text{Sign}\,(G, X)$ on an element $g \in G$, we obtain a real number which we denote by $\text{Sign}\,(g, X)$. Note that $\text{Sign}\,(g, X)$ *is determined entirely by the action of* g *on the real cohomology of* X. Thus $\text{Sign}\,(g, X)$ depends only on the connected component of G containing g. On the other hand it can be computed by the use of harmonic forms. In fact G acts on the space H^{2k}_R of real harmonic forms in dimension $2k$, preserving both the indefinite bilinear form $\int \alpha \wedge \alpha$ and the inner product $\int \alpha \wedge *\alpha$. Hence

$$\text{Sign}\,(g, X) = \text{Trace}_{\mathbf{R}}\,(g \mid (H^{2k}_{\mathbf{R}})^+) - \text{Trace}_{\mathbf{R}}\,(g \mid (H^{2k}_{\mathbf{R}})^-) ,$$

[10] An alternative more direct argument is to observe that the H_+ space of one inner product is always complementary to the H_- space of any other inner product. However we want a general argument which will apply equally to the case when l is odd.

where $(H_R^{2k})^{\pm}$ are the eigenspaces of $*$ on H_R^{2k}. As we saw before

$$(H^{2k})^+ = (H_R^{2k})^+ \otimes_R C = (\text{Ker } D^+) \cap H^{2k}$$
$$(H^{2k})^- = (H_R^{2k})^- \otimes_R C = (\text{Ker } D^-) \cap H^{2k} ,$$

so that Sign (g, X) is the contribution of H^{2k} to the Lefschetz number $L(g, D^+)$. On the ther hand, if we put

$$V^q = H^q \oplus H^{4k-q} \qquad 0 \leqq q < 2k ,$$

and recall that the involution τ switches the two factors in this decomposition, we see that we have G-module iscmorphisms

$$(V^q)^+ \cong H^q \cong (V^q)^- ,$$

and so V^q contributes nothing to $L(g, D^+)$. Thus we have

$$\text{Sign } (g, X) = L(g, D^+) \tag{6.8}$$

when $L(g, D^+)$ is the Lefschetz number of g for the operator D^+.

We return now and consider the case when l is odd. In this case the operator A is skew-adjoint, and if $(AA^*)^{1/2}$ denotes the positive square root of AA^*, the operator $J = A/(AA^*)^{1/2}$ satisfies $J^2 = -1$, and so defines a complex structure on H^l. Since J commutes with the action of G, we obtain in this way a complex representation ρ of G. For the same reasons as in the We case of even l, we see that ρ is independent of the choice of inner product. now define the G-signature of the G-manifold X by

$$\text{Sign } (G, X) = \rho - \rho^* \in R(G) . \tag{6.9}$$

By evaluating the character of Sign (G, X) on an element $g \in G$, we obtain a purely imaginary number which we denote by Sign (g, X). Thus

$$\text{Sign } (g, X) = \rho(g) - \rho^*(g) = 2i \text{ Im } \rho(g) .$$

Note that Sign (g, X) *depends only on the action of* g *on the real cohomology* of X. As in the even case, we can compute it by harmonic forms. We identify $H^l(X; R)$ with the space H_R^l of real harmonic l-forms. G preserves both the skew form $\int \alpha \wedge \beta$ and the inner product $(\alpha, \beta) = \int \alpha \wedge *\beta$. The skew-adjoint operator A is given by.

$$(\alpha, A\beta) = \int \alpha \wedge \beta = (-1)^l \int \alpha \wedge **\beta \qquad \text{since } *^2 = (-1)^l$$
$$= -(\alpha, *\beta) \qquad \text{since } l \text{ is odd} ,$$

and so $A = -*$. Since

$$AA^* = -A^2 = -*^2 = 1 ,$$

it follows that the complex structure J is given by $J = -*$. Now in the com-

plexification

$$H^l = H^l_{\mathbb{R}} \otimes \mathbb{C} ,$$

we defined the involution τ by

$$\tau = i^{l^2} * = i * .$$

Hence on the $(+1)$-eigenspace H^l_+ of τ, we have $-* = i$, and so we have an isomorphism of *complex* vector spaces

$$H^l_+ \cong H^l_{\mathbb{R}} ,$$

where $H^l_{\mathbb{R}}$ is endowed with the complex structure of J above. Similarly

$$H^l_- \cong \overline{H^l_{\mathbb{R}}}$$

and both isomorphisms are compatible with the action of G. Hence

$$\text{Sign}\,(g, X) = \text{Trace}\,(g\,|H^l_+) - \text{Trace}\,(g\,|\,H^l_-) . \tag{6.10}$$

Now exactly as in the case of even l, we see that the spaces H^q for $q \neq l$ contribute nothing to the Lefschetz number of D^+, and so we obtain

$$\text{Sign}\,(g, X) = L(g, D^+) .$$

Remarks. 1. Like the Hirzebruch signature our G-signature is multiplicative, i.e.,

$$\text{Sign}\,(G, X)\cdot\text{Sign}\,(G, Y) = \text{Sign}\,(G, X \times Y) .$$

This is valid for all dimensions provided we put $\text{Sign}\,(G, X) = 0$ if $\dim X$ is odd.

2. The G-signature defined on H^l is a purely algebraic notion, and it has the following important property. If M is any real G-module and M^* is its dual, then $M \oplus M^*$ has two natural bilinear forms; a symmetric one B^+, and a skew-symmetric one B^-, and both of these have zero G-signature. To see this, observe that multiplication by -1 on M^* defines an involution α of $M \oplus M^*$ which commutes with G, but anti-commutes with $B^\pm$. In other words α takes $B^\pm$ into $-B^\pm$ and so

$$\begin{aligned} \text{Sign}\,(G, B^\pm) &= \text{Sign}\,(G, -B^\pm) \\ &= -\text{Sign}\,(G, B^\pm) \qquad \text{(from the definitions)} , \end{aligned}$$

showing that $\text{Sign}\,(G, B^\pm) = 0$.

Having identified $L(g, D^+)$ as a homology-invariant (for both l even and l odd) we proceed to apply the Lefschetz theorem (3.9) to calculate $L(g, D^+)$ in terms of the fixed point set X^g of X. Observe first that each component of X^g is even-dimensional. This follows from the fact that g preserves orientation, so that the number of $+1$ (or -1) eigenvalues in the action of g on the

tangent space of X at any point $x \in X^g$ is necessarily even. If u denotes the symbol of D^+ then, using the multiplicative formula (6.3), we find that ch $i^*(u)(g)$ is the product of the following terms

$$A = \psi\left\{\prod \left(\frac{e^{-x_j} - e^{x_j}}{x_j}\right)(TX^g)\right\}$$

$$B = \prod (-e^{-x_j} + e^{x_j})(N^g(-1))$$

$$C^\theta = \prod (e^{-x_j - i\theta} - e^{x_j + i\theta})(N^g(\theta)) .$$

Here ψ: $H^*(X^g; \tilde{\mathbf{Q}}) \to H^*(TX^g; \mathbf{Q})$ is the Thom isomorphism, where $\tilde{\mathbf{Q}}$ denotes the local coefficient system isomorphic to $\mathbf{Q}$ defined by the local orientations of X^g. If X^g is orientable, then once we choose an orientation, we get an isomorphism $\tilde{\mathbf{Q}} \cong \mathbf{Q}$. If X^g is not orientable, then we have to work with these "twisted" coefficients. Note that, since X and the $N^g(\theta)$ are all oriented, the bundles TX^g and $N^g(-1)$ have isomorphic twisted coefficients. Thus if $\tilde{\mathbf{Q}}$ is as above, the Euler class $e(N^g(-1)) \in H^*(X^g; \tilde{\mathbf{Q}})$.

The evalution on the fundamental class $[TX^g]$ in (3.9) can, as usual, be replaced by an evalution on the "twisted" fundamental class $[X^g]$. We simply replace A by $\psi^{-1}A$, and multiply by $(-1)^k$ (where $\dim X^g = 2k$) to allow for the difference between the two orientations of TX^g. If we cancel the common factors occurring in $\psi^{-1}A, B, C^\theta$ on the one hand, and $\mathcal{I}, \mathcal{R}, \mathcal{S}^\theta$, and $\det(1 - g \mid N^g)$ on the other, we find

$$L(g, D^+) = \{A_1 B_1 \textstyle\prod_\theta C_1^\theta\}[X] ,$$

where

$$A_1 = \prod \frac{x_j}{\tanh x_j/2}(TX^g) = 2^t \mathfrak{L}(X^g), \qquad 2t = \dim X^g$$

$$B_1 = \prod \tanh \frac{x_j}{2}(N^g(-1)) = 2^{-r} \mathfrak{L}(N^g(-1))^{-1} \cdot e(N^g(-1)), \qquad 2r = \dim N^g(-1)$$

$$C_1^\theta = \prod \frac{1}{\tanh \dfrac{x_j + i\theta}{2}}(N^g(\theta)) .$$

In order to express C_1^θ by a stable characteristic class of the unitary group, we define

$$\mathfrak{M}^\theta = \sum \mathfrak{M}_r^\theta(c_1, \cdots, c_r) = \prod \frac{\tanh i\theta/2}{\tanh \dfrac{x_j + i\theta}{2}} \tag{6.11}$$

so that

$$C_1^\theta = (i \tan \theta/2)^{-s(\theta)} \mathfrak{M}^\theta(N^g(\theta)) , \qquad s(\theta) = \dim_{\mathbf{C}}(N^g(\theta)) .$$

Putting all this together we obtain:

G-SIGNATURE THOREM (6.12). *Let X be a compact oriented manifold of dimension $2l$, and let the compact Lie group G act on X preserving the orientation. Then G acts on $H^l(X;\mathbf{R})$ preserving the bilinear form. Let* Sign (G, X) *be the character of G defined from this action by* (6.7) *for l even, and by* (6.9) *for l odd. Let* Sign (g, X) *be the value of* Sign (G, X) *on an element $g \in G$. Let X^g be the fixed-point set of g, N^g the normal bundle of X^g in X, and*

$$N^g = N^g(-1) \oplus \textstyle\sum_{0<\theta<\pi} N^g(\theta)$$

the decomposition of N^g determined by the eigenvalues of g. Then $N^g(-1)$ is a real vector bundle of even dimension and $N^g(\theta)$ is a complex vector bundle. Let[11] $2t = \dim X^g$, $2r = \dim N^g(-1)$, $s(\theta) = \dim_{\mathbf{C}} N^g(\theta)$. *Finally let $\mathfrak{L}$ be the stable characteristic class of the orthogonal group given by* (6.5), *and $\mathfrak{M}^\theta$ the stable characteristic class of the unitary group given by* (6.11). *Then we have*

$$\text{Sign}\,(g, X) = \{2^{t-r} \textstyle\prod_{0<\theta<\pi} (i\tan\theta/2)^{-s(\theta)} \mathfrak{L}(X^g)\mathfrak{L}\big(N^g(-1)\big)^{-1} e\big(N^g(-1)\big) \prod_{0<\theta<\pi} \mathfrak{M}^\theta\big(N^g(\theta)\big)\}[X^g]\,.$$

Here $e\big(N^g(-1)\big)$ denotes the "twisted" Euler class of $N^g(-1)$, and $[X^g]$ is the "twisted" fundamental class of X^g, both twistings being defined by the local coefficient system of orientations of X^g.

This theorem, like the special case of it discussed in [2] when X^g is zero-dimensional, provides the most interesting application of our general Lefschetz theorem to differential topology. We shall therefore spend a little time discussing various special cases and corollaries. For brevity the expression on the right-hand side of the formula in (6.12) will be denoted by $L(g, X)$. Thus $L(1, X) = L(X)$ is the Hirzebruch L-genus.

First let us observe that the Euler class $e\big(N^g(-1)\big)$ is in dimension $2r$ and $\dim X^g = 2t$. Thus if $r > t$, the formula in (6.12) shows that Sign $(g, X) = 0$. Thus we have

COROLLARY (6.13). *Assume, in the situation of* (6.12) *that (for all components of X^g) we have $r > t$. Then* Sign $(g, X) = 0$.

We shall now examine the particular case of an involution. If $\dim X = 4k + 2$, then Sign (G, X) is twice the imaginary part of a character, but for G of order 2 the characters are all real, and so Sign $(G, X) = 0$. Thus only the case $\dim X = 4k$ is of interest, and we apply (6.12) for this case. Since the only normal eigenvalue is now -1, we obtain

$$\text{Sign}\,(g, X) = \{2^{t-r}\mathfrak{L}(X^g)\mathfrak{L}(N^g)^{-1} e(N^g)\}[X^g]\,. \tag{6.14}$$

To simplify this expression let us observe that, if $Y \subset X$ is any closed

[11] These numbers depend of course on the component of X^g. In order to keep the formula within bounds, we have not made this explicit.

submanifold, we can define its "self-intersection manifold". To do this, we take the inclusion map i: $Y \to X$ and replace it, in the manner of Thom, by a homotopic map f: $Y \to X$ transverse regular along $Y \subset X$. The inverse image $Z = f^{-1}(Y)$ is then the required self-intersection. Clearly the normal bundle of Z in Y is isomorphic to $N \mid Z$, where N is the normal bundle of Y in X. Hence the normal bundle of Z in X is isomorphic to $(N \oplus N) \mid Z$, and so has a natural orientation. Thus if X is oriented, so is Z. The oriented bordism class of Y represented by j: $Z \to Y$ is independent of the choice of the map f. In particular the oriented cobordism class of Z is independent of f. We call this the self-intersection of Y, and denote it by Y^2. Also the homology class of Y defined by Z is independent of f. In fact it is dual of the twisted Euler class $e(N)$. Hence for any $\xi \in H^*(Y)$, we have

$$\{\xi e(N)\}[Y] = j^*(\xi)[Z] .$$

We now return to our involution g, and apply this formula with $Y = X^g$, $\xi = 2^{t-r}\mathfrak{L}(X^g)\mathfrak{L}(N^g)^{-1}$. Since $\mathfrak{L}$ is multiplicative, and since $j^*(N^g)$ is the normal bundle of Z in X^g, we have

$$\begin{aligned}\{2^{t-r}\mathfrak{L}(X^g)\mathfrak{L}(N^g)^{-1}e(N^g)\}[Y] &= 2^{t-r}\mathfrak{L}(Z)[Z] \\ &= L(Z) \qquad (\text{since } \dim Z = 2(t-r)) .\end{aligned}$$

Combined with (6.14), this gives the following rather simple result.

PROPOSITION (6.15). *Let X be a compact oriented manifold of dimension $4k$, and let g be an orientation preserving involution with fixed point set X^g. Let $(X^g)^2$ denote the oriented cobordism class of the self-intersection of X^g in X. Then*

$$\text{Sign}\,(g, X) = \text{Sign}\,((X^g)^2) .$$

This rather attractive formulation of (6.12) for involutions was pointed out to us by F. Hirzebruch on the basis of the 8-dimensional case which we had explicitly computed.

Since $g^2 = 1$, we have

$$\begin{aligned}\text{Sign}\,(g, X) &\equiv \dim H^{2k}(X, \mathbf{R}) \mod 2 \\ &\equiv E(x) \mod 2 ,\end{aligned}$$

where $E(X)$ is the Euler characteristic of X. Thus from (6.15) or (6.13), we deduce the following result proved differently by Conner and Floyd [12; (27.4)].

COROLLARY (6.16). *Let X be a compact oriented manifold of* dim $4k$ *with odd Euler characteristic. Let g be an involution of X preserving the orientation. Then* dim $X^g \geqq \frac{1}{2}$ dim X (i.e., *at least one component of X^g has such a dimension*).

Although Sign $(G, X) = 0$ when $\dim X \equiv 2 \bmod 4$, and G has order 1 or 2, this is certainly not so for other groups. The simplest example of a non-trivial G-signature in these dimensions is obtained as follows. Let w be a primitive cube root of unity, $X = \mathbf{C}/\{1, w\}$ the elliptic curve with periods $1, w$. Then $z \mapsto wz$ gives an automorphism g ("complex multiplication") of X of period 3. $H^1(X; \mathbf{C})$ is generated by the differentials dz, $d\bar{z}$, and we have

$$*dz = -idz \qquad *d\bar{z} = id\bar{z} .$$

Hence calculating Sign (g, X) by harmonic forms, we see that

$$\text{Sign}\,(g, X) = w - \bar{w} = i\sqrt{3} \neq 0 .$$

For illustrative purposes, we shall now consider a few special cases of Theorem (6.12) in low dimensions.

As a simple check on (6.15) we can take $X = P_4(\mathbf{C})$ with either of the involutions

$$(x_0, x_1, x_2, x_3, x_4) \longmapsto (-x_0, x_1, x_2, x_3, x_4)$$
$$(x_0, x_1, x_2, x_3, x_4) \longmapsto (-x_0, -x_1, x_2, x_3, x_4) .$$

In both cases Sign $(g, X) = 1$ because g acts trivially on the cohomology. In the first case, we have $X^g = P_3(\mathbf{C}) \cup P_0(\mathbf{C})$ so that $(X^g)^2 = P_2(\mathbf{C})$ and Sign $((X^g)^2) = 1$; in the second case, $X^g = P_2(\mathbf{C}) \cup P_1(\mathbf{C})$ and $(X^g)^2 =$ point, has signature 1.

Returning now to Theorem (6.12), let us suppose that g is of *odd order*. Then $N^g(-1) = 0$, and so the formula for Sign (g, X) simplifies to

$$\text{(6.17)} \quad \text{Sign}\,(g, X) = \{2^t \textstyle\prod_{0<\theta<\pi} (i \tan \theta/2)^{-s(\theta)} \mathfrak{L}(X^g) \prod_{0<\theta<\pi} \mathfrak{M}^\theta(N^g(\theta))\}[X^g] .$$

Note that in this case X^g has a natural orientation induced from that of X and the complex orientation of the bundles $N^g(\theta)$. One must be careful however not to confuse this orientation of X^g with other "natural" orientations. For example, if $\dim X^g = 0$ so that X^g consists of points, the orientation to be used in (6.17) may not coincide with the usual orientation of a point. Suppose now that X is 4-dimensional, (and connected), and that g is non-trivial. The fixed-point set X^g then consists of points $\{P_j\}$ and 2-manifolds $\{Y_k\}$. The contribution of P_j to Sign (g, X) is then

$$-\varepsilon \cot \frac{\alpha_j}{2} \cot \frac{\beta_j}{2} ,$$

where $\exp(\pm i\alpha_j)$, $\exp(\pm i\beta_j)$ are the eigenvalues of $g \mid T_{P_j}$, $0 < \alpha_j < \pi$, $0 < \beta_j < \pi$, and $\varepsilon = \pm 1$ is the difference between the two "natural" orientations of T_{P_j}. In other words if, in a basis of T_{P_j} (oriented relative to the orientation of X), $g \mid T_{P_j}$ is represented by the matrix

$$\begin{pmatrix} \cos \alpha_j & -\sin \alpha_j \\ \sin \alpha_j & \cos \alpha_j \end{pmatrix} \oplus \begin{pmatrix} \cos \beta_j & -\sin \beta_j \\ \sin \beta_j & \cos \beta_j \end{pmatrix} ,$$

then the contribution of P_j to Sign (g, X) is

$$-\cot\frac{\alpha_j}{2}\cot\frac{\beta_j}{2}.$$

Since

$$\mathfrak{M}_\theta = 1 + \frac{ic_1}{\sin\theta} + \cdots,$$

the contribution of the component Y_k to Sign (g, X) is

$$\left\{2\cdot -i\cot\theta_k/2\frac{ic_1(N_k^g)}{\sin\theta_k}\right\}[Y_k] = \frac{Y_k^2}{\sin^2\theta_k/2},$$

where N_k^g is the complex normal bundle of Y_k and Y_k^2 is the self-intersection number of Y_k. Thus we have established

PROPOSITION (6.18). *Let X be a connected compact oriented 4-manifold, g an automorphism of X of odd order (necessarily preserving orientation). Let the fixed-point set X^g consist of isolated points $\{P_j\}$ and connected 2-manifolds $\{Y_k\}$. For each j, let the action of g on the tangent space at P_j be given by the matrix*

$$\begin{pmatrix}\cos\alpha_j & -\sin\alpha_j\\ \sin\alpha_j & \cos\alpha_j\end{pmatrix}\oplus\begin{pmatrix}\cos\beta_j & -\sin\beta_j\\ \sin\beta_j & \cos\beta_j\end{pmatrix}$$

relative to an oriented basis. For each k, let $\exp(\pm i\theta_k)$ denote the normal eigenvalues of g along Y_k, and let Y_k^2 denote the self-intersection number of Y_k. Then

$$\text{Sign}\,(g, X) = \textstyle\sum_j -\cot\frac{\alpha_j}{2}\cdot\cot\frac{\beta_j}{2} + \sum_k \operatorname{cosec}^2\frac{\theta_k}{2}\cdot Y_k^2.$$

As a check on (6.18), consider the transformation $(x_0, x_1, x_2)\mapsto(x_0, wx_1, wx_2)$ where $w = \exp(i\alpha)$ is a primitive q^{th} root of unity (q odd). Then we have one fixed point which contributes $-(\cot\alpha/2)^2$, and one fixed line which contributes $(\operatorname{cosec}\alpha/2)^2$. Thus

$$\text{Sign}\,(g, X) = -\left(\cot\frac{\alpha}{2}\right)^2 + \left(\operatorname{cosec}\frac{\alpha}{2}\right)^2 = 1$$

which is correct, because g acts trivially on the cohomology of X.

We conclude this section by pointing out that one can also consider the operator D^+ extended (using a connection) to act on differential forms with coefficients in any vector bundle V. This no longer has any connection with ordinary cohomology, and so it is not as interesting as the case we have been considering. However one gets formulas generalizing (6.6) and (6.12). If we denote the Lefschetz number of the extension of D^+ to V by Sign (g, X, V),

then we have

$$\text{Sign}\,(g, X, V) = \{\text{ch}\,(V|X^g)(g)A\}[X^g]\,, \tag{6.19}$$

where $\{A\}$ is the expression occurring in the formula of (6.12). The right hand side of (6.19) will for brevity be denoted by $L(g, X, V)$. For $g = 1$, we put $L(1, X, V) = L(X, V)$, and $\text{Sign}\,(1, X, V) = \text{Sign}\,(X, V)$.

Formula (6.19) is of interest because, as we vary V, it provides a whole set of "integrality theorems" for Pontrjagin numbers.

Finally we should perhaps point out that if X is a spin-manifold, then the operators of this section are closely related to the Dirac operator of § 5. This connection arises from the isomorphism of Spin $(2l)$-modules

$$(-1)^l(\Delta^+ - \Delta^-) = \Lambda_+(R^{2l}) - \Lambda_-(R^{2l})\,.$$

7. Invariants for free actions

It is well-known that the characteristic numbers of a manifold are invariants of the appropriate cobordism group. Thus the Pontrjagin numbers are invariants of oriented cobordism (SO-cobordism), and the Chern numbers are invariants of almost complex cobordism (U-cobordism). For a G-manifold, one can similarly define G-characteristic numbers using the various fixed-point sets of elements of G, and these will be G-cobordism invariants. For example the formulas for the index-character of the various operators of § 4, 5, 6 are all expressed in terms of such G-characteristic numbers. Moreover the fact that these particular combinations of characteristic numbers are characters of G provides "integrality theorems" analogous to the well-known integrality theorems of [4] and [8] for the usual characteristic numbers.

These remarks apply to even-dimensional manifolds, but by a change of view-point, they can be exploited for odd-dimensional manifolds on which G acts *freely*. Suppose for example that X is a compact oriented free G-manifold of dimension $2n - 1$, and suppose we can find an oriented G-manifold Y (not necessarily G-free) of dimension $2n$, with boundary X. Let $v \in K_{SO(2n)}(R^{2n})$ be any "universal symbol class", and let $u \in K_G(TY)$ be the element defined by v. If Y were *without boundary*, then we could for any $g \in G$ apply the index theorem in the form (2.17) and obtain an explicit expression, say $v(g, Y)$, involving the evaluation of certain characteristic classes on the fixed-point set Y^g of g. Since in our case Y has a boundary, this does not apply, but since X is G-free, we have

$$Y^g \cap \partial Y = \varnothing \qquad \text{for } g \neq 1$$

and so the expression $v(g, Y)$ still makes sense. We obtain in this way a

function

$$g \longmapsto v(g, Y)$$

from the non-identical elements of G to C. We denote this function by $v(Y)$. Note that it is just a complex-valued function, and not necessarily given by a character of G. Suppose now that Y' is another oriented G-manifold with boundary X, and that we put

$$Z = Y \cup_X (-Y')$$

(where $-Y'$ is Y' with the opposite orientation), so that Z is a closed oriented manifold without boundary. Because $v(g, Y)$ is computed from Y^g, and because

$$Z^g = Y^g + (-Y')^g \qquad \text{(disjoint)}$$

we have

$$v(g, Z) = v(g, Y) - v(g, Y') .$$

Now $v(g, Z)$ *is* given by a character because we can apply (2.17) to the *closed* G-manifold Z. Hence the residue class of $v(Y)$ *modulo characters* depends only on X, and we denote it by $v(X)$. It is clearly an invariant of free G-cobordism. Formally the reader will notice the close analogy with the invariant originally used by Milnor to distinguish exotic 7-spheres.

The invariants $v(X)$ we have just defined are definitely non-trivial as simple examples show. They can be used to derive many of the results of Conner-Floyd [12] on G-cobordism (cf. [2]). In fact we reverse the procedure of [12] in which G-cobordism is developed first, and then applied to fixed-point theory.

The detailed application of the invariants $v(X)$ to G-cobordism will not be developed here. Instead we shall concentrate on one particular case where much stronger results can be obtained. If we take v to be the universal symbol (say L) giving rise to the operator D^+ of § 6, then for a closed manifold Y of dimension $2n$ we have

$$L(g, Y) = \text{Sign}\,(g, Y) ,$$

and this can be computed from the action of G on $H^n(Y; \mathbf{R})$. Because of its connection with cohomology, we can use L to define a more refined invariant than before.

First of all however we need to discuss an additivity property of the signature due to S. P. Novikov.[12] Suppose Y is an oriented G-manifold of of dimension $2n$ with boundary X, and let $\hat{H}^n(Y)$ denote the image of the

[12] We are indebted to Hirzebruch for drawing our attention to Novikov's result.

natural homomorphism

$$\varphi: H^n(Y, X) \longrightarrow H^n(Y) .$$

By Poincaré duality $H^n(Y)$ is dual to $H^n(Y, X)$, and hence the bilinear form B on $\hat{H}^n(Y)$ defined by

$$B(\varphi(a), \varphi(b)) = ab[Y]$$

is non-degenerate. B is symmetric for n even, skew-symmetric for n odd, and, as for closed manifolds, we can now define Sign (G, Y). Suppose now Y' is another oriented G-manifold with boundary $-X$ and that $Z = Y \cup_X Y'$. The additivity property is then

PROPOSITION (7.1). Sign (G, Z) = Sign (G, Y) + Sign (G, Y').

PROOF. Consider the cohomology sequences of (Z, Y) and (Z, Y')

$$H^n(Y', X) \xrightarrow{\alpha'} H^n(Z) \xrightarrow{\beta} H^n(Y)$$
$$H^n(Y') \xleftarrow{\beta'} H^n(Z) \xleftarrow{\alpha} H^n(Y, X)$$

where we replace $H^n(Z, Y)$ by $H^n(Y', X)$, and similarly with Y, Y' interchanged. By Poincaré duality, these two sequences are duals of each other. Thus $A = \operatorname{Im}\alpha$ and $A' = \operatorname{Im}\alpha'$ are mutual annihilators for the bilinear form $B(Z)$ on $H^n(Z)$. Hence $A \cap A'$ is the annihilator of $A + A'$, and so $H^n(Z)/A + A' \cong (A \cap A')^*$. On the other hand

$$\begin{aligned}(A + A')/A \cap A' &\cong A/A \cap A' \oplus A'/A \cap A' \cong \operatorname{Im}\beta\alpha \oplus \operatorname{Im}\beta'\alpha' \\ &\cong \hat{H}^n(Y) \oplus \hat{H}^n(Y') .\end{aligned}$$

Thus, splitting the filtration $A \cap A' \subset A + A' \subset H^n(Z)$ (G-invariantly), we get a decomposition of G-modules

$$H^n(Z) \cong (A \cap A') \oplus \hat{H}^n(Y) \oplus \hat{H}^n(Y') \oplus (A \cap A')^* ,$$

and with respect to this decomposition the bilinear form $B(Z)$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & B(Y) & 0 & * \\ 0 & 0 & B(Y') & * \\ (-1)^n & * & * & * \end{pmatrix} .$$

By a transformation $T'(\ )T$, i.e., by a change of splitting, all the terms $*$ can be eliminated. Then

$$B(Z) \cong B(Y) \oplus B(Y') \oplus C ,$$

where C is the natural bilinear form on $(A + A') \oplus (A + A')^*$. As remarked

in § 5, the G-signature of such a C is always zero. Hence

$$\text{Sign}\,(G, Z) = \text{Sign}\,(G, Y) + \text{Sign}\,(G, Y')$$

as required.

Suppose now that X is a closed oriented G-manifold of dimension $2n-1$, and assume that we can find an oriented G-manifold Y with boundary X. Then, for any element $g \in G$ *having no fixed-points* on X, we can define

$$\sigma(g, X) = L(g, Y) - \text{Sign}\,(g, Y)\,.$$

We shall now show that this is independent of the choice of Y. Suppose then that Y' is another choice, and that $Z = Y \cup_X (-Y')$ is the closed oriented G-manifold obtained by attaching Y and $-Y'$ along their common boundary X. By (7.1) we have

$$\text{Sign}\,(g, Z) = \text{Sign}\,(g, Y) - \text{Sign}\,(g, Y') \tag{7.2}$$

and, since Z^g is the disjoint sum of Y^g and $(-Y')^g$, we have

$$L(g, Z) = L(g, Y) - L(g, Y')\,. \tag{7.3}$$

Subtracting (7.3) from (7.2), we obtain

$$\begin{aligned}\{L(g, Y) - \text{Sign}\,(g, Y)\} - \{L(g, Y') - \text{Sign}\,(g, Y')\} &= L(g, Z) - \text{Sign}\,(g, Z)\\ &= 0 \qquad \text{by (6.12)}\,.\end{aligned}$$

Thus our invariant $\sigma(g, X)$ is well-defined. It is a G-diffeomorphism invariant. In fact, a minor modification in the proof we have just given shows it is an invariant in a rather stronger sense. Thus suppose X' is another G-manifold (on which g has no fixed point), and suppose that X and X' are equivalent in the following sense:

(A) there exists an oriented G-manifold W of dimension $2n$ with $\partial W = X - X'$, $W^g = \varnothing$, and $\hat{H}^*(W; \mathbf{R}) = 0$.

Then if $\partial Y = X, \partial Y' = X'$ we put

$$Z = Y \cup_X W \cup_{X'} (-Y')\,.$$

Computing as before we find that W contributes nothing either to L or to Sign, so that we have

$$\begin{aligned}\sigma(g, X) - \sigma(g, X') &= L(g, Z) - \text{Sign}\,(g, Z)\\ &= 0\,.\end{aligned}$$

We state this in a theorem.

THEOREM (7.4). *Let G be a compact Lie group, and let X be a closed oriented G-manifold of dimension $(2n-1)$ which bounds an oriented G-*

manifold Y. For any $g \in G$ having no fixed point on X, we define the complex number $\sigma(g, X)$ by

$$\sigma(g, X) = L(g, Y) - \text{Sign}\,(g, Y)\,,$$

where $L(g, Y)$ is the number occurring on the right side of the formula in (6.12) *(with X replaced by Y), and* Sign (g, Y) *is defined by the action of G on $\hat{H}^n(Y; \mathbf{R})$ (the image of $H^n(Y, X; \mathbf{R}) \to H^n(Y; \mathbf{R})$)—endowed with its bilinear form—as in* (6.7) *and* (6.9). *Then $\sigma(g, X)$ depends only on X and not on Y. Moreover if X' is another G-manifold, and if there exists an oriented G-manifold W with $\partial W = X - X'$, $W^g = \varnothing$ and $\hat{H}^n(W; \mathbf{R}) = 0$, then*

$$\sigma(g, X) = \sigma(g, X')\,.$$

Theorem (7.4) is particularly interesting when G is finite, X is simply-connected and G acts freely on X, so that X is the universal covering space of $M = X/G$. If $M' = X'/G$ is h-cobordant to M (i.e., if there exists an oriented V with $\partial V = M - M'$ and with $M \to V$, $M' \to V$ homotopy equivalences), then the universal covering space W of V provides an equivalence as in (7.4) so that $\sigma(g, X) = \sigma(g, X')$ for $g \neq 1$. Hence we have

COROLLARY (7.5). *Let M be an oriented $(2n - 1)$-manifold with finite fundamental group G. Assume its universal covering space X bounds some oriented G-manifold. Then $\sigma(g, X)$, for $g \neq 1$, is an h-cobordism invariant of M.*

Remarks. 1. It has been pointed out to us by F. Hirzebruch and C. T. C. Wall that, for free actions of finite groups, the invariant σ can be defined without the hypothesis in (7.5). To do this, one has to appeal to the free cobordism theory of Conner-Floyd [12] according to which (when $\dim X = 2n - 1$) some multiple NX always bounds an oriented free G-manifold Y. We can then define

$$\sigma(g, X) = \frac{-1}{N}\,\text{Sign}\,(g, Y)\,. \qquad (g \neq 1)\,.$$

Although this is a simple and more general definition, the fixed-point version is still very useful for computing σ, as is shown by the case of lens spaces (see below).

2. When X is a homotopy $(4k - 1)$-sphere, and g is a fixed-point-free involution, F. Hirzebruch has shown that our invariant $\sigma(g, X)$ coincides with the invariant of Browder-Livesay [11].

A specially simple application of (7.5) is to the case of lens spaces, when X is a sphere and G acts orthogonally. This is the case studied in detail in [2] where it is shown that our invariant σ distinguishes the different lens

spaces. Of course for lens spaces we can take Y to be a ball, there is a unique fixed point and so we do not need the general fixed-point theory developed in this paper.

Since a connected group acts trivially on cohomology, the main interest in Sign (g, X) is with finite groups. For connected groups however one obtains some interesting identities which will be discussed in the next section. In the meantime we shall prove

PROPOSITION (7.6). *The function σ in* (7.5) *is an analytic function on the open set $U \subset G$ consisting of elements having no fixed point on X. If G is connected, so that $R(G)$ is an integral domain, then σ is defined by a unique element of the field of fractions of $R(G)$.*

PROOF. For elements $g \in G$ with the same fixed-point set in Y (where $\partial Y = X$ as in (7.5)), the explicit nature of the formula

$$\sigma(g, X) = L(g, Y) - \text{Sign}\,(g, Y)$$

shows at once that σ is analytic in g. To deal with the general case where the fixed-point set changes, we must go back to the properties of the index given in [7]. The excision property implies that we have a well-defined homomorphism

$$\text{ind}^{\dot{Y}}: K_G(T\dot{Y}) \longrightarrow R(G)$$

where $\dot{Y}$ is the open manifold $Y - X$. Now let γ be a conjugacy class in G having no fixed points in X, and consider the inclusions

$$(TY)^\gamma \xrightarrow{i} T\dot{Y} \xrightarrow{j} TY\,.$$

By the general localization theorem[13] of [20], j induces an isomorphism $(j_*)_\gamma$ in the following diagram

$$\begin{array}{ccc} K_G(T\dot{Y})_\gamma & \xrightarrow{(j_*)_\gamma} & K_G(TY)_\gamma \\ \Big\downarrow{\scriptstyle \text{ind}_\gamma^{\dot{Y}}} & \swarrow{\scriptstyle \text{ind}_\gamma^{Y}} & \\ R(G)_\gamma & & \end{array}$$

and hence ind_γ^Y can be defined to make the diagram commute. If, however, we restrict from G to the closed subgroup generated by $g \in \gamma$, the homomor-

[13] Here we need the localization theorem for general groups G, whereas in [5] we only gave the proof for abelian groups.

phism $\mathrm{ind}_T^{V^0}$ can be calculated in terms of the fixed point manifold Y^g. This was how we obtained our general Lefschetz formula. In particular, if we let $a_\gamma \in K_G(TY)_\gamma$ be the element defined by the symbol class $a \in K_G(TY)$ of the operator D^+ of § 6, we have

$$\mathrm{ind}_T^V a_\gamma(g) = L(g, Y)\ , \tag{7.7}$$

where the left side is obtained by the map $R(G)_\gamma \to \mathbf{C}$ (evaluation at g or γ). This alternative expression for $L(g, Y)$ is better for investigating the dependence on g. In fact our proposition now follows rather formally as we shall see. For brevity, we put

$$R = R(G), \quad M = K_G(T\mathring{Y}), \quad N = K_G(TY)$$

so that M, N are R-modules. We now consider the continuous map $G \xrightarrow{\varphi}$ Spec $R(G)$, where Spec $R(G)$ is the affine scheme of $R(G)$ and $\varphi(g)$ is the prime ideal defined by g. We let $\tilde{R}$ be the sheaf on G induced by φ from the structure sheaf on $R(G)$. Thus the stalk $\tilde{R}_g$ is just the local ring $R(G)_\gamma$ (where γ is the conjugacy class of g), and $\tilde{R}$ may be identified with a subsheaf of the sheaf of germs of analytic functions on G. The R-modules M, N define $\tilde{R}$-sheaves $\tilde{M}, \tilde{N}$ on G, and the R-homomorphisms

$$\begin{array}{ccc} M & \xrightarrow{j_*} & N \\ \big\downarrow{\scriptstyle \mathrm{ind}} & & \\ R & & \end{array}$$

induce $\tilde{R}$-homomorphisms

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{j}_*} & \tilde{N} \\ \big\downarrow{\scriptstyle \widetilde{\mathrm{ind}}} & & \\ \tilde{R}\ . & & \end{array}$$

The localization theorem implies that $\tilde{j}_*$ *is an isomorphism over the open set* $U \subset G$, so that restricting to U we get a diagram

$$\begin{array}{ccc} \tilde{M}_U & \xrightarrow{\simeq} & \tilde{N}_U \\ \big\downarrow{\scriptstyle \widetilde{\mathrm{ind}}_U} & & \\ \tilde{R}_U\ . & & \end{array}$$

The element $a \in N$ defines a section of $\tilde{N}$, and hence a section a_U of $\tilde{M}_U$. Let L_U be the section of $\tilde{R}_U$ given by

$$L_U = \widetilde{\mathrm{ind}}_U\,(a_U)\ .$$

Then L_U is an analytic function on U. On the other hand, by (7.7) it is clear that the value of L_U at $g \in U$ is just $L(g, Y)$. Thus $L(g, Y)$ is an analytic function of g in U. Since the other term Sign (g, Y) in the definition of $\sigma(g, X)$ is a character of G, it follows that σ is analytic in U. To prove the last part of the proposition when G is connected, we take an element $g \in U$ which generates a maximal torus T (i.e., the powers of g are dense in T). Since $R(G) \to R(T)$ is injective, it follows that only the zero of $R(G)$ vanishes at g. Hence, if γ denotes the conjugacy class of g, the local ring $\tilde{R}_g = R(G)_\gamma$ is just the full field $F(G)$ of fractions of $R(G)$. In the neighborhood of g, the analytic function $L(\ , Y)$ is therefore given by an element of $F(G)$. If U were connected, we could appeal to analytic continuation to get the uniqueness of this element. In fact U need not be connected.[14] But we can argue as follows. Let $V \subset U \cap T$ be the set of all elements having the same fixed points as g (i.e., as T). Then V is open (because it consists of elements having no fixed points in $Y - Y^g$) and dense (because it contains all generators of T). For any element $h \in V$, the explicit fixed-point expression for $L(h, Y)$ shows at once that this is given by a unique element f of the field $F(T)$. Since V is open and dense the analytic function L is entirely determined by $f \in F(T)$. Since, in a neighborhood of g, L is given by an element in $F(G)$, it follows that $f \in F(G)$, and the proof is complete.

As the simplest illustration of (7.6), let us consider circle actions. Thus we assume the circle G acts on Y and has no fixed points (for the whole group) on the boundary X. Then we have

$$R(G) = \mathbf{Z}[t, t^{-1}]$$
$$F(G) = \mathbf{Q}(t) ,$$

and we can interpret t as the coordinate $e^{i\theta}$ of a "general point" of G. Let $Z \subset Y$ be the points fixed under the whole group G. The normal bundle N of Z in Y can be written as a (finite) direct sum $N = \sum_{k>0} N_k$, where N_k is the complex vector bundle on which t acts as t^k. The manifold Z inherits then a natural orientation from those of N and Y. Since the connected group G acts trivially on the cohomology of Y, the Lefschetz formula (6.12), applied to the general element t of G, gives

$$\sigma(t, X) = \left\{2^m \prod_k \left\{\prod_j \frac{t^k e^{x_j} + 1}{t^k e^{x_j} - 1}(N_k)\right\} \mathfrak{L}(Z)\right\}[Z] - \text{Sign}\,(Y) , \tag{7.7}$$

where, as usual, the elementary symmetric functions of the $x_j(N_k)$ are the Chern classes of N_k, and $2m = \dim Z$. Formula (7.7) shows clearly that $\sigma(t, X)$

[14] If we had introduced the complexification G^c of G, which is an affine algebraic group, all our sheaves could have been defined over a Zariski open set W containing U, and L_U would have been the restriction of a rational function G^c having no poles in W.

is a rational function of t (with coefficients in Q), and that its denominator is a product of factors $1 - t^k$. Thus $\sigma(t, X)$ can only have a pole at a point $t = \alpha \in G$ which has a larger fixed point set than Z. Proposition (7.6) however gives a stronger result, namely that α cannot be a pole if $Y^\alpha \cap X = \emptyset$. Moreover the value $\sigma(\alpha, X)$ at such a point α is given by applying the formula (6.12) to the fixed point set Y^α.

The function $\sigma(t, X)$ appears to be quite an interesting invariant of the circle action. A simple deduction from (7.7), pointed out by F. Hirzebruch, is that $\sigma(t, X)$ is finite as $t \to \infty$, and that its value there is

$$\begin{aligned} \sigma(\infty, X) &= 2^m \mathfrak{L}(Z)[Z] - \operatorname{Sign}(Y) \\ &= \operatorname{Sign}(Z) - \operatorname{Sign}(Y) . \end{aligned} \tag{7.8}$$

If the circle acts *freely* on X, we can take Y to be the associated disc bundle with Z as zero-section. If $x \in H^2(Z)$ denotes the first Chern class of the circle bundle $X \to Z$, the formula (7.7) for $\sigma(t, X)$ reduces to

$$\sigma(t, X) = \left\{ 2^{n-1} \left(\frac{te^x + 1}{te^x - 1} \right) \mathfrak{L}(Z) \right\}[Z] - \operatorname{Sign}(Y) . \tag{7.9}$$

Moreover, since $H^n(Y, X) \to H^n(Y)$ can be identified with the homomorphism $H^{n-2}(Z) \to H^n(Z)$ given by multiplication with x, we see that $\operatorname{Sign}(Y)$ is just the signature of the degenerate form on $H^{n-2}(Z)$ given by $(u, v) \mapsto xuv[Z]$. It is zero for odd n, and for even n, it can be interpreted as the signature of the quadratic form on $H^{n-2}(Z^2)$ restricted to the image of $H^{n-2}(Z)$. Here Z^2 denotes a self-intersection manifold of Z in Y.

The only possible pole of $\sigma(t, X)$ in (7.9) is at $t = 1$. The value for $t = -1$ is, as we have seen in (6.15), given by

$$\sigma(-1, X) = \operatorname{Sign}(Z^2) - \operatorname{Sign}(Y) .$$

For general circle actions on X without fixed points (but not necessarily free), it is not clear that we can find a G-manifold Y with $\partial Y = X$. However as we have remarked before, the restriction σ_f of σ to all finite subgroups of G can be defined without assuming the existence of Y. Since the points of finite order are dense on the circle, it follows that there is at most one analytic function σ (defined for $t \neq 1$) equal to σ_f at all points of finite order. However it is conceivable that σ_f may not extend to an analytic (or even continuous) function. This would prove that X cannot bound a G-manifold. It would be interesting to know if such a situation can actually occur. The same questions arise of course for any compact Lie group, the elements of finite order being always dense.

8. Vector fields

Our general Lefchetz theorem, when applied to one-parameter groups, leads to interesting identities on characteristic classes which we shall now describe. There are two cases of special interest, namely the riemannian and hermitian.

Let X be a compact oriented riemannian manifold of dimension $2l$. We recall [18] that the group of isometries of X is a compact Lie group G. Suppose now that A is a vector field on X which is an infinitesimal isometry. This means that the corresponding one-parameter group $\exp(tA)$ is a subgroup of G. The zero set X^A of A is fixed under the whole group $\exp(tA)$. Moreover for $0 < |t| < \varepsilon$ the fixed-point set of $\exp(tA)$ is precisely X^A.

The operator D^+ of § 6 is invariant under the group G of isometries of X and in particular therefore it is invariant under the one-parameter group $\exp(tA)$. Applying the G-signature theorem (6.12) to the element $g_t = \exp(tA)$, we get

$$\text{Sign}(g_t, X) = L(g_t, X) . \tag{8.1}$$

More generally if V is any vector bundle associated to the riemannian structure, we apply (6.19) and obtain

$$\text{Sign}(g_t, X, V) = L(g_t, X, V) . \tag{8.2}$$

We now consider both sides of this formula as functions of t for $0 < t < \varepsilon$. Since $\text{Sign}(g_t, X, V)$ is given by a character of G, it is an analytic function of t, and as $t \to 0$

$$\text{Sign}(g_t, X, V) \longrightarrow \text{Sign}(1, X, V) = \text{Sign}(X, V) . \tag{8.3}$$

On the other hand, $L(g_t, X, V)$ is given by evaluating a cohomology class over X^A and, if we examine the explicit formula for $L(g_t, X, V)$, we see that it is analytic in t but has a pole as $t \to 0$. More precisely, each component of X^A of dimension $2m$ has a pole of order (at most) $n - m$. Thus we have a Laurent series

$$L(g_t, X, V) = \sum_{i=-n}^{\infty} a_i t^i , \tag{8.4}$$

where the coefficients a_i are given by evaluating certain explicit expressions in characteristic classes over X^A. From (8.2), (8.3), and (8.4), we deduce

$$\begin{cases} a_i = 0 & -n \leqq i \leqq -1 \\ a_0 = \text{Sign}(X, V) . \end{cases} \tag{8.5}$$

On the other hand, putting $t = 0$ in (8.1), we have $\text{Sign}(X, V) = L(X, V)$ and so

$$a_0 = L(X, V) . \tag{8.6}$$

Since both sides of (8.6) are linear in V, it follows that (8.6) still holds if we replace V by an element of $K_G(X)$ (associated to the riemmannian structure). The most convenient elements to take are those given by the following lemma.

LEMMA (8.7). *Let $f(t_1, \cdots, t_l)$ be any symmetric polynomial in l variables with integer coefficients. Then there exists $u_f \in R(SO(2l))$ so that in $H^*_{SO(2l)}(\mathbf{Q})$*

$$\operatorname{ch} u_f = f(x_1^2, \cdots, x_l^2) + \text{higher terms}.$$

PROOF. Since the Chern character is a ring homomorphism $R(SO(2l)) \to H^*_{SO(2l)}(\mathbf{Q})$, it is sufficient to show that u exists when f is an elementary symmetric function σ_k of degree k. We now define u_k to be the coefficient of t^k in

$$\textstyle\sum_{i=0}^{2l} \lambda^i t^i (1-t)^{2l-i},$$

where $\lambda^i = \lambda^i(\mathbf{C}^{2l}) = \lambda^i(\mathbf{R}^{2l}) \otimes_{\mathbf{R}} \mathbf{C}$ is regarded as an element of $R(SO(2l))$. Then we have

$$\textstyle\sum_{k=0}^{2l} u_k t^k = (1-t)^{2l} \sum \lambda^i \left(\frac{t}{1-t}\right)^i$$

and so

$$\begin{aligned}\operatorname{ch}\left(\textstyle\sum u_k t^k\right) &= (1-t)^{2l} \textstyle\prod_{i=1}^{l} \left(1 + \frac{t}{1-t} e^{x_i}\right)\left(1 + \frac{t}{1-t} e^{-x_i}\right) \\ &= \textstyle\prod \left(1 + t(e^{x_i} - 1)\right)\left(1 + t(e^{-x_i} - 1)\right).\end{aligned}$$

Hence equating coefficients

$$\begin{aligned}u_k &= \sigma_k(e^{x_1} - 1, \cdots, e^{x_l} - 1, e^{-x_1} - 1, \cdots, e^{-x_l} - 1) \\ &= \sigma_k(x_1, \cdots, x_l, -x_1, \cdots, -x_l) + \text{higher terms}.\end{aligned}$$

Thus $(-1)^k u_{2k} = \sigma_k(x_1^2, \cdots, x_l^2) + \cdots$, and so this is the required element of $R(SO(2l))$.

Now assume $l = 2k$ so that $\dim X = 4k$, and let f, u_f be as above with $\deg f = k$. We consider the element

$$u_f(TX) \in K_G(X)$$

associated to the tangent bundle of X. To calculate $L(g_t, X, u_f(TX))$ we shall need to calculate

$$\operatorname{ch}\left(i^* \cdot u_f(TX)\right)(g_t),$$

(where $g_t = \exp tA$). The normal eigenvalues $e^{\pm i\theta}$ of g_t are now of the form $e^{\pm it\alpha}$ where $\pm i\alpha$ are the eigenvalues of the skew adjoint transformation of the normal plane induced by the vector field A. Note in particular that (for small t) the eigenvalue -1 does not occur, so that $N^g(-1) = 0$. Now to calculate $\operatorname{ch}\left(i^* u_f(TX)\right)(g)$ on a component of X^A of dimension $2m$, we have only to adopt the following prescription. We take ch u_f, replace the last $l - m$ of the variables x_j by $y_j + it\alpha_j$, and then make the symmetric functions of $(x_1^2, \cdots, x_m^2)$

act on TX^A, while the symmetric functions of the y_j corresponding to a given α act on $N_\alpha(X^A)$, the part of the normal bundle defined by α.

Thus we obtain the formula

$$
\begin{aligned}
&L(g_t, X, u_f(TX)) \\
&\quad = \left\{ 2^m \frac{\operatorname{ch} u_f(x_1, \cdots, x_m, \cdots, y_j + it\alpha_j, \cdots)}{\prod \tanh \dfrac{y_j + it\alpha_j}{2}} \mathfrak{L}(X^A) \right\}[X^A] \\
&\quad = \left\{ 2^l \frac{\operatorname{ch} u_f(x_1, \cdots, x_m, \cdots, y_j + it\alpha_j, \cdots)}{t^{l-m}} \prod \left(i\alpha_j + \frac{y_j}{t} \right)^{-1} Q(t) \right\}[X^A]
\end{aligned} \tag{8.8}
$$

where $Q(t)$ is a power series in t with constant term 1. Now the term in $\{\ \}$ of the right degree in (x, y) (namely of degee m) is easily seen to be holomorphic in t. Moreover the value for $t = 0$ involves only the constant term of $Q(t)$ and the first term, namely f, in ch u_f. Thus putting $t = 0$, we deduce

$$
\begin{aligned}
L(X, u_f(TX)) = \{ 2^l f(x_1^2, \cdots, x_m^2, \cdots (y_j + i\alpha_j)^2, \cdots) \\
\prod (i\alpha_j + y_j)^{-1} \}[X^A] .
\end{aligned} \tag{8.9}
$$

On the other hand computing $L(X, u_f(TX))$ directly, we find

$$
L(X, u_f(TX)) = 2^l f(x_1^2, \cdots, x_m^2)[X] . \tag{8.10}
$$

Suppose now that we take f to be a polynomial of degree $q < k$. Then we can proceed as above to calculate

$$
L(g_t, X, \rho \cdot u_f(TX))
$$

where $\rho \in R(T)$ is any character of the torus T, generated by $\exp(tA)$, such that $\rho(g_t)$ is divisible by precisely $t^{2(k-q)}$. We then obtain a formula just like (8.9), except that the left-hand side is now zero. Formally this fits in with (8.10) because, when $\deg f < k$, we have

$$
f(x_1^2, \cdots, x_m^2)[X] = 0 .
$$

Thus we have established the following theorem.

THEOREM (8.11). *Let X be a compact oriented manifold of dimension $4k$, and let A be an infinitesimal isometry. Let X^A denote the zero-set of A, and let $\pm i\alpha_j$ be the eigenvalues of the skew-adjoint transformation N_A induced by A in the normal bundle of X^A (the α_j will be constant on each component). We orient the normal bundle so that, for an oriented basis, N_A is given by the matrix $\bigoplus_j \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}$ where $\alpha_j > 0$. We then take the induced orientation on X^A. Now let $f(t_1, \cdots, t_l)$ be a symmetric polynomial homogeneous of degree $q \leqq k$. Then we have*

$$\{f(\cdots, x_i^2, \cdots, (y_j + i\alpha_j)^2, \cdots) \prod (i\alpha_j + y_j)^{-1}\}[X^A]$$
$$= f(x_1^2, \cdots, x_{2k}^2)[X] \qquad \text{for } q = k$$
$$= 0 \qquad \text{for } q < k .$$

Here the symmetric functions in the x_i^2 act on TX^A (and give its Pontrjagin classes), while the symmetric functions in the y_j (for a fixed α) give the Chern classes of $N_\alpha(X^A)$.

This theorem gives an expression for the various Pontrjagin numbers of X in terms of Pontrjagin numbers and normal eigenvalues of X^A. The simplest case occurs when X^A consists of a finite set of points $\{P\}$ so that (8.11) becomes

$$\sum \varepsilon(P)(-1)^k \frac{f(\alpha_1^2(P), \cdots, \alpha_{2k}^2(P))}{\prod \alpha_j(P)}$$
$$= f(x_1^2, \cdots, x_{2k}^2)[X] \qquad \text{for } \deg f = k \tag{8.12}$$
$$= 0 \qquad \text{for } \deg f < k$$

where $\varepsilon(P) = 1$ if our convention for orientation of T_P agrees with that of X and -1 otherwise.

Formula (8.12) has been proved by direct differential-geometric methods by Bott [9], and similar methods [10] also yield (8.11). Naturally Bott uses the curvature-description of characteristic classes (cf. § 1). Our proof is algebro-topological, because, despite appearances, we have in fact used no analysis. Essentially (8.11) is a simple consequence of the localisation theorem of [5], and computations with characteristic classes.

We pass now to the case of complex structures. If we have a vector field A preserving a complex structure on X, the one-parameter group $\exp(tA)$ lies in the group of all holomorphic automorphisms of X. Unfortunately the connected component of this group is a Lie group but not in general compact. We cannot therefore apply our methods without making a further restriction, whereas the differential-geometric methods of Bott [9] still work. On the other hand, our methods use only the *almost complex structure*: integrability is irrelevant for our topological considerations. Using the formula of (4.6) and calculations entirely analogous to those above, we obtain

PROPOSITION (8.13). *Let X be an almost complex hermitian manifold of dimension l, and let A be a vector field preserving the metric and the almost complex structure. Let X^A denote the zero-set of A. Then the tangent and normal bundles of X^A have natural complex structures, and the skew-hermitian transformation N_A induced by A on the normal bundle of X^A has eigenvalues $i\alpha_j$. Let $f(t_1, \cdots, t_l)$ be a symmetric polynomial homogeneous of degree $q \leqq l$. Then we have*

$$\{f(x_1, \cdots, y_j + i\alpha_j, \cdots) \prod (i\alpha_j + y_j)^{-1}\}[X^A]$$
$$= f(x_1, \cdots, x_l)[X] \qquad \text{for } q = l$$
$$= 0 \qquad \text{for } q < l .$$

Here the symmetric functions in the x_i act on TX^A (and give its Chern classes), while the symmetric functions in the y_i (for fixed α) give the Chern classes of $N_\alpha(X^A)$.

In addition to results like (8.12) and (8.13), there are more precise results which use the analysis of [7]. Thus if we return to (8.1) and recall that Sign (g_t, X) depends only on the action of g_t on the cohomology of X, we see that

$$\text{Sign}\,(g_t, X) = \text{Sign}\,(1, X) = \text{Sign}\,(X)$$

is actually *independent of* t. Hence, in addition to the identities of (8.5), we have in this case an infinite sequence of identities obtained by equating to zero all positive powers of t in the expansion (8.4) for $V = 1$.

Similar remarks apply to Kähler manifolds. Thus if G is a compact connected group of automorphisms of a Kähler manifold, then G acts trivially on the sheaf cohomology group $H^q(X, \mathcal{O})$—because these are canonically isomorphic to subspaces of the complex cohomology groups $H^q(X; \mathbf{C})$.

9. Miscellaneous special cases

The special cases of the index and Lefschetz theorems we have discussed in the preceding sections were all ones arising from the geometry of some structure. In this section we discuss a number of special cases not of this type.

Consider first the case of *differential operators on odd-dimensional manifolds*. The symbol σ of a differential operator of order r satisfies the symmetry condition

$$\sigma(\alpha(\xi)) = (-1)^r \sigma(\xi) , \tag{9.1}$$

where ξ is a cotangent vector, and $\alpha(\xi) = -\xi$ is the antipodal map on TX. For the class $[\sigma]$ in $K(TX)$, the factor $(-1)^r$ can be ignored; we have

$$\alpha^*[\sigma] = [\sigma] \qquad \in K(TX)$$

and so

$$\text{ch}\,\alpha^*[\sigma] = \text{ch}\,[\sigma] \qquad \in H^*(TX; \mathbf{Q}) .$$

Hence by the Index Theorem (2.12), we have

$$\begin{aligned}\text{index } \sigma &= (-1)^n \text{ ch } [\sigma]\cdot \mathcal{J}(X)[TX] \\ &= (-1)^n \alpha^* \text{ ch } [\sigma]\cdot \mathcal{J}(X)[\alpha_*(TX)] \\ &= (-1)^n \text{ ch } [\sigma]\cdot \mathcal{J}(X)\cdot(-1)^n[TX] \\ &= -\text{index } \sigma\ ,\end{aligned}$$

and so index $\sigma = 0$. The crucial point in this is of course that the map α on TX changes the orientation if n is odd. Thus we have established

PROPOSITION (9.2). *The index of an elliptic differential operator on an odd dimensional compact manifold is zero.*

Remark. The full index theorem is not needed to prove (9.2). It is not difficult to prove in fact that, for a differential operator d on an odd-dimensional manifold, the symbol gives an element of finite order in $K(TX)$. As soon as one knows that the index is a homomorphism $K(X) \to \mathbf{Z}$, this implies that index $d = 0$.

For Lefschetz numbers, the analogue of (9.2) does not necessarily hold. In fact the map $\theta \mapsto -\theta$ of the circle has a non-trivial Lefschetz number (equal to 2) for the de Rham complex, i.e., for the ordinary derivative $f \mapsto df$. However if we consider only orientation preserving actions, the situation is different. In fact if X is oriented, and $g\colon X \to X$ preserves the orientation, then the bundle $N^g(-1)$ (i.e., the part of the normal bundle to X^g with eigenvalue -1) has even dimension. Since all the bundles $N^g(\theta)$ corresponding to other eigenvalues always have even dimension, it follows that

$$\dim X^g \equiv \dim X \mod 2\ .$$

If now u is the symbol of a G-invariant differential operator d, then the restriction i^*u to TX^g will still satisfy the symmetry condition (9.1). Hence, if $\dim X$ is odd, (9.2), together with the Lefschetz theorem (3.9), implies that the Lefschetz number $L(g, u) = 0$. Thus we have

PROPOSITION (9.3). *Let X be a compact oriented manifold of odd dimension, G a compact Lie group of orientation preserving actions of X, and d an elliptic G-invariant differential operator on X. Then the G-index of d is zero.*

In the remainder of this section we shall consider examples in which the vector bundles are trivial. In other words, we consider elliptic operators acting on *systems of functions*. The symbol of such an operator is then a continuous map

$$\sigma\colon S(X) \longrightarrow GL(N, \mathbf{C})\ ,$$

where $S(X)$ is the unit sphere bundle of X, and N is the rank of the system.

We can therefore consider the induced homomorphism in cohomology[15]

$$\sigma^*: H^*(GL(N, \mathbf{C})) \longrightarrow H^*(S(X)) .$$

We recall now a few facts about the cohomology of $GL(N, \mathbf{C})$ and the relation of σ^* with characteristic classes.

First we recall that $U(N)$ is deformation retract of of $GL(N, \mathbf{C})$ so that

$$H^*(GL(N, \mathbf{C})) \cong H^*(U(N)) .$$

Then $H^*(U(N))$ is an exterior algebra generated by elements

$$h_i^N \in H^{2i-1}(U(N)) \qquad i = 1, \cdots, N .$$

These elements have the following additional properties:

(i) the restriction homomorphism takes h_i^N to h_i^{N-1} (for $i < N$),

(ii) $h_N^N = \pi^*(u_N)$ where $\pi: U(N) \to U(N)/U(N-1) = S^{2N-1}$ is the natural map and $u_N \in H^{2N-1}(S^{2N-1})$ is the natural generator[16].

In view of (i) we write h_i instead of h_i^N.

Since the harmonic forms on $U(N)$ are just the bi-invariant forms, one can easily write explicit differential forms w_i representing h_i.

Suppose now that B is a closed subspace of a compact space A and that

$$f: B \longrightarrow GL(N, \mathbf{C})$$

is a continuous map. Then (extending f to a map of A into $\mathrm{End}(\mathbf{C}^n)$), we get a complex on A exact on B, and so an element

$$[f] \in K(A, B) .$$

The characteristic classes of $[f]$ are related to the classes f^*h_i by $c_i[f] = \delta f^*h_i$ where $\delta: H^*(B) \to H^*(A, B)$ is the coboundary homomorphism. This then implies that

$$\mathrm{ch}\,[f] = \delta\left\{\sum_{i=1}^{N} \frac{(-1)^{i-1} f^*h_i}{(i-1)!}\right\} .$$

Returning now to our symbol σ we see that

$$\mathrm{ch}\,[\sigma] = \delta\left\{\sum_{i=1}^{N} \frac{(-1)^{i-1} \sigma^*h_i}{(i-1)!}\right\} \qquad \in H^*(TX)$$

where

$$\delta: H^*(SX) \longrightarrow H^*(BX, SX) \cong H^*(TX)$$

is the coboundary, and BX is the unit ball bundle of TX. Substituting this formula for $\mathrm{ch}\,[\sigma]$ in the index theorem (2.12), and using the formula

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[15] Although $GL(N, \mathbf{C})$ is not compact, we take cohomology here with arbitrary (not compact) supports.

[16] S^{2N-1} is oriented as the boundary of the ball in $\mathbf{C}^N$.

$$\delta u[TX] = u[SX]$$

for any $u \in H^*(SX)$, we deduce

PROPOSITION (9.4). *Let $\sigma: SX \to GL(N, \mathbf{C})$ be the symbol of an elliptic $N \times N$-system d. Then*

$$\text{index } d = (-1)^n \left\{ \left(\sum_{i=1}^{N} \frac{(-1)^{i-1} \sigma^* h_i}{(i-1)!} \right) \mathcal{J}(X) \right\} [SX]$$

where $h_i \in H^{2i-1}(GL(N, \mathbf{C}))$ are the generators normalized as in (ii) *above, $\mathcal{J}(X)$ denotes the Index class of X and* $\dim X = n$.

As a simple example, suppose X is a hypersurface in $\mathbf{R}^{n+1}$. Then all Pontrjagin classes of Y are zero, and so $\mathcal{J}(X) = 1$. Hence as a Corollary of (9.4), we have

COROLLARY (9.5). *Let X be a hypersurface in $\mathbf{R}^{n+1}$, σ the symbol of an elliptic $N \times N$-system d on X. Then*

$$\text{index } d = (-1)^{N+n-1} \frac{\sigma^* h_n}{(n-1)!} [SX] \qquad \text{if } N \geqq n$$

$$= 0 \qquad \text{if } N < n .$$

Remarks. 1. If $N = n$ then, by property (ii) of h_n, we see that $\sigma^* h_n[SX]$ can be interpreted as the degree of the composite map

$$\pi \circ \sigma: SX \longrightarrow S^{2n-1}$$

so that (9.5) takes the simple form

$$\text{index } d = \frac{-\text{degree } (\pi \circ \sigma)}{(n-1)!} . \tag{9.6}$$

2. Since the first Pontrjagin class occurs in dimension 4, it follows that $\mathcal{J}(X) = 1$ if $\dim X \leqq 3$. Hence (9.5) and (9.6) also hold for any X of dimension $\leqq 3$.

In (9.4) the evaluation on $[SX]$ can be reduced as usual to an evaluation on the twisted fundamental class $[X]$. For this we introduce the homomorphism

$$\pi_*: H^*(SX; \mathbf{Q}) \longrightarrow H^*(X; \tilde{\mathbf{Q}})$$

usually called "integration over the fibre". Here $\tilde{\mathbf{Q}}$ denotes twisted coefficients, and π_* lowers dimension by $n - 1$. One way of defining π_* is as the composition

$$H^*(SX; \mathbf{Q}) \xrightarrow{\delta} H^*(TX; \mathbf{Q}) \xrightarrow{\psi^{-1}} H^*(X; \tilde{\mathbf{Q}})$$

where δ is the coboundary of the pair (BX, SX), and ψ is the Thom isomorphism. The formula of (9.4) can now be rewritten

$$\text{index } d = (-1)^n\left\{\left(\sum \frac{(-1)^{i-1}\pi_*\sigma^* h_i}{(i-1)!}\right)\mathcal{J}(X)\right\}[X] . \tag{9.7}$$

Since

$$\dim \pi_*\sigma^* h_i = 2i - n ,$$

the summation in (9.7) need only be taken over $n/2 \leqq i \leqq N$. In particular, if $N \leqq (n-1)/2$, the summation is empty and so index $d = 0$. If $n = 2N$, we have just one term namely $\pi_*\sigma^* h_N$ in dimension 0. Since $\mathcal{J}(X)$ involves only dimensions divisible by 4, it follows that index $d = 0$ unless n is divisible by 4. If n is divisible by 4, we consider the exact cohomology sequence

$$\begin{array}{ccccc} H^{n-1}(SX; \mathbf{Q}) & \xrightarrow{\delta} & H^n(TX; \mathbf{Q}) & \xrightarrow{\alpha} & H^n(BX; \mathbf{Q}) \\ & & \uparrow \psi & & \| \\ & & H^0(X; \tilde{\mathbf{Q}}) & & H^n(X; \mathbf{Q}) . \end{array}$$

Assuming that X is connected, we have

$$\begin{aligned} H^0(X; \tilde{\mathbf{Q}}) &\cong \mathbf{Q} && \text{if } X \text{ is orientable} \\ &= 0 && \text{if } X \text{ is non-orientable} . \end{aligned}$$

Moreover if X is oriented, and 1 denotes the generator of $H^0(X; \mathbf{Q})$, we have

$$\alpha\psi(1) = e(X) ,$$

where $e(X)$ is as usual, the Euler class of X. Hence if $e(X) \neq 0$, we have

$$\text{Im } \delta = \text{Ker } \alpha = 0 ,$$

and so $\delta\sigma^* h_N = 0$. Thus

$$\pi_*\sigma^* h_N = \psi^{-1}\delta\sigma^* h_N = 0 ,$$

and so index $d = 0$. We therefore have the following Corollary of (9.4) giving conditions under which index $d = 0$.

COROLLARY (9.8). *Let d be an elliptic $N \times N$ system on a compact n-manifold where $N \leqq n/2$. Then* index $d = 0$ *unless $n = 2N = 4k$, X is orientable, and has Euler number zero.*

Corollary (9.8) applies in particular when $N = 1$ and $n > 1$. Thus *when* $\dim X > 1$, *the index of an elliptic operator acting on functions is always zero.*

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